Sets of Conductivity and Elasticity Tensors Stable under Lamination

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Abstract

A complete characterization is obtained of any set of constitutive (conductivity or elasticity) tensors that is stable under lamination between any two of the elements of the set. The conditions are local and are expressed in terms of the curvature and tangent plane at points on the boundary of the set. They are checked to hold for several well-known examples of sets stable under a more general process of homogenization. A companion paper investigates the link between these conditions and quasiconvexity. © 1994 John Wiley & Sons, Inc.

1. Introduction

This paper is devoted to a central issue in the field of composite materials: laminate microstructures and the range their effective tensors can take as the microstructure is varied. Laminates are one of the most elementary kind of composite and have been the subject of extensive theoretical and experimental studies.

The importance of lamination is multifold. Explicit formulae are now available for the effective tensors of multi-phase laminates and even for particular classes of two-phase higher rank laminates (by higher rank laminates we mean laminates of laminates) in the context of both conductivity and elasticity; see, for example, Tartar, [31] and [32], Francfort-Murat, [8], Lurie and Cherkaev, [18], and Milton, [20]. The knowledge of these formulae is relevant to many applications, but especially to bounding possible effective behaviors of two phase mixtures: it provides a convenient tool for the generation of whole sets of effective tensors. Optimality of a given bound is usually checked against such sets.

An example that demonstrates the wide range of material properties achievable through lamination is given by Milton in [21] where it is shown that by multiple rank layering of two isotropic elastic materials with positive Poisson's ratios one can generate isotropic composites with Poisson's ratios close to -1. This is surprising because the Poisson's ratio, which measures the transverse expansion of a rod under lengthwise compression, is almost invariably positive in naturally occurring materials.

In all known solvable examples the set of effective tensors generated as the microstructure of the composite varies over all possible configurations has been shown to coincide with the set of effective tensors of laminate microstructures

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(see [19] for a discussion). But it is only a conjecture that this is universal. If true, the impact of such a result would be tremendous because it would provide a relatively easy method to numerically generate the set of all possible effective tensors. One is, however, tempted to conclude the opposite in light of the recent counterexample of Šverák (see [29]) to the famous conjecture of the calculus of variations: "Is rank-1 convexity a sufficient condition for quasiconvexity"; see Ball, [3], for a review, and Kohn and Strang, [15], for comments on the relation with the lamination conjecture.

In any case a complete understanding of the role of lamination would impact on many subjects such as austenite-martensite phase transitions (James, [13], Ball-James, [4], Kohn, [14]), optimal design (Murat-Tartar, [25], Kohn-Strang, [15], Lurie-Cherkaev, [16] and [18], Allaire-Kohn, [1]) and damage mechanics (Francfort-Marigo, [10]).

Our goal in this paper is to be seen as a modest first step in that direction. We propose to characterize sets of effective tensors that are stable under lamination. In this context stability means that any single-rank lamination process performed on any two tensors in the set never generates an effective tensor outside the set. The stability of such sets under multiple rank lamination is then ensured since multiple rank laminations result from a sequence of single rank laminations.

Our analysis will be mathematically elementary and could be considered as an exercise in a calculus course: we use the simplest lamination formulae (in conductivity and elasticity) and draw the consequences of stability under such a formula. In a companion paper by one of us (see [22]) it will be shown that the resulting conditions yield in a natural way a tensorial object of higher complexity (namely a "quasiconvex translation") which can be used to produce bounds on the set of all effective tensors, and not just those obtained through lamination. Such bounds could potentially be used to establish stability of the set under homogenization.

The paper is organized as follows. In Section 2 we set up the framework for our future investigations and recall a few known results. Section 3 is devoted to the derivation of necessary and sufficient conditions for a set of tensors to be stable under lamination. The key observation is to notice that by making an appropriate fractional linear transformation of the tensor space, dependent on the direction of lamination, lamination reduces to a linear average. In this representation stability under lamination in the chosen direction is equivalent to convexity and therefore is a local condition only pertaining to the curvature of the boundary of the set. The corresponding conditions in the original representation (Theorems 3.1, 3.2, and Corollary 3.1) are derived from the standard lamination formula.

Along the way it is established in Remark 3.5 that part of the boundary of the set of all effective tensors is characterized by a minimization of a sum of energies each corresponding to a fixed but arbitrary applied field. Actually the whole boundary of the set of effective tensors can be obtained in such a manner at the expense of adding a sum of dual energies to the previous sum of energies. This latter remark will be expanded upon in the companion paper; see Milton, [22].

In Section 4 the conditions for stability under lamination are explicitly checked in two specific examples pertaining to conductivity: two-phase mixtures at fixed orientation and fixed volume fraction in d dimensions; and the d-dimensional polycrystal. The example of two-phase mixtures has been studied by many authors following the pioneering work of Hashin and Shtrikman (see [12]) who characterized the range of values the effective conductivity can take as the microstructure is varied when the composite and components are isotropic. The example of the three-dimensional polycrystal has also received considerable attention following the notable contribution of Schulgasser (see [27]) who introduced laminate materials to establish the optimality of the arithmetic mean bound (which states that the effective conductivity of an isotropic polycrystal is not greater than the arithmetic average of the three principal conductivities).

2. Setting of the Problem

Throughout the study d will be the spatial dimension and N the dimension of the vector space on which the investigated tensors act: in the conductivity case $N = d$ whereas in the elasticity case the elastic tensors act on symmetric matrices on $\mathbb{R}^d$ so that $N = d(d+1)/2$. Other cases such as thermoelectricity, piezoelectricity, or thermoelasticity could easily be studied with minor modifications to the ensuing analysis. It is only for simplicity that we restrict our attention to the illustrative examples of conductivity and elasticity. We denote by

- $L_s(\mathbb{R}^p)$ the set of symmetric linear mappings on $\mathbb{R}^p$; thus $L_s(\mathbb{R}^N)$ represents the space of our tensors,
- S^{d-1} the unit sphere in $\mathbb{R}^d$, representing the space of the directions of lamination,
- $\mathcal{M} = \{\mathbf{C} \in L_s(\mathbb{R}^N) \mid \alpha\mathbf{I} \leqq \mathbf{C} \leqq \beta\mathbf{I}\}$ where $\leqq$ is to be understood throughout as holding true in the sense of quadratic forms on $\mathbb{R}^N$, and α and β are positive constants [these are introduced to ensure uniform (very strong) ellipticity of the conductivity (elasticity) equations when the conductivity (elasticity) tensor field $\mathbf{C}(\mathbf{x})$ is restricted to lie in $\mathcal{M}$ for all $\mathbf{x}$]. The notion of effective behavior is attached to statements of convergence associated with sequences of tensor fields $\mathbf{A}^\varepsilon$ (with $\mathbf{A}^\varepsilon(\mathbf{x}) \in \mathcal{M}$ for all $\mathbf{x}$) as the parameter ε (to be thought of as a scaling parameter) tends to zero. Specifically we appeal to the notion of H-convergence defined as follows:

