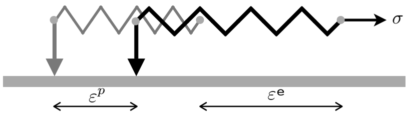
Periodic Homogenization in Elasto-Plasticity

G. F. , Paris and
A. Giacomini, Brescia

International Conference on Nonlinear and Multiscale Partial Differential Equations: Theory, Numerics and Applications
dedicated to Luc Tartar

0/11. Small strain elasto-plasticity – the rheology

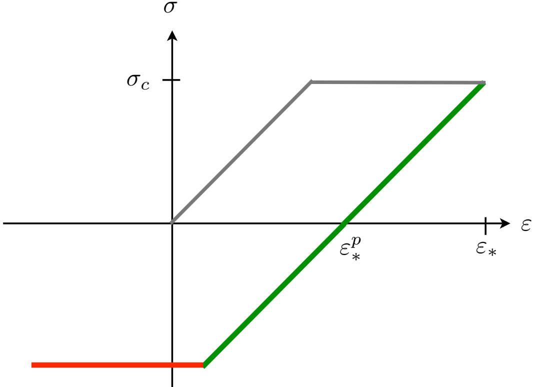
- A model with brake and spring:



$$|\sigma| \leq \sigma_c$$

$$\text{with} \begin{cases} \dot{\varepsilon}^p \geq 0 & \sigma = \sigma_c \\ \dot{\varepsilon}^p = 0 & |\sigma| < \sigma_c \\ \dot{\varepsilon}^p \leq 0 & \sigma = -\sigma_c \end{cases}$$

- Response:



Introduction

1/11. Small strain elasto-plasticity – the formulation

$$\mathbb{M}^{N\times N}_{dev} := \{\tau \text{ symmetric}: \ \text{tr}\ \tau = 0\},\ \tau = \frac{\text{tr}\ \tau}{N}\mathfrak{i} + \tau_D$$

- $Eu := \dfrac{Du + Du^t}{2} = e + p$

$p \in \mathbb{M}^{N\times N}_{dev}$
$\sigma = Ae;\ \operatorname{div}\sigma = 0 \quad \text{in } \Omega$
A : Hooke's law

$\sigma_D \in K := \{\tau \text{ dev.}: f(\tau) \leq 0\}$
with K closed convex
(f conv., $f(0) < 0$, $f \overset{|\tau|\nearrow\infty}{\nearrow} \infty$)
set of admissible stresses

1/11. Small strain elasto-plasticity – the formulation

$$\mathbb{M}^{N\times N}_{dev} := \{\tau \text{ symmetric}: \ \text{tr}\ \tau = 0\},\ \tau = \frac{\text{tr}\ \tau}{N}\mathfrak{i} + \tau_D$$

- $Eu := \dfrac{Du + Du^t}{2} = e + p$

$p \in \mathbb{M}^{N\times N}_{dev}$

$\sigma = Ae;\ \operatorname{div} \sigma = 0 \quad \text{in } \Omega$

A : Hooke's law

$\sigma_D \in K := \{\tau \text{ dev.} : f(\tau) \le 0\}$
with K closed convex
(f conv., $f(0) < 0$, $f \overset{|\tau|\nearrow\infty}{\nearrow} \infty$)
set of admissible stresses

- Flow rule:

$\dot{p}(t) \in \mathcal{A}\sigma :=$

$$\left\{\tau \in \mathbb{M}^{N\times N}_{dev} : \exists\lambda \ge 0 \text{ s.t. } \tau = \lambda\frac{\partial f}{\partial \tau}(\sigma_D(t)) \text{ and } \lambda f(\sigma_D(t)) = 0\right\}$$

$$\Updownarrow$$

$\dot{p}(t) \in N_K(\sigma_D(t))$, the normal cone to K at $\sigma_D(t) \in \partial K(t)$

1/11. Small strain elasto-plasticity – the formulation

$$\mathbb{M}^{N\times N}_{dev} := \{\tau \text{ symmetric}: \ \text{tr } \tau = 0\},\ \tau = \frac{\text{tr } \tau}{N}\mathfrak{i} + \tau_D$$

- $Eu := \dfrac{Du + Du^t}{2} = e + p$

$p \in \mathbb{M}^{N\times N}_{dev}$

$\sigma = Ae;\ \operatorname{div} \sigma = 0 \quad \text{in } \Omega$

A : Hooke's law

$\sigma_D \in K := \{\tau \text{ dev.} : f(\tau) \leq 0\}$
with K closed convex
(f conv., $f(0) < 0$, $f \overset{|\tau|\nearrow\infty}{\nearrow} \infty$)
set of admissible stresses

- Flow rule:

$\dot{p}(t) \in \mathcal{A}\sigma :=$

$$\left\{\tau \in \mathbb{M}^{N\times N}_{dev} : \exists\lambda \geq 0 \text{ s.t. } \tau = \lambda\frac{\partial f}{\partial \tau}(\sigma_D(t)) \text{ and } \lambda f(\sigma_D(t)) = 0\right\}$$

$$\Updownarrow$$

$\dot{p}(t) \in N_K(\sigma_D(t))$, the normal cone to K at $\sigma_D(t) \in \partial K(t)$