DEFINITION 2.1. A sequence $\mathbf{A}^\varepsilon$, with $\mathbf{A}^\varepsilon \in L^\infty(\mathbb{R}^d, \mathcal{M})$ for all ε , is said to H-converge to an element $\mathbf{A}^*$ of $L_s(\mathbb{R}^N)$ if and only if for every bounded domain Ω in $\mathbb{R}^d$ and every $\mathbf{f}$ in $(H^{-1}(\Omega))^r$ the solution $(\mathbf{u}^\varepsilon, \mathbf{s}^\varepsilon)$, unique in $(H_0^1(\Omega))^r \times (L^2(\Omega))^N$ of

$$\begin{aligned} \mathbf{s}^\varepsilon &= \mathbf{A}^\varepsilon \underline{\nabla} \mathbf{u}^\varepsilon && \text{in } \Omega\,, \\ \operatorname{div} \mathbf{s}^\varepsilon &= \mathbf{f} && \text{in } \Omega\,, \\ \mathbf{u}^\varepsilon &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{2.1}$$

converges weakly in $(H_0^1(\Omega))^r \times (L^2(\Omega))^N$ as $\varepsilon \to 0$ to the solution $(\mathbf{u}, \mathbf{s})$, unique in $(H_0^1(\Omega))^r \times (L^2(\Omega))^N$, of

$$
\begin{aligned}
\mathbf{s} &= \mathbf{A}^* \underline{\nabla} \mathbf{u} && \text{in } \Omega\,, \\
\operatorname{div} \mathbf{s} &= \mathbf{f} && \text{in } \Omega\,, \\
\mathbf{u} &= 0 && \text{on } \partial\Omega\,.
\end{aligned}
\tag{2.2}
$$

In the above definition $r = 1$ if $N = d$ and $r = d$ if $N = d(d+1)/2$ while $\underline{\nabla}$ identifies with the gradient if $N = d$ and with the strain tensor (symmetrized gradient) if $N = d(d+1)/2$.

In Definition 2.1 the tensor $\mathbf{A}^*$ is said to be the effective tensor of the sequence $\mathbf{A}^\varepsilon$ and we write $\mathbf{A}^\varepsilon \overset{H}{\rightharpoonup} \mathbf{A}^*$. The existence of such a tensor is guaranteed through the following compactness result due to Tartar (Tartar [30]; see also Spagnolo [28]).

THEOREM 2.1. *For any sequence* $\mathbf{A}^\varepsilon$ *satisfying*

$$
\mathbf{A}^\varepsilon \in L^\infty(\mathbb{R}^d; \mathcal{M})\,, \tag{2.3}
$$

there exists a subsequence $\mathbf{A}^{\varepsilon'}(\mathbf{x})$ *and an element* $\mathbf{A}^*$ *of* $L^\infty(\mathbb{R}^d, \mathcal{M})$ *such that*

$$
\mathbf{A}^{\varepsilon'} \overset{H}{\rightharpoonup} \mathbf{A}^*\,, \tag{2.4}
$$

as ε' *tends to zero.*

Considerable attention has been focused on characterizing the set of all possible tensor fields $\mathbf{A}^*(\mathbf{x})$ when $\mathbf{A}^\varepsilon(\mathbf{x})$ is restricted to take values within a set $U \subset \mathcal{M}$ representing the tensors of available component materials. According to the results of Dal-Maso and Kohn in [6] $\mathbf{A}^*(\mathbf{x})$ is a possible effective tensor field if and only if $\mathbf{A}^*(\mathbf{x})$ belongs for every $\mathbf{x}$ to a fixed subset of $\mathcal{M}$ denoted by GU in the notation of Lurie and Cherkaev; see [16]. This set corresponds to the closure of the set of possible effective tensors of composites with periodic structure; see [5].

When $GU = U$ we say U is stable under homogenization. In particular GU itself is always stable under homogenization, i.e., $G(GU) = GU$. This leads to an alternative characterization of GU as the smallest set containing U that is stable under homogenization. Thus an understanding of the sets of tensors which are stable under homogenization would provide a key to the characterization of GU given U. A first step in this direction is to understand which sets of tensors are stable under a specific homogenization process such as lamination. This is the object of the paper.

3. Stability under Lamination: A Local Necessary and Sufficient Condition

Laminate materials refer to special sequences $\mathbf{A}^\varepsilon(\mathbf{x})$. For example a simple laminate of two materials $\mathbf{A}$ and $\mathbf{B}$, layered in direction $\mathbf{n}$, is represented by the sequence

$$
\mathbf{A}^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x}\cdot\mathbf{n})\mathbf{A} + (1 - \chi^\varepsilon(\mathbf{x}\cdot\mathbf{n}))\mathbf{B}\,, \tag{3.1}
$$

where $\chi^\varepsilon(y)$ is a sequence of characteristic functions each taking only the values zero and one. A common choice of $\chi^\varepsilon(y)$ is the periodic function

$$\chi^\varepsilon(y) \begin{array}{ll} = 1 & \text{if } [y/\varepsilon] \leqq \theta\,, \\ = 0 & \text{otherwise}\,, \end{array} \tag{3.2}$$

where $[z]$ denotes the difference between z and the largest integer less than z, and $\theta \in (0,1)$ gives the proportion of material $\mathbf{A}$ in the laminate. A set L of tensors is said to be stable under lamination if each pair of elements, $\mathbf{A}$ and $\mathbf{B}$, in this set, when layered together, always produce an effective tensor $\mathbf{A}^*$ also in this set, i.e., $\mathbf{A}^* \in L$ for all sequences $\mathbf{A}^\varepsilon(\mathbf{x})$ of the form (3.1) with $\chi^\varepsilon(y)$ given by (3.2), $\mathbf{n} \in S^{d-1}$ and $\mathbf{A}, \mathbf{B} \in L$. Stability under lamination is necessary to ensure stability under homogenization. The important question of whether it is also sufficient to ensure stability under homogenization remains open.

Often we need to keep track of a scalar parameter (or several scalar parameters) in addition to the effective tensor. For example this scalar may represent the mass density of the materials, or in the case of two phase composites it may represent the volume fraction of one of the phases in the composite. To this end we consider sequences of tensor-scalar pairs,

$$\mathscr{A}^\varepsilon(\mathbf{x}) = (\mathbf{A}^\varepsilon(\mathbf{x}), a^\varepsilon(\mathbf{x}))\,, \tag{3.3}$$

with

$$\mathbf{A}^\varepsilon(\mathbf{x}) \in \mathscr{M}\,, \qquad a^\varepsilon(\mathbf{x}) \in [\alpha', \beta'] \quad \forall\ \mathbf{x}\,, \tag{3.4}$$

such that as $\varepsilon \to 0$, $\mathscr{A}^\varepsilon(\mathbf{x})$ converges to

$$\mathscr{A}^*(\mathbf{x}) = (\mathbf{A}^*(\mathbf{x}), a^*(\mathbf{x}))\,, \tag{3.5}$$

in the sense that $\mathbf{A}^\varepsilon(\mathbf{x})$ H-converges to $\mathbf{A}^*(\mathbf{x})$ and $a^\varepsilon(\mathbf{x})$ converges weak* in $L^\infty(\mathbb{R}^d)$ to $a^*(\mathbf{x})$. Again one seeks the characterization of the set $G\mathscr{U}$ of all possible limits $\mathscr{A}^*(\mathbf{x})$ when $\mathscr{A}^\varepsilon(\mathbf{x})$ is restricted to take values within a set $\mathscr{U} \subset \mathscr{M} \times [\alpha', \beta']$ representing the tensors of the available component materials together with their associated scalar parameter (such as their mass density). We say $\mathscr{U}$ is stable under homogenization when $G\mathscr{U} = \mathscr{U}$.

For the sequence (3.1) representing a simple laminate of two materials $\mathbf{A}$ and $\mathbf{B}$, with associated scalar parameters a and b layered in direction $\mathbf{n}$, there is an associated scalar parameter sequence

$$a^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x} \cdot \mathbf{n})a + (1 - \chi^\varepsilon(\mathbf{x} \cdot \mathbf{n}))b\,, \tag{3.6}$$

which converges weak* to the constant function

$$a^* = \theta a + (1 - \theta)b = b + \theta(a - b)\,, \tag{3.7}$$

when $\chi^\varepsilon(y)$ is given by the periodic function (3.2). A set $\mathscr{L}$ of tensor-scalar pairs is said to be stable under lamination if and only if $(\mathbf{A}^*, a^*) \in \mathscr{L}$ for all sequences

$\mathbf{A}^\varepsilon(\mathbf{x})$ and $a^\varepsilon(\mathbf{x})$ of the form (3.1) and (3.6) with $\chi^\varepsilon(y)$ given by (3.2), $\mathbf{n} \in S^{d-1}$ and $(\mathbf{A}, a), (\mathbf{B}, b) \in \mathscr{L}$.

Our goal in this section is to identify which sets $\mathscr{L}$ of tensor-scalar pairs are stable under lamination. For simplicity we will only consider sets $\mathscr{L}$ with second-order differentiable smooth boundaries. The analysis yields, as a corollary, an identification of sets L of tensors that are stable under lamination, since L is stable if and only if the associated set

$$\mathscr{L}' \equiv \{(\mathbf{A}, a) \mid \mathbf{A} \in L\,, \quad a = k\} \tag{3.8}$$

is stable where $k \in [\alpha, \beta]$ is a fixed but arbitrary scalar, constant throughout $\mathscr{L}'$. Note that $\mathscr{L}'$ is a degenerate set where a only takes a single value.

Remark 3.1. It should be emphasized that the cross sections at constant a of an arbitrary set $\mathscr{L} \subset \mathscr{M} \times [\alpha', \beta']$ stable under lamination are themselves stable under lamination. Stability of the cross sections, however, does not ensure stability of $\mathscr{L}$. In other words, the lamination closure of a set $\mathscr{U}$ may be larger than the union of the lamination closure of the cross sections of $\mathscr{U}$.

We assume that $\mathscr{L}$ is a connected compact set with smooth boundary $\partial\mathscr{L}$. If $\mathscr{L}$ is to be stable under lamination then for any two points

$$\mathscr{A} = (\mathbf{A}, a)\,, \qquad \mathscr{B} = (\mathbf{B}, b) \tag{3.9}$$

within the set $\mathscr{L}$, and for any direction of lamination $\mathbf{n} \in S^{d-1}$, the trajectory of points $\mathscr{A}^*(\theta, \mathbf{n}) \equiv (A^*(\theta, \mathbf{n}), a^*(\theta))$ resulting from the lamination in direction $\mathbf{n}$ of $\mathscr{A}$ with $\mathscr{B}$ in proportion θ must remain within $\mathscr{L}$ as θ varies between 0 and 1.

To this effect we recall, for any elements $\mathbf{A}$ and $\mathbf{B}$ of $\mathscr{M}$ the formula for the effective tensor $\mathbf{A}^*$ obtained through lamination of $\mathbf{A}$ and $\mathbf{B}$ in proportions θ and $1 - \theta$, with layering direction $\mathbf{n}$ (see Tartar, [32], and Francfort-Murat, [8]),

$$(1 - \theta)(\mathbf{A}^* - \mathbf{A})^{-1} = (\mathbf{B} - \mathbf{A})^{-1} + \theta\mathbf{\Gamma}_{\mathbf{A}}(\mathbf{n})\,, \tag{3.10}$$

where in the conductivity problem

$$\mathbf{\Gamma}_{\mathbf{A}}(\mathbf{n}) = \mathbf{n} \otimes \mathbf{n}/(\mathbf{A} \cdot \mathbf{n} \cdot \mathbf{n})\,, \tag{3.11}$$

whereas in the elasticity problem $\mathbf{\Gamma}_{\mathbf{A}}$ is obtained through its action on any $\mathbf{e}$ in $L_s(\mathbb{R}^d)$

$$\mathbf{\Gamma}_{\mathbf{A}}(\mathbf{n})\mathbf{e} = \frac{1}{2}\{(\mathbf{q}(\mathbf{n})\mathbf{e}\mathbf{n}) \otimes \mathbf{n} + \mathbf{n} \otimes (\mathbf{q}(\mathbf{n})\mathbf{e}\mathbf{n})\}\,, \tag{3.12}$$

in which $\mathbf{q}(\mathbf{n}) \in L_s(\mathbb{R}^d)$ is defined via

$$\eta \cdot (\mathbf{q}(\mathbf{n}))^{-1}\zeta = \mathbf{A} \cdot (\zeta \otimes \mathbf{n}) \cdot (\eta \otimes \mathbf{n})\,, \qquad \forall\, \eta, \zeta \in \mathbb{R}^d\,. \tag{3.13}$$