- b.c. : $u(x,t) = w(x,t) \in AC([0,T]; H^{\frac{1}{2}}(\partial_d\Omega; \mathbb{R}^3))$

$\partial_d\Omega$ Dirichlet bdary: open / $\partial_t\Omega := \partial\Omega \setminus \overline{\partial_d\Omega}$: open, no forces

1/11. Small strain elasto-plasticity – the formulation

$$\mathbb{M}^{N\times N}_{dev} := \{\tau \text{ symmetric}: \ \text{tr}\ \tau = 0\},\ \tau = \frac{\text{tr}\ \tau}{N}\mathfrak{i} + \tau_D$$

- $Eu := \dfrac{Du + Du^t}{2} = e + p$

$p \in \mathbb{M}^{N\times N}_{dev}$

$\sigma = Ae;\ \operatorname{div} \sigma = 0 \quad \text{in } \Omega$

A : Hooke's law

$\sigma_D \in K := \{\tau \text{ dev.} : f(\tau) \le 0\}$

with K closed convex

$(f \text{ conv.},\ f(0) < 0,\ f \overset{|\tau|\nearrow\infty}{\nearrow} \infty)$

set of admissible stresses

- Flow rule:

$\dot{p}(t) \in \mathcal{A}\sigma :=$

$$\left\{\tau \in \mathbb{M}^{N\times N}_{dev} : \exists \lambda \ge 0 \text{ s.t. } \tau = \lambda \frac{\partial f}{\partial \tau}(\sigma_D(t)) \text{ and } \lambda f(\sigma_D(t)) = 0\right\}$$

$$\Updownarrow$$

$\dot{p}(t) \in N_K(\sigma_D(t))$, the normal cone to K at $\sigma_D(t) \in \partial K(t)$

- b.c. : $u(x,t) = w(x,t) \in AC([0,T]; H^{\frac{1}{2}}(\partial_d \Omega; \mathbb{R}^3))$

  $\partial_d \Omega$ Dirichlet bdary: open / $\partial_t \Omega := \partial\Omega \setminus \overline{\partial_d \Omega}$: open, no forces

- Existence of an evolution known under C^2-smoothness for $\partial\Omega +$ C^2-smoothness of $\partial_{\partial\Omega}[\partial_d \Omega]$:
  - by viscoplastic approx. (Suquet 1978)
  - through var. evolutions (Dal Maso-De Simone-Mora 2004)

1/11. Small strain elasto-plasticity – the formulation

$$\mathbb{M}^{N\times N}_{dev} := \{\tau \text{ symmetric}: \ \text{tr } \tau = 0\}, \ \tau = \frac{\text{tr } \tau}{N}\mathfrak{i} + \tau_D$$

- $Eu := \dfrac{Du + Du^t}{2} = e + p$

$p \in \mathbb{M}^{N\times N}_{dev}$

$\sigma = Ae; \ \operatorname{div} \sigma = 0 \quad \text{in } \Omega$

A : Hooke's law

$\sigma_D \in K := \{\tau \text{ dev.} : f(\tau) \le 0\}$
with K closed convex
$(f \text{ conv.}, \ f(0) < 0, \ f \overset{|\tau|\nearrow\infty}{\nearrow} \infty)$
set of admissible stresses

- Flow rule:

$\dot{p}(t) \in \mathcal{A}\sigma :=$
$$\left\{\tau \in \mathbb{M}^{N\times N}_{dev} : \exists \lambda \ge 0 \text{ s.t. } \tau = \lambda \frac{\partial f}{\partial \tau}(\sigma_D(t)) \text{ and } \lambda f(\sigma_D(t)) = 0\right\}$$
$$\Updownarrow$$
$\dot{p}(t) \in N_K(\sigma_D(t))$, the normal cone to K at $\sigma_D(t) \in \partial K(t)$

- b.c. : $u(x,t) = w(x,t) \in AC([0,T]; H^{\frac{1}{2}}(\partial_d\Omega; \mathbb{R}^3))$

$\partial_d\Omega$ Dirichlet bdary: open / $\partial_t\Omega := \partial\Omega \setminus \overline{\partial_d\Omega}$: open, no forces

$E(u) = e + p$ kin. compatibility
$$\begin{cases} u \in AC(0,T; BD(\Omega)) \\ e \in AC(0,T; L^2(\Omega; \mathbb{R}^N)) \\ p \in AC(0,T; \mathcal{M}_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{N\times N}_{dev})) \end{cases}$$

1/11. Small strain elasto-plasticity – the formulation

$$\mathbb{M}^{N\times N}_{dev} := \{\tau \text{ symmetric : } \operatorname{tr}\tau = 0\},\ \tau = \frac{\operatorname{tr}\tau}{N}\mathfrak{i} + \tau_D$$

- $Eu := \dfrac{Du + Du^t}{2} = e + p$

$p \in \mathbb{M}^{N\times N}_{dev}$

$\sigma = Ae;\ \operatorname{div}\sigma = 0 \quad \text{in } \Omega$

A : Hooke's law

$\sigma_D \in K := \{\tau \text{ dev. } : f(\tau) \le 0\}$

with K closed convex

$(f \text{ conv.}, f(0) < 0, f \overset{|\tau|\nearrow\infty}{\nearrow} \infty)$

set of admissible stresses

- Flow rule:

$\dot{p}(t) \in \mathcal{A}\sigma :=$

$$\left\{\tau \in \mathbb{M}^{N\times N}_{dev} : \exists\lambda \ge 0 \text{ s.t. } \tau = \lambda\frac{\partial f}{\partial\tau}(\sigma_D(t)) \text{ and } \lambda f(\sigma_D(t)) = 0\right\}$$

$$\Updownarrow$$

$\dot{p}(t) \in N_K(\sigma_D(t))$, the normal cone to K at $\sigma_D(t) \in \partial K(t)$

- b.c. : $u(x,t) = w(x,t) \in AC([0,T]; H^{\frac{1}{2}}(\partial_d\Omega;\mathbb{R}^3))$

  $\partial_d\Omega$ Dirichlet bdary: open / $\partial_t\Omega := \partial\Omega \setminus \overline{\partial_d\Omega}$: open, no forces

b.c. on $\partial_d\Omega$ has been relaxed: $p = [w - u] \odot \nu,\ w - u \perp \nu$

2/11. A remark about stress admissibility – Lipschitz domain Ω

- From div $\sigma = 0 \;+\; \sigma \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym} \cap K)$, we get:

$$(\sigma_D \nu)_\tau (\text{the tangential part of } \sigma\nu) \in (K\nu)_\tau$$

$\uparrow$ a priori well defined as an element of $H_{00}^{-\frac{1}{2}}(\partial_d \Omega; \mathbb{R}^N)$

2/11. A remark about stress admissibility – Lipschitz domain Ω

• From div $\sigma = 0 \; + \; \sigma \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym} \cap K)$, we get:

$$(\sigma_D \nu)_\tau (\text{the tangential part of } \sigma\nu) \in (K\nu)_\tau$$

$\uparrow$ a priori well defined as an element of $H_{00}^{-\frac{1}{2}}(\partial_d\Omega; \mathbb{R}^N)$

$\Downarrow$

• Since $\dot{p} = [\dot{w}(t) - \dot{u}(t)] \odot \nu$ on $\partial_d\Omega$, with $[\dot{w}(t) - \dot{u}(t)] \perp \nu$, we expect a boundary flow rule:

$$[\dot{w}(t) - \dot{u}(t)] \in N_{(K\nu)_\tau}((\sigma_D\nu)_\tau) \text{ on } \partial_d\Omega$$

in fact well defined as an element of $L^\infty(\partial_d\Omega; \mathbb{R}^N)$ $\uparrow$ maybe not unique unless bdary is C^2

2/11. A remark about stress admissibility – Lipschitz domain Ω

- From div $\sigma = 0 \; + \; \sigma \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym} \cap K)$, we get:

$$(\sigma_D \nu)_\tau (\text{the tangential part of } \sigma\nu) \in (K\nu)_\tau$$

$\uparrow$ a priori well defined as an element of $H_{00}^{-\frac{1}{2}}(\partial_d \Omega; \mathbb{R}^N)$

$$\Downarrow$$

- Since $\dot{p} = [\dot{w}(t) - \dot{u}(t)] \odot \nu$ on $\partial_d\Omega$, with $[\dot{w}(t) - \dot{u}(t)] \perp \nu$, we expect a boundary flow rule:

$$[\dot{w}(t) - \dot{u}(t)] \in N_{(K\nu)_\tau}((\sigma_D \nu)_\tau) \text{ on } \partial_d\Omega$$

in fact well defined as an element of $L^\infty(\partial_d\Omega; \mathbb{R}^N)$ $\uparrow$ maybe not unique unless bdary is C^2

$$\not\Uparrow$$

bulk flow rule

3/11. Variational evolution for elasto-plasticity in a nutshell

- Define:

– diss. pot. : $H(p) := \sup\{\sigma_D \cdot p : \sigma \in K\}$

– dissipation: $\mathcal{H}(q) := \int_{\Omega \cup \partial_d \Omega} H\left(\frac{q}{|q|}(x)\right) d|q|$

– total energy: $E(t) := 1/2 \int_\Omega Ae(t) \cdot e(t) dx + \int_0^t \mathcal{H}(\dot{p}(s))\, ds$

At each time t, $(u(t), e(t), p(t))$ compatible triplet is abs. cont. and satisfies

- Global min.: $1/2 \int_\Omega Ae(t) \cdot e(t) dx \leq 1/2 \int_\Omega A\eta \cdot \eta dx + \mathcal{H}(q - p(t))$ for every compatible test triplet (v, η, q) with respect to $w(t)$

- Energy cons.: $\frac{dE}{dt}(t) = \int_{\partial_D \Omega} \sigma(t) \nu \cdot \dot{w}(t) d\mathcal{H}^{N-1}$ (with $\sigma(t) := Ae(t)$)

3/11. Variational evolution for elasto-plasticity in a nutshell

- Define:

– diss. pot. : $H(p) := \sup\{\sigma_D \cdot p : \sigma \in K\}$

– dissipation: $\mathcal{H}(q) := \int_{\Omega \cup \partial_d \Omega} H\left(\frac{q}{|q|}(x)\right) d|q|$

– total energy: $E(t) := 1/2 \int_\Omega Ae(t) \cdot e(t) dx + \int_0^t \mathcal{H}(\dot{p}(s))\, ds$

At each time t, $(u(t), e(t), p(t))$ compatible triplet is abs. cont. and satisfies

- Global min.: $1/2 \int_\Omega Ae(t) \cdot e(t) dx \leq 1/2 \int_\Omega A\eta \cdot \eta dx + \mathcal{H}(q - p(t))$ for every compatible test triplet (v, η, q) with respect to $w(t)$

- Energy cons.: $\frac{dE}{dt}(t) = \int_{\partial_D \Omega} \sigma(t)\nu \cdot \dot{w}(t) d\mathcal{H}^{N-1}$ (with $\sigma(t) := Ae(t)$)

- Proof through time discretisation: Find (u_i, e_i, p_i) kin. compatible solving

$$\min \left\{ 1/2 \int_\Omega Ae \cdot e dx + \mathcal{H}(p - p_{i-1}) \right\}$$

The lower semi-continuity of $\mathcal{H}$ is ensured by Reshetnyak's lower semi-continuity theorem

4/11. Why minimality & relaxation of boundary condition in a nutshell

For (v, η, q) kin. admissible with $w(t)$,

$\mathcal{H}^{hom}(q - p(t)) \geq \int_\Omega \sigma_D(t) \cdot (q - p(t)) = -\int_\Omega \sigma(t) \cdot (\eta - e(t)) =$
$-\int_\Omega Ae(t) \cdot \eta + \int_\Omega Ae(t) \cdot e(t) \geq \frac{1}{2} \int_\Omega Ae(t) \cdot e(t) - \frac{1}{2} \int_\Omega A\eta \cdot \eta \Rightarrow$

$$(u(t), e(t), p(t)) \text{ minimizes } \frac{1}{2} \int_\Omega A\eta \cdot \eta + \mathcal{H}^{hom}(q - p(t))$$

4/11. Why minimality & relaxation of boundary condition in a nutshell

For (v, η, q) kin. admissible with $w(t)$,

$\mathcal{H}^{hom}(q - p(t)) \geq \int_\Omega \sigma_D(t) \cdot (q - p(t)) = -\int_\Omega \sigma(t) \cdot (\eta - e(t)) =$

$-\int_\Omega Ae(t) \cdot \eta + \int_\Omega Ae(t) \cdot e(t) \geq \frac{1}{2} \int_\Omega Ae(t) \cdot e(t) - \frac{1}{2} \int_\Omega A\eta \cdot \eta \Rightarrow$

$$(u(t), e(t), p(t)) \text{ minimizes } \frac{1}{2} \int_\Omega A\eta \cdot \eta + \mathcal{H}^{hom}(q - p(t))$$

• time discret. at t_{i+1}: $\Rightarrow \min_{(v,\eta,q)} \frac{1}{2} \int_\Omega A\eta \cdot \eta + \mathcal{H}^{hom}(q - p_i)$

min. seq. (u_n, e_n, p_n) with $u_n = w(t_{i+1})$ bd. in $W^{1,1} \times L^2 \times L^1$

$$\Downarrow$$

$$\liminf \frac{1}{2} \int_\Omega Ae_n \cdot e_n \geq \frac{1}{2} \int_\Omega Ae_{i+1} \cdot e_{i+1}$$

$$\int_\Omega H(p_n - p_i) \geq \mathcal{H}(p_{i+1} - p_i) = \text{“} \int_\Omega H(p_{i+1} - p_i) + \int_{\partial_d \Omega} H(p_{i+1} - p_i) \text{”}$$

$$\Downarrow$$

Hence the relaxation of the boundary cond.

5/11. Recovering a "classical" evolution: Issues of duality

- Recovering flow rule necessitates good definition of $\langle \sigma_D, p \rangle$:

5/11. Recovering a "classical" evolution: Issues of duality

- Recovering flow rule necessitates good definition of $\langle \sigma_D, p \rangle$:

– below σ and p are arbitrary provided that σ satisfies eqm. + Neumann b.c.+stress adm. & p is assd. to (u, e, p) kin. compatible with w as b.c. –

- First define $\langle \sigma_D, p \rangle$ as a distribution:

$$\langle \sigma_D, p \rangle(\varphi) = -\int_\Omega \varphi\sigma \cdot (e - Ew)\, dx - \int_\Omega \sigma \cdot [(u - w) \odot \nabla\varphi]\, dx$$

$\uparrow$ *OK since* $\sigma \in L^N$

5/11. Recovering a "classical" evolution: Issues of duality

- Recovering flow rule necessitates good definition of $\langle \sigma_D, p \rangle$:

– below σ and p are arbitrary provided that σ satisfies eqm. + Neumann b.c.+stress adm. & p is assd. to (u, e, p) kin. compatible with w as b.c. –