Let us also recall the linear relation (3.7) giving a^* in terms of a and b. The set $\mathscr{L}$ is not stable under lamination if and only if for some choice of $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{L}$ there exists a $\bar{\theta} \in (0,1)$ and a lamination direction $\mathbf{n} \in S^{d-1}$ such that

$$\mathscr{A}^*(\bar{\theta}, \bar{\mathbf{n}}) \notin \mathscr{L} . \tag{3.14}$$

From (3.7) and (3.10) it is clear that $\mathscr{A}^*(\theta, \mathbf{n})$ varies continuously and smoothly with θ (and with $\mathbf{n}$). Since we have

$$\mathscr{A}^*(1, \bar{\mathbf{n}}) = \mathscr{A} \in \mathscr{L} , \qquad \mathscr{A}^*(0, \bar{\mathbf{n}}) = \mathscr{B} \in \mathscr{L} , \tag{3.15}$$

the continuous smooth dependence of $\mathscr{A}^*(\theta, \bar{\mathbf{n}})$ on θ in conjunction with the fact that $\mathscr{L}$ is closed implies that there exist $\theta_-, \theta_+ \in [0,1]$ with $\theta_- < \theta_+$ such that

$$\begin{aligned} &\mathscr{A}^*(\theta_-, \bar{\mathbf{n}}) \in \partial\mathscr{L} , \qquad &&\mathscr{A}^*(\theta_+, \bar{\mathbf{n}}) \in \partial\mathscr{L} , \\ &\mathscr{A}^*(\theta, \bar{\mathbf{n}}) \notin \mathscr{L} , \qquad &&\forall\, \theta \in (\theta_-, \theta_+) . \end{aligned} \tag{3.16}$$

Now consider the effective tensors $\mathscr{A}^*_-$ and $\mathscr{A}^*_+$ associated with $\mathscr{A}^*(\theta_-, \bar{\mathbf{n}})$ and $\mathscr{A}^*(\theta_+, \bar{\mathbf{n}})$ respectively. The points $\mathscr{A}(\theta, \bar{\mathbf{n}})$ with $\theta_+ > \theta > \theta_-$ may be viewed as resulting from the layering of $\mathscr{A}^*_-$ and $\mathscr{A}^*_+$ This is easily seen as a consequence of the following alternative approach to layering.

Let us introduce a reference material $\sigma_0\mathbf{I}$ where $\sigma_0 < \alpha$ or $\sigma_0 > \beta$. We then consider the mapping $\mathbf{S}$ from $\mathscr{M}$ into L_s defined as

$$\mathbf{S}(\mathbf{A}) = \sigma_0(\sigma_0\mathbf{I} - \mathbf{A})^{-1} , \tag{3.17}$$

and further introduce, for any $\mathbf{n} \in S^{d-1}$, the Fourier component $\mathbf{\Gamma}(\mathbf{n})$ in direction $\mathbf{n}$ of the Green's function for the underlying equilibrium equations, given by

$$\begin{gathered} \mathbf{\Gamma}(\mathbf{n}) = \mathbf{n} \otimes \mathbf{n} \quad \text{if} \quad N = d , \\ \mathbf{\Gamma}(\mathbf{n})\mathbf{e} = \mathbf{en} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{en} - (\mathbf{e} \cdot \mathbf{n} \cdot \mathbf{n})\mathbf{n} \otimes \mathbf{n} , \ \forall\ \mathbf{e} \in L_s(\mathbb{R}^d) \text{ if } N = d(d+1)/2 . \end{gathered} \tag{3.18}$$

The reader is invited to refer to Milton, [20], Section 2, for further details. Note that $\mathbf{\Gamma}(\mathbf{n})$ satisfies

$$\mathbf{\Gamma}(\mathbf{n})^2 = \mathbf{\Gamma}(\mathbf{n}) . \tag{3.19}$$

Remark 3.2. It can readily be verified (Milton, [20], Sections 4 and 11) that in both settings,

$$\mathbf{\Gamma}_A(\mathbf{n}) = \mathbf{\Gamma}(\mathbf{n})[\mathbf{\Gamma}(\mathbf{n})\mathbf{A}\mathbf{\Gamma}(\mathbf{n})]^{-1}\mathbf{\Gamma}(\mathbf{n}) , \tag{3.20}$$

where the bracketed inverse is to be understood as taken on the range of $\mathbf{\Gamma}(\mathbf{n})$.

Finally we consider the mapping $W_\mathbf{n}$ from $\mathscr{M}$ into $\mathscr{M}$ defined as

$$W_\mathbf{n}(\mathbf{A}) = (\mathbf{S}(\mathbf{A}) - \mathbf{\Gamma}(\mathbf{n}))^{-1} . \tag{3.21}$$

If $\mathbf{A}^*$ is obtained through lamination of $\mathbf{A}$ and $\mathbf{B}$ in proportions $\theta, 1-\theta$ with layering direction $\mathbf{n}$, then according to Milton, [20], equation (4.11), the resulting tensor $\mathbf{A}^*$ satisfies

$$W_{\mathbf{n}}(\mathbf{A}^*) = \theta W_{\mathbf{n}}(\mathbf{A}) + (1-\theta) W_{\mathbf{n}}(\mathbf{B}) \,. \tag{3.22}$$

Thus in the right "coordinates," namely the coordinates given through the transformation $W_{\mathbf{n}}$ (which are dependent on the particular direction of lamination $\mathbf{n}$) the lamination formula (3.10) reduces to a simple arithmetic average. In the context of conductivity a related observation was made by Tartar (see [31]) who noted that certain combinations of the matrix elements of the conductivity tensor average linearly under lamination.

Associated with the transformation $W_{\mathbf{n}}$ is a transformation $\mathcal{W}_{\mathbf{n}}$ from $\mathcal{M} \times \mathbb{R}$ into $L_s(\mathbb{R}^N) \times \mathbb{R}$ defined via

$$\mathcal{W}_{\mathbf{n}}(\mathcal{A}) = \mathcal{W}_{\mathbf{n}}(\mathbf{A}, a) = (W_{\mathbf{n}}(\mathbf{A}), a) \,. \tag{3.23}$$

According to (3.22) and (3.7) stability under lamination in direction $\mathbf{n}$ is equivalent to the requirement that the set $\mathcal{W}_{\mathbf{n}}(\mathcal{L})$ be convex. We have thus established a direct link between stability under lamination and convexity: $\mathcal{L}$ is stable under lamination if and only if $\mathcal{W}_{\mathbf{n}}(\mathcal{L})$ is convex for all $\mathbf{n}$.

Remark 3.3. This generalizes a simple observation of Lurie and Cherkaev in [16]. For the conductivity problem the set G is invariant under rotations in the sense that for any given rotation matrix $\mathbf{R}$, $\mathbf{R}^T\mathbf{A}^*\mathbf{R} \in G$ if and only if $\mathbf{A}^* \in G$. Hence G can be represented by a region in $\mathbb{R}^d$, where each point $(\lambda_1, \lambda_2 \dots \lambda_d)$ and its permutations represents the set of eigenvalues of a possible tensor $\mathbf{A}^*$. If two materials, represented by points $(\lambda_1^A, \lambda_2^A \dots \lambda_d^A)$, $(\lambda_1^B, \lambda_2^B, \dots \lambda_d^B)$ have their principle axes of conductivity aligned and are laminated, in proportions θ and $1-\theta$, in the direction of one of these axes, say the j-th axis, then it is well known (and follows from the lamination formula (3.10)) that the eigenvalues of the conductivity tensor $\mathbf{A}^*$ of the resulting laminate are given by the arithmetic mean

$$\lambda_i^* = \theta\lambda_i^A + (1-\theta)\lambda_i^B \qquad \text{for } i \neq j \,, \tag{3.24}$$

and in the j-th direction by the harmonic mean,

$$1/\lambda_j^* = \theta/\lambda_j^A + (1-\theta)/\lambda_j^B \,. \tag{3.25}$$

In [16] Lurie and Cherkaev noted that the set G, being stable under this restricted type of lamination, must be a convex set when represented in the co-ordinates $(\lambda_1, \lambda_2, \dots, \lambda_{j-1}, 1/\lambda_j, \lambda_{j+1}, \dots \lambda_d)$ for each $j \in \{1, 2, \dots, d\}$. When $d = 2$ this convexity condition, turns out to be sufficient to guarantee the stability of G under homogenization, and in particular under other more general types of lamination [see Francfort and Murat, [9], Francfort and Milton, [7], and Lurie and Cherkaev, [16]]. When $d = 3$ the work of Nesi (see [26]) on the effective conductivity of polycrystals, strongly suggests that it is necessary to laminate in other directions, not just in the directions of principal conductivity. In other words, when $d \geqq 3$, the convexity conditions of Lurie and Cherkaev in [16] are necessary but most likely not sufficient to guarantee stability under lamination for the conductivity problem.

From (3.22) we conclude that if $\theta_- < \theta < \theta_+$ and if we set

$$\theta = \lambda\theta_- + (1 - \lambda)\theta_+ \,, \tag{3.26}$$

then we have

$$\begin{aligned} \mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{A}(\theta, \bar{\mathbf{n}})) &= \theta\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{A}) + (1 - \theta)\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{B}) \\ &= \lambda(\theta_-\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{A}) + (1 - \theta_-)\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{B})) \\ &\quad + (1 - \lambda)(\theta_+\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{A}) + (1 - \theta_+)\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{B})) \\ &= \lambda\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{A}(\theta_-, \bar{\mathbf{n}})) + (1 - \lambda)\mathcal{W}_{\bar{\mathbf{n}}}(\mathscr{A}(\theta_+, \bar{\mathbf{n}})) \,, \end{aligned} \tag{3.27}$$

i.e., $\mathscr{A}(\theta, \bar{\mathbf{n}})$ results from the layering of $\mathscr{A}(\theta_-, \bar{\mathbf{n}})$ and $\mathscr{A}(\theta_+, \bar{\mathbf{n}})$ in volume fractions λ and $1 - \lambda$ along the direction $\bar{\mathbf{n}}$.

We have thus proved that a necessary and sufficient condition for $\mathscr{L}$ to be stable under lamination is that for any pair $\mathscr{A} = (\mathbf{A}, a)$ and $\mathscr{B} = (\mathbf{B}, b)$ of $\partial\mathscr{L}$, the layering in any direction $\mathbf{n}$ of S^{d-1} of $\mathscr{A}$ and $\mathscr{B}$ remain within $\mathscr{L}$.

Remark 3.4. It is easily verified that the set $\mathcal{W}_{\mathbf{n}}(\mathscr{L})$ is stable under lamination if and only if for every $\mathbf{m}$ of S^{d-1}, for any elements

$$\mathscr{A}_0 = (\mathscr{A}_0, a_0) \,, \qquad \mathscr{B}_0 = (\mathscr{B}_0, b_0) \tag{3.28}$$

of $\mathcal{W}_{\mathbf{n}}(\mathscr{L})$, and for any θ in [0,1], one has

$$\mathscr{C}_0 = (\mathbf{C}_0, \theta a_0 + (1 - \theta)b_0) \in \mathcal{W}_{\mathbf{n}}(\mathscr{L}) \,, \tag{3.29}$$

with $\mathbf{C}_0$ given by

$$\begin{aligned} \theta[\mathbf{A}_0^{-1} + \mathbf{\Gamma}(\mathbf{n}) - \mathbf{\Gamma}(\mathbf{m})]^{-1} + (1 - \theta)[\mathbf{B}_0^{-1} + \mathbf{\Gamma}(\mathbf{n}) - \mathbf{\Gamma}(\mathbf{m})]^{-1} \\ = [\mathbf{C}_0^{-1} + \mathbf{\Gamma}(\mathbf{n}) - \mathbf{\Gamma}(\mathbf{m})]^{-1} \,. \end{aligned} \tag{3.30}$$

This characterization, however, will not be explored any further.

Actually the convex (and connected) character of $\mathcal{W}_{\mathbf{n}}(\mathscr{L})$ (see (3.27)) permits us to further restrict the necessary and sufficient condition for stability under lamination. Indeed convexity is a local condition on the boundary of $\mathcal{W}_{\mathbf{n}}(\mathscr{L})$: namely it suffices to consider points on $\partial\mathcal{W}_{\mathbf{n}}(\mathscr{L})$ in a neighborhood of $\mathcal{W}_{\mathbf{n}}(\mathscr{A})$ and to express convexity for those points.

An appropriate choice of σ_0, such as $\sigma_0 > \beta$, (see equations (3.17) and (3.21)) guarantees that the mapping $\mathcal{W}_{\mathbf{n}}$ is a homeomorphism on a neighborhood of $\mathscr{L}$. Thus in particular the boundary $\partial\mathscr{L}$ of $\mathscr{L}$ is mapped onto the boundary $\partial\mathcal{W}_{\mathbf{n}}(\mathscr{L})$

of $\mathscr{W}_{\mathbf{n}}(\mathscr{L})$. Hence stability under lamination will be ensured upon considering pairs $\mathscr{A}$ and $\mathscr{B}$ both on $\partial\mathscr{L}$ *and* close to one another.