- First define $\langle \sigma_D, p \rangle$ as a distribution:

$$\langle \sigma_D, p \rangle(\varphi) = -\int_\Omega \varphi\sigma \cdot (e - Ew)\, dx - \int_\Omega \sigma \cdot [(u - w) \odot \nabla\varphi]\, dx$$

$\uparrow$ *OK since* $\sigma \in L^N$

- Find sufficient conditions on $\partial_{\partial\Omega}[\partial_d\Omega]$ for $\langle \sigma_D, p \rangle$ to be a finite Radon meas. on all of $\mathbb{R}^N$ with

$$|\langle \sigma_D, p \rangle| \leq \|\sigma_D\|_{L^\infty} |p|, \ \langle \sigma_D, p \rangle_a = \sigma_D \cdot p_a$$

We then say that $\partial_{\partial\Omega}[\partial_d\Omega]$ is admissible

5/11. Recovering a "classical" evolution: Issues of duality

- Recovering flow rule necessitates good definition of $\langle \sigma_D, p\rangle$:

– below σ and p are arbitrary provided that σ satisfies eqm. + Neumann b.c.+stress adm. & p is assd. to (u, e, p) kin. compatible with w as b.c. –

- First define $\langle \sigma_D, p\rangle$ as a distribution:

$$\langle \sigma_D, p\rangle(\varphi) = -\int_\Omega \varphi\sigma \cdot (e - Ew)\, dx - \int_\Omega \sigma \cdot [(u - w) \odot \nabla\varphi]\, dx$$

$\uparrow$ *OK since* $\sigma \in L^N$

- Find sufficient conditions on $\partial_{\partial\Omega}[\partial_d\Omega]$ for $\langle \sigma_D, p\rangle$ to be a finite Radon meas. on all of $\mathbb{R}^N$ with

$$|\langle \sigma_D, p\rangle| \leq \|\sigma_D\|_{L^\infty}|p|, \ \langle \sigma_D, p\rangle_a = \sigma_D \cdot p_a$$

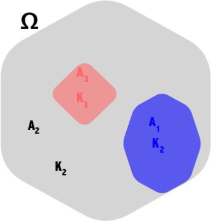
We then say that $\partial_{\partial\Omega}[\partial_d\Omega]$ is admissible

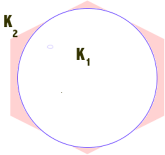
$\Downarrow$

- We recover a bulk flow rule, in Ω and a boundary flow rule on $\partial_d\Omega$

$$\dot{p}_a \in N_K(\sigma_D) \text{ in } \Omega, \quad [\dot{w}(t) - \dot{u}(t)] \in N_{(K\nu)_\tau}((\sigma_D\nu)_\tau) \text{on } \partial_d\Omega$$

6/11. Heterogeneous elasto-plasticity: A multi-phase domain

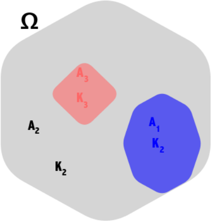


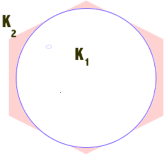


No ordering property of the K_i's

We will need C^2 interfaces

6/11. Heterogeneous elasto-plasticity: A multi-phase domain



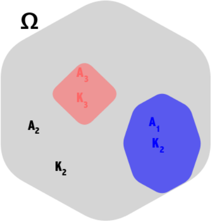


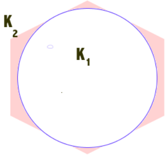
No ordering property of the K_i's
We will need C^2 interfaces

- Define the dissipation :

$H(x,p) := H_i(p) = \sup\{\sigma_D \cdot p : \sigma_D \in K_i\}$ in each phase i. Since we expect p to be a measure, how do we define H on $\bar{\Omega}_i \cap \bar{\Omega}_j$?

6/11. Heterogeneous elasto-plasticity: A multi-phase domain

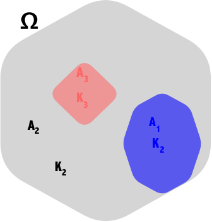


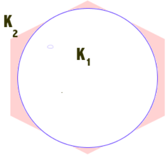


No ordering property of the K_i's
We will need C^2 interfaces

- $\inf\{H_i, H_J\}$?

6/11. Heterogeneous elasto-plasticity: A multi-phase domain





No ordering property of the K_i's
We will need C^2 interfaces

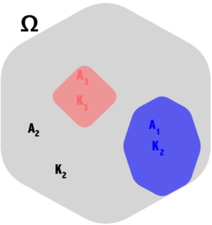
• ~~$\inf\{H_i, H_j\}$~~ because destroys convexity $\Rightarrow$ Inf-convolution:

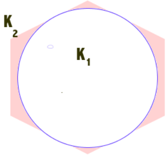
$$H(x,\xi) := \begin{cases} \inf\{H_i(a \odot \nu(x)) + H_j(-b \odot \nu(x)); a - b = c\}, & \text{if } \xi = c \odot \nu(x) \\ \infty, & \text{else} \end{cases}$$