Assuming smoothness the convexity condition can be expressed in terms of constraints on the first and second derivatives of the surface $\partial\mathscr{W}_{\mathbf{n}}(\mathscr{L})$. In principle we could determine these constraints, and then by inverting the transformations (3.17) and (3.21) map them back to constraints on the first and second derivatives of the surface $\partial\mathscr{L}$. (The chain rule of differentiation ensures that higher derivatives of the surface $\partial\mathscr{L}$ will not enter into these constraints.) In practice it is easier to work with the lamination formula (3.10) with points $\mathscr{A}$ and $\mathscr{B}$ on ∂L. What these considerations show, however, is that restrictions on third and higher derivatives of the surface $\partial\mathscr{L}$ are not needed.

A local parametrization H in a neighborhood $\mathscr{V}$ of a point $\mathscr{B} = (\mathbf{B}, b)$ on the smooth surface $\partial\mathscr{L}$ of $\mathscr{L}$ is chosen so that

$$\text{(3.31)}\qquad \begin{aligned} &H(\mathbf{C},c) \geqq 0 \quad \text{iff} \quad (\mathbf{C},c) \in \mathscr{V} \cap \mathscr{L}\,,\\ &H(\mathbf{C},c) = 0 \quad \text{iff} \quad (\mathbf{C},c) \in \mathscr{V} \cap \partial\mathscr{L} \equiv \mathscr{V}'\,,\\ &\partial H/\partial \mathbf{C} \ \text{ and } \ \partial H/\partial c \ \text{ are not both zero when } \ (\mathbf{C},\ c) \in \mathscr{V}'\,, \end{aligned}$$

with H smooth. We consider a second point $\mathscr{A} = (\mathbf{A}, a) \in \mathscr{L}$ and apply the lamination formula. From (3.10) it is deduced that for θ small enough we have

$$\text{(3.32)}\qquad \begin{aligned} (\mathbf{A}^* - \mathbf{A})/(1-\theta) &= (\mathbf{I} + \theta(\mathbf{B}-\mathbf{A})\boldsymbol{\Gamma}_{\mathbf{A}}(\mathbf{n}))^{-1}(\mathbf{B}-\mathbf{A})\\ &= \mathbf{B} - \mathbf{A} - \theta(\mathbf{B}-\mathbf{A})\boldsymbol{\Gamma}_{\mathbf{A}}(\mathbf{n})(\mathbf{B}-\mathbf{A}) + 0(\theta^2)\,, \end{aligned}$$

which implies

$$\text{(3.33)}\qquad \mathbf{A}^* = \mathbf{B} - \theta(\mathbf{B}-\mathbf{A})[\mathbf{I} + \boldsymbol{\Gamma}_{\mathbf{A}}(\mathbf{n})(\mathbf{B}-\mathbf{A})] + 0(\theta^2)\,.$$

Since $\mathbf{B}$ and its associated scalar parameter b are such that $(\mathbf{B}, b)$ belongs to $\mathscr{V}$ we have

$$\text{(3.34)}\qquad H(\mathbf{B}, b) = 0\,.$$

We now express the stability of the set $\mathscr{L}$ under lamination by imposing that $(\mathbf{A}^*, a^*)$ belongs to $\mathscr{V} \cap \mathscr{L}$ for θ sufficiently small. According to the previous considerations this condition is a necessary and sufficient conditions for stability under lamination. Thus we require

$$\text{(3.35)}\qquad H(\mathbf{A}^*, a^*) \geqq 0\,.$$

In view of (3.7), (3.33) this last inequality reads as

$$\text{(3.36)}\quad H(\mathbf{B},b) - \theta\frac{\partial H}{\partial \mathbf{C}} \cdot [(\mathbf{B}-\mathbf{A})(\mathbf{I} + \boldsymbol{\Gamma}_{\mathbf{A}}(\mathbf{n})(\mathbf{B}-\mathbf{A}))] - \theta\frac{\partial H}{\partial c}(b-a) + 0(\theta^2) \geqq 0\,,$$

where $\partial H/\partial \mathbf{C}$ and $\partial H/\partial c$ denote the partial derivates of $H(\mathbf{C}, c)$ at $(\mathbf{B}, b)$. By virtue of (3.34) and since θ is positive we obtain

$$\text{(3.37)}\qquad \frac{\partial H}{\partial \mathbf{C}} \cdot [(\mathbf{B}-\mathbf{A})(\mathbf{I} + \boldsymbol{\Gamma}_{\mathbf{A}}(\mathbf{n})(\mathbf{B}-\mathbf{A}))] + \frac{\partial H}{\partial c}(b-a) \leqq 0\,,$$

and this must hold true for any element $(\mathbf{A}, a)$ of $\mathcal{L}$ and for any $\mathbf{n} \in S^{n-1}$. Given $(\mathbf{B}, b)$ on the surface $\mathcal{V}'$ and the corresponding derivatives $\partial H/\partial \mathbf{C}$ and $\partial H/\partial c$ of the surface at that point, then (3.37) places global constraints on the location of any other point $(\mathbf{A}, a)$ of $\mathcal{L}$.

Remark 3.5. At any point $\mathcal{B}$ on $\partial\mathcal{L}$ where $\partial H/\partial \mathbf{C}$ is positive semidefinite the term $\frac{\partial H}{\partial \mathbf{C}} \cdot [(\mathbf{B} - \mathbf{A})\mathbf{\Gamma}_\mathbf{A}(\mathbf{n})(\mathbf{B} - \mathbf{A})]$ is non-negative, which implies that

$$\frac{\partial H}{\partial \mathbf{C}} \cdot \mathbf{A} + \frac{\partial H}{\partial c} a \geqq \frac{\partial H}{\partial \mathbf{C}} \cdot \mathbf{B} + \frac{\partial H}{\partial c} b \tag{3.38}$$

for all $(\mathbf{A}, a) \in \mathcal{L}$. Thus $\mathcal{L}$ lies entirely on one side of the tangent hyperplane to $\mathcal{L}$ at $\mathcal{B}$, and in particular this implies the "local convexity" of $\mathcal{L}$ at $\mathcal{B}$.

Set $\partial H/\partial \mathbf{C} = \sum_{i=1}^{N} \mathbf{h}_i \otimes \mathbf{h}_i$, where each $\mathbf{h}_i$ is an eigenvector of $\partial H/\partial \mathbf{C}$ and $\mathbf{h}_i \cdot \mathbf{h}_i$ is the associated (non-negative) eigenvalue. Also set $\partial H/\partial c = \lambda$, and introduce the function

$$E(\mathbf{h}_1, \ldots, \mathbf{h}_N, \lambda, \mathbf{A}, a) = \sum_{i=1}^{N} \mathbf{h}_i \cdot \mathbf{A}\mathbf{h}_i + \lambda a \, . \tag{3.39}$$

Then the equation (3.38) reads as

$$E(\mathbf{h}_1, \ldots, \mathbf{h}_N, \lambda, \mathbf{B}, b) = \min_{(\mathbf{A},a) \in \mathcal{L}} E(\mathbf{h}_1, \ldots, \mathbf{h}_N, \lambda, \mathbf{A}, a) \, . \tag{3.40}$$

In other words, the point $(\mathbf{B}, b)$ realizes the minimum of a sum of energies added together with the scalar parameter multiplied by λ. The significance of this result is as follows. Suppose that we are given the set $\mathcal{U}$ of initial component materials together with their associated scalar parameter and are interested in characterizing the set $G\mathcal{U}$. Then the points on the boundary of $G\mathcal{U}$ which satisfy $\partial H/\partial \mathbf{C} \geqq 0$ (in which H is now a local parametrization of $\partial G\mathcal{U}$) coincide with those points $(\mathbf{B}, b)$ where the function $E(\mathbf{h}_1, \ldots, \mathbf{h}_N, \lambda, \mathbf{A}, a)$ is minimized over all $(\mathbf{A}, a) \in \partial G\mathcal{U}$ as the vectors $\mathbf{h}_1, \ldots, \mathbf{h}_N$ and the parameter λ take all possible values.

We have yet to choose $\mathbf{A}$ and $\mathbf{B}$ close to one another. This is performed by letting $\mathbf{A}$ tend to $\mathbf{B}$ on $\partial\mathcal{L}$; in other words we set

$$\mathcal{A}(t) = (\mathbf{A}(t), a(t)) \qquad 0 \leqq t \leqq 1 \, , \tag{3.41}$$

with $\mathcal{A}(t)$ a twice differentiable trajectory satisfying

$$\mathcal{A}(0) = \mathcal{B} \, , \qquad \mathcal{A}(t) \in \mathcal{V}' \ \forall \ t \in [0, 1] \, , \tag{3.42}$$

with end point derivatives

$$\mathcal{A}'(0) \equiv (\mathbf{A}', a') \, , \qquad \mathcal{A}''(0) \equiv (\mathbf{A}'', a'') \, . \tag{3.43}$$

Since for $0 \leqq t \leqq 1$

$$H(\mathcal{A}(t)) = 0 \, , \tag{3.44}$$

we have

$$(3.45)\quad \begin{aligned} &\frac{\partial H}{\partial \mathscr{A}}(\mathscr{A}(t)) \cdot \mathscr{A}'(t) = 0, \\ &\frac{\partial H}{\partial \mathscr{A} \partial \mathscr{A}}(\mathscr{A}(t)) \cdot \mathscr{A}'(t) \cdot \mathscr{A}'(t) + \frac{\partial H}{\partial \mathscr{A}}(\mathscr{A}(t)) \cdot \mathscr{A}''(t) = 0 , \end{aligned}$$

which, specialized to the time $t = 0$, yields

$$\frac{\partial H}{\partial \mathbf{C}}(\mathscr{B}) \cdot \mathbf{A}' + \frac{\partial H}{\partial c}(\mathscr{B}) a' = 0 ,$$

$$(3.46)\quad \begin{aligned} &\frac{\partial^2 H}{\partial \mathbf{C} \partial \mathbf{C}}(\mathscr{B}) \cdot \mathbf{A}' \cdot \mathbf{A}' + 2 \frac{\partial H}{\partial \mathbf{C} \partial c}(\mathscr{B}) \cdot \mathbf{A}' a' + \frac{\partial^2 H}{\partial c \partial c}(\mathscr{B}) a' a' \\ &\qquad + \frac{\partial H}{\partial \mathbf{C}}(\mathscr{B}) \cdot \mathbf{A}'' + \frac{\partial H}{\partial c}(\mathscr{B}) a'' = 0 . \end{aligned}$$

We set

$$(3.47)\quad \begin{aligned} H_{\mathbf{C}} &= \frac{\partial H}{\partial \mathbf{C}}(\mathscr{B}) , & H_c &= \frac{\partial H}{\partial c}(\mathscr{B}) , \\ H_{\mathbf{CC}} &= \frac{\partial^2 H}{\partial \mathbf{C}} \partial \mathbf{C}(\mathscr{B}) , & H_{cc} &= \frac{\partial H}{\partial c \partial c}(\mathscr{B}) , \quad H_{\mathbf{C}c} = \frac{\partial^2 H}{\partial \mathbf{C} \partial c}(\mathscr{B}) , \end{aligned}$$

so that (3.46) reads

$$(3.48)\quad H_{\mathbf{C}} \cdot \mathbf{A}' + H_c a' = 0 ,$$

$$(3.49)\quad H_{\mathbf{C}} \cdot \mathbf{A}'' + H_c a'' = -[H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' + 2 H_{\mathbf{C}c} \cdot \mathbf{A}' a' + H_{cc}(a')^2] .$$

Equations (3.48) and (3.49) determine the possible $(\mathbf{A}', a')$ and $(\mathbf{A}'', a'')$ when $\mathscr{A}(t)$ is constrained to remain on the boundary of $\mathscr{L}$.

Upon recalling (3.37) we obtain

$$(3.50)\quad \begin{aligned} &H_{\mathbf{C}} \cdot \left[\left(-t\mathbf{A}' - \frac{t^2}{2}\mathbf{A}'' + 0(t^3) \right) \left(\mathbf{I} + \mathbf{\Gamma}_{\mathbf{A}(t)}(\mathbf{n}) \left(-t\mathbf{A}' - \frac{t^2}{2}\mathbf{A}'' + 0(t^3) \right) \right) \right] \\ &\qquad - H_c \left(t a' + \frac{t^2}{2} a'' + 0(t^3) \right) \leqq 0 . \end{aligned}$$

Expanding this to second order in t, and noting from (3.48) that the terms which are linear in t vanish, gives

$$(3.51)\quad H_{\mathbf{C}} \cdot (\mathbf{A}' \mathbf{\Gamma}_{\mathbf{B}}(\mathbf{n}) \mathbf{A}') - \tfrac{1}{2}(H_{\mathbf{C}} \cdot \mathbf{A}'' + H_c a'') \leqq 0,$$

which may be rewritten with the help of (3.49) as

$$H_{\mathbf{C}} \cdot (\mathbf{A}'\boldsymbol{\Gamma}_{\mathbf{B}}(\mathbf{n})\mathbf{A}') + \frac{1}{2}H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' + H_{\mathbf{C}c} \cdot \mathbf{A}'a' + \frac{1}{2}H_{cc}(a')^2 \leqq 0 \,. \tag{3.52}$$

Since the trajectory $\mathscr{A}(t)$ can be freely chosen, subject only to the restrictions (3.42), the inequality (3.52) must hold for all $\mathbf{A}'$ in $L_s(\mathbb{R}^N)$ and a' in $\mathbb{R}$ satisfying (3.48). These are the desired constraints on the second-order derivatives of the surface $H(\mathscr{C}) = 0$.