$\Downarrow$

destroys l.s.c./ Need to re-establish l.s.c. of $\mathcal{H}$

6/11. Heterogeneous elasto-plasticity: A multi-phase domain





No ordering property of the K_i's
We will need C^2 interfaces

• ~~$\inf\{H_i, H_j\}$~~ because destroys convexity $\Rightarrow$ Inf-convolution:

$$H(x,\xi) := \begin{cases} \inf\{H(a \odot \nu(x)) + H(-b \odot \nu(x)); a - b = c\}, & \text{if } \xi = c \odot \nu(x) \\ \infty, & \text{else} \end{cases}$$

$\Downarrow$

destroys l.s.c./ Need to re-establish l.s.c. of $\mathcal{H}$

• Existence of a variational evolution : We recover all results of homogeneous case + interfacial flow rule:

$$\dot{u}_i - \dot{u}_j \in N_{(K_i\nu)_\tau \cap (K_j\nu)_\tau}((\sigma_D \nu)_\tau)$$

Homogenization

7/11. Periodic Homogenization

• Rescaled heterogeneous variational evolution: x replaced by x/ε for multiphase torus $\mathcal{Y}$ with C^2 interfaces:

$$A(x/\varepsilon) \text{ Hooke's law, } \; K(x/\varepsilon) \text{ admissibility set,}$$

$$H(x/\varepsilon, \cdot) \text{ dissipation pot.}$$

• $\partial_{\partial\Omega}[\partial_d\Omega]$ admissible

• Method used: 2-scale convergence

8/11. Two-scale kinematics

- Two-scale limits of sequences of $BD(\Omega)$-functions:

$$u_n \overset{*}{\rightharpoonup} u \quad \text{weakly* in } BD(\Omega) \quad \Rightarrow$$

$$Eu_n \overset{w^*\text{-}2}{\rightharpoonup} Eu \otimes \mathcal{L}_y^N + E_y\mu \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}_{sym}^{N\times N})$$

with
$$\begin{cases} \mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^N)), E_y\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}_{sym}^{N\times N}), \\ \mu(F \times \mathcal{Y}) = 0, \ \forall F \text{ Borel} \subseteq \Omega. \end{cases}$$

Moreover,
$$\begin{cases} \mu = \mu_x(y)\eta \otimes \mathcal{L}_y^N, \ \mu_x \in BD(\mathcal{Y}), \ \int_{\mathcal{Y}} \mu_x dy = 0, \ \eta \in \mathcal{M}_b^+(\Omega) \\ E_y\mu = \eta \overset{gen.}{\otimes} E_y\mu_x, \ |E_y\mu_x|(\mathcal{Y}) \neq 0. \end{cases}$$

8/11. Two-scale kinematics

- Two-scale limits of sequences of $BD(\Omega)$-functions:

$$u_n \overset{*}{\rightharpoonup} u \quad \text{weakly* in } BD(\Omega) \quad \Rightarrow$$

$$Eu_n \overset{w^*\text{-}2}{\rightharpoonup} Eu \otimes \mathcal{L}^N_y + E_y\mu \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{N\times N}_{sym})$$

with $\begin{cases} \mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^N)), E_y\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{N\times N}_{sym}), \\ \mu(F \times \mathcal{Y}) = 0, \ \forall F \text{ Borel} \subseteq \Omega. \end{cases}$

Moreover,

$$\begin{cases} \mu = \mu_x(y)\eta \otimes \mathcal{L}^N_y, \ \mu_x \in BD(\mathcal{Y}), \ \int_{\mathcal{Y}} \mu_x dy = 0, \ \eta \in \mathcal{M}^+_b(\Omega) \\ E_y\mu = \eta \overset{gen.}{\otimes} E_y\mu_x, \ |E_y\mu_x|(\mathcal{Y}) \neq 0. \end{cases}$$

- Limit kinematics:

If $\begin{cases} u_n \overset{*}{\rightharpoonup} u & \text{weakly* in } BD(\Omega') \\ e_n \overset{w\text{-}2}{\rightharpoonup} E & \text{two-scale weakly in } L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^{N\times N}_{sym}) \\ p_n \overset{w^*\text{-}2}{\rightharpoonup} P & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D) \end{cases}$

then, $E(x,y)\, \mathcal{L}^N_x \otimes \mathcal{L}^N_y + P - Eu \otimes \mathcal{L}^N_y = E_y\mu$ in $\Omega' \times \mathcal{Y}$.

 Two-scale kinematics

- Two-scale limits of sequences of $BD(\Omega)$-functions:

$$u_n \overset{*}{\rightharpoonup} u \quad \text{weakly* in } BD(\Omega) \quad \Rightarrow$$

$$Eu_n \overset{w^*\text{-}2}{\rightharpoonup} Eu \otimes \mathcal{L}^N_y + E_y\mu \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{N\times N}_{sym})$$

$$\text{with } \begin{cases} \mu \in \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{R}^N)), E_y\mu \in \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{N\times N}_{sym}), \\ \mu(F\times\mathcal{Y}) = 0, \ \forall F \text{ Borel} \subseteq \Omega. \end{cases}$$

Moreover,

$$\begin{cases} \mu = \mu_x(y)\eta \otimes \mathcal{L}^N_y, \ \mu_x \in BD(\mathcal{Y}), \ \int_\mathcal{Y} \mu_x dy = 0, \ \eta \in \mathcal{M}^+_b(\Omega) \\ E_y\mu = \eta \overset{gen.}{\otimes} E_y\mu_x, \ |E_y\mu_x|(\mathcal{Y}) \neq 0. \end{cases}$$

- Limit kinematics:

$$\text{If } \begin{cases} u_n \overset{*}{\rightharpoonup} u & \text{weakly* in } BD(\Omega') \\ e_n \overset{w\text{-}2}{\rightharpoonup} E & \text{two-scale weakly in } L^2(\Omega'\times\mathcal{Y};\mathbb{M}^{N\times N}_{sym}) \\ p_n \overset{w^*\text{-}2}{\rightharpoonup} P & \text{two-scale weakly* in } \mathcal{M}_b(\Omega'\times\mathcal{Y};\mathbb{M}^N_D) \end{cases}$$

then, $E(x,y)\,\mathcal{L}^N_x \otimes \mathcal{L}^N_y + P - Eu \otimes \mathcal{L}^N_y = E_y\mu$ in $\Omega'\times\mathcal{Y}$.

- Further,

$$\mathcal{H}^{hom}(P) := \int_{(\Omega\cup\partial_d\Omega)\times\mathcal{Y}} H(y, \tfrac{P}{|P|})d|P| \leq \liminf_n \mathcal{H}_\varepsilon(p_\varepsilon).$$

9/11. Two-scale quasi-static evolutions

• Under appropriate i.c.'s, there exists a subsequence $\{\varepsilon_n\}$ such that, for all $t \in [0, T]$,

$(u_{\varepsilon_n}(t), e_{\varepsilon_n}(t), p_{\varepsilon_n}(t))$ two-scale converges to $(u(t), E(t), P(t))$

• Under appropriate i.c.'s, there exists a subsequence $\{\varepsilon_n\}$ such that, for all $t \in [0, T]$,
$(u_{\varepsilon_n}(t), e_{\varepsilon_n}(t), p_{\varepsilon_n}(t))$ two-scale converges to $(u(t), E(t), P(t))$

• $(u(t), E(t), P(t))$ is a two-scale quasistatic evolution:

♦ $t \mapsto (u(t), E(t), P(t))$ with correct limit kinematics is abs. cont. and satisfies

♦ Global stability: for every (v, Ξ, Q) with correct limit kinematics

$$\mathcal{Q}^{hom}(E(t)) := \int_{\Omega\times\mathcal{Y}} A(y)E(t)\cdot E(t)dxdy \le \mathcal{Q}^{hom}(\Xi) + \mathcal{H}^{hom}(Q - P(t))$$

♦ Energy equality:

$$\mathcal{Q}^{hom}(E(t)) + \int_0^t \mathcal{H}^{hom}(0, t; \dot{P}(s))ds = \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\partial_D\Omega} \sigma(\tau)\cdot\dot{w}(\tau)\, d\mathcal{H}^{N-1}d\tau$$

where $\sigma(t, x) := \int_{\mathcal{Y}} A(y)E(t, x, y)\, dy$ for a.e. $x \in \Omega$

• Under appropriate i.c.'s, there exists a subsequence $\{\varepsilon_n\}$ such that, for all $t \in [0, T]$,
$(u_{\varepsilon_n}(t), e_{\varepsilon_n}(t), p_{\varepsilon_n}(t))$ two-scale converges to $(u(t), E(t), P(t))$

• $(u(t), E(t), P(t))$ is a two-scale quasistatic evolution:

♦ $t \mapsto (u(t), E(t), P(t))$ with correct limit kinematics is abs. cont. and satisfies

♦ Global stability: for every (v, Ξ, Q) with correct limit kinematics

$$\mathcal{Q}^{hom}(E(t)) := \int_{\Omega\times\mathcal{Y}} A(y)E(t)\cdot E(t)dxdy \leq \mathcal{Q}^{hom}(\Xi) + \mathcal{H}^{hom}(Q - P(t))$$

♦ Energy equality:

$$\mathcal{Q}^{hom}(E(t)) + \int_0^t \mathcal{H}^{hom}(0, t; \dot{P}(s))ds = \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\partial_D\Omega} \sigma(\tau)\cdot\dot{w}(\tau)\, d\mathcal{H}^{N-1}d\tau$$

where $\sigma(t, x) := \int_{\mathcal{Y}} A(y)E(t, x, y)\, dy$ for a.e. $x \in \Omega$

• Not possible to eliminate y-dependence

10/11. Recovering an elasto-plastic model?

- Easily obtained that

$$\mathrm{div}_y \Sigma = 0 \text{ on } \Omega \times \mathcal{Y}, \quad \Sigma_D(x,y) \in K(y) \text{ for } \mathcal{L}^N_x \otimes \mathcal{L}^N_y\text{-a.e. } (x,y) \in \Omega \times \mathcal{Y}$$

$$\mathrm{div}_x \sigma = 0 \text{ in } \Omega, \quad \sigma \cdot \nu = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}_d$$

10/11. Recovering an elasto-plastic model?

- Easily obtained that

$$\mathrm{div}_y \Sigma = 0 \text{ on } \Omega \times \mathcal{Y}, \quad \Sigma_D(x,y) \in K(y) \text{ for } \mathcal{L}^N_x \otimes \mathcal{L}^N_y\text{-a.e. } (x,y) \in \Omega \times \mathcal{Y}$$
$$\mathrm{div}_x \sigma = 0 \text{ in } \Omega, \quad \sigma \cdot \nu = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}_d$$

- What about a flow rule: relates the disintegrated $\dot{P}(t) = \eta(t) \overset{gen.}{\otimes} \dot{P}_x(t,\cdot)$ to $\Sigma_D(t,x,\cdot)$ as in the heterogeneous case.

10/11. Recovering an elasto-plastic model?

- Easily obtained that

$$\mathrm{div}_y \Sigma = 0 \text{ on } \Omega \times \mathcal{Y}, \quad \Sigma_D(x,y) \in K(y) \text{ for } \mathcal{L}^N_x \otimes \mathcal{L}^N_y\text{-a.e. } (x,y) \in \Omega \times \mathcal{Y}$$
$$\mathrm{div}_x \sigma = 0 \text{ in } \Omega, \quad \sigma \cdot \nu = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}_d$$

- What about a flow rule: relates the disintegrated $\dot{P}(t) = \eta(t) \overset{gen.}{\otimes} \dot{P}_x(t,\cdot)$ to $\Sigma_D(t,x,\cdot)$ as in the heterogeneous case.

– bulk flow rule –

For $\mathcal{L}^N_x$-a.e. $x \in \Omega$:

$$\frac{\dot{P}_x(t)}{|\dot{P}_x(t)|}(y) \in N_{K(y)}(\Sigma_D(t,x,y)) \qquad \text{for } \mathcal{L}^N_y\text{-a.e. } y \in \{|\dot{P}_x(t)| > 0\}$$

10/11. Recovering an elasto-plastic model?

- Easily obtained that

$$\mathrm{div}_y \Sigma = 0 \text{ on } \Omega \times \mathcal{Y}, \quad \Sigma_D(x,y) \in K(y) \text{ for } \mathcal{L}^N_x \otimes \mathcal{L}^N_y\text{-a.e. } (x,y) \in \Omega \times \mathcal{Y}$$

$$\mathrm{div}_x \sigma = 0 \text{ in } \Omega, \quad \sigma \cdot \nu = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}_d$$

- What about a flow rule: relates the disintegrated $\dot{P}(t) = \eta(t) \overset{gen.}{\otimes} \dot{P}_x(t,\cdot)$ to $\Sigma_D(t,x,\cdot)$ as in the heterogeneous case.

– bulk flow rule –

For $\mathcal{L}^N_x$-a.e. $x \in \Omega$:

$$\frac{\dot{P}_x(t)}{|\dot{P}_x(t)|}(y) \in N_{K(y)}(\Sigma_D(t,x,y)) \qquad \text{for } \mathcal{L}^N_y\text{-a.e. } y \in \{|\dot{P}_x(t)| > 0\}$$

– interfacial flow rule –

For $\mathcal{L}^N_x$-a.e. $x \in \Omega$ and for every $i \neq j$:

$$\frac{\dot{\mu}^i_x(t,y) - \dot{\mu}^j_x(t,y)}{|\dot{\mu}^i_x(t,y) - \dot{\mu}^j_x(t,y)|} \in \vec{N}_{K_\Gamma(y)}((\Sigma_D(t,x,\cdot)\nu)_\tau(y)) \ \mathcal{H}^{N-1}\text{-a.e.in } \{\dot{\mu}^i_x(t) \neq \dot{\mu}^j_x(t)\}$$

where

$\dot{\mu}_x(t)$ disintegration of $\dot{\mu}(t)$, meas. assoc. with $(\dot{u}(t), \dot{E}(t), \dot{P}(t))$,
$\dot{\mu}^i_x(t)$ and $\dot{\mu}^j_x(t)$ traces on Γ_{ij} of the restrictions of $\dot{\mu}_x(t)$ on $\mathcal{Y}_i$ and $\mathcal{Y}_j$ respectively,
$\vec{N}_{K_\Gamma(y)}(\tau)$ denotes the normal cone (a cone of vectors) to $K_\Gamma(y)$ at a vector $\tau \perp \nu(y)$

A bit complex because relies on various disintegration arguments of both kinematic and static fields

11/11. Recovering an elasto-plastic model bis?

• In essence, $P_x(\cdot,\cdot)$ is, for $\mathcal{L}^N_x$-a.e. x, an internal var. satisfying a flow rule in y that expresses normality at the micro level.......
Well not quite because $\dot{P}_x \neq \frac{d}{dt} P_x$!

11/11. Recovering an elasto-plastic model bis?

• In essence, $P_x(\cdot,\cdot)$ is, for $\mathcal{L}^N_x$-a.e. x, an internal var. satisfying a flow rule in y that expresses normality at the micro level....... *Well not quite because* $\dot{P}_x \neq \frac{d}{dt} P_x$!

• Worse, $P_x(y,t)$ is the internal variable, whereas, thermo- mechanically, it should be $P_y(x,t)$. *Not possible to switch disintegration around!*