Finally since $H(\mathbf{C}, c) = 0$ is locally (in $\mathscr{V}$) the equation of the boundary $\partial\mathscr{L}$ of $\mathscr{L}$, and because

$$\mathscr{V}_0' \equiv \mathscr{W}_n(\mathscr{V}') \tag{3.53}$$

forms part of the boundary $\mathscr{W}_{\mathbf{n}}(\mathscr{L})$ any point $\mathscr{C}_0 = (\mathbf{C}_0, c_0)$ of $\mathscr{V}_0 \equiv \mathscr{W}_{\mathbf{n}}(\mathscr{V})$ belongs to $\mathscr{V}_0'$ if and only if

$$H(\mathscr{W}_{\mathbf{n}}^{-1}(\mathscr{C}_0)) = 0 \,. \tag{3.54}$$

As discussed earlier, the convexity of $\mathscr{W}_{\mathbf{n}}(\mathscr{L})$ is equivalent to the stability of $\mathscr{L}$ under lamination in direction $\mathbf{n}$. It suffices to express this convexity in terms of restrictions on the local parameterization (3.53) of $\mathscr{V}_0'$ that only involve derivatives of up to second order of $H\mathscr{W}_{\mathbf{n}}^{-1}$. In view of this the local parameterization of $\mathscr{V}_0$ should only involve derivatives up to second order of H, which permits us to assert that (3.52) is the only restriction on H that needs to be considered (for all $\mathbf{A}', a'$ satisfying (3.48)). Indeed any term of order higher than two in t in (3.50) would involve derivatives of order higher than two in H.

Thus, the following theorem has been established:

THEOREM 3.1. *A connected compact set $\mathscr{L}$ of tensor-scalar pairs with smooth boundary is stable under lamination if and only if, whenever H is a locally smooth parametrization of $\mathscr{L}$ in a neighborhood $\mathscr{V}$, that is*

$$\begin{aligned} &H(\mathscr{C}) \geqq 0 \quad \text{iff} \quad \mathscr{C} \in \mathscr{V} \cap \mathscr{L}\,, \\ &H(\mathscr{C}) = 0 \quad \text{iff} \quad \mathscr{C} \in \mathscr{V} \cap \partial\mathscr{L} \equiv \mathscr{V}'\,, \\ &\partial H/\partial \mathbf{C} \ \text{ and } \ \partial H/\partial c \ \text{ are not both zero when } \ (\mathbf{C}, c) \in \mathscr{V}'\,, \end{aligned} \tag{3.55}$$

then, at any point $\mathscr{B} = (\mathbf{B}, b)$ of $\mathscr{V}'$ and for any $\mathbf{A}'$ in $L_s(\mathbb{R}^N)$ and any a' in $\mathbb{R}$ satisfying

$$H_{\mathbf{C}} \cdot \mathbf{A}' + H_c a' = 0 \,, \tag{3.56}$$

we have

$$H_{\mathbf{C}} \cdot (\mathbf{A}'\boldsymbol{\Gamma}_{\mathbf{B}}(\mathbf{n})\mathbf{A}') + \tfrac{1}{2}\{H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' + H_{cc}(a')^2 + 2H_{\mathbf{C}c} \cdot \mathbf{A}'a'\} \leqq 0 \,, \tag{3.57}$$

for any $\mathbf{n}$ in S^{d-1}, where $H_{\mathbf{C}}, H_c \ldots$ respectively stand for $\frac{\partial H}{\partial \mathbf{C}}(\mathscr{B})$, $\frac{\partial H}{\partial c}(\mathscr{B}), \ldots$.

Remark 3.6. The constraint (3.56) can be regarded as restricting $\mathbf{A}'$ and a' to the tangent plane of the surface at $\mathscr{B}$. Let us introduce the inner product

$$\langle \mathscr{C}, \mathscr{A} \rangle = \mathbf{C} \cdot \mathbf{A} + ca \tag{3.58}$$

between any two tensor scalar pairs $\mathscr{C} = (\mathbf{C}, c)$ and $\mathscr{A} = (\mathbf{A}, a)$, and norm

$$|\mathscr{C}| = \langle \mathscr{C}, \mathscr{C} \rangle^{1/2} = (\mathbf{C} \cdot \mathbf{C} + c^2)^{1/2} . \tag{3.59}$$

Then (3.56) can be rewritten as

$$\langle \mathscr{N}, \mathscr{A}' \rangle = 0 \qquad \text{with } \mathscr{A}' = (\mathbf{A}', a') , \tag{3.60}$$

where

$$\mathscr{N} = (H_{\mathbf{C}}, H_c)/|(H_{\mathbf{C}}, H_c)| \tag{3.61}$$

is independent of the parametrization H, and represents the normal to the surface $H(\mathscr{C}) = 0$.

Remark 3.7. The theorem is easily extended to sets $\mathscr{L}$ with piecewise smooth boundaries. In any neighborhood $\mathscr{V}$ containing only a single smooth boundary $H(\mathscr{C}) = 0$ it is clear that (3.57) must hold for all $\mathbf{A}'$ and a' satisfying (3.56). Consider a point $\mathscr{B}$ at the junction of m smooth boundaries with normals at $\mathscr{B}$ denoted by $\mathscr{N}_1, \mathscr{N}_2, \ldots, \mathscr{N}_m$. Then the convex character of $\mathscr{W}_n(\mathscr{L})$ for all $\mathbf{n}$ implies that the $\mathscr{N}_i$'s point on one side of a hyperplane, and that the set of $\mathscr{C}$ satisfying

$$\langle \mathscr{N}_i, \mathscr{C} \rangle \geqq 0 \qquad \forall\ i \tag{3.62}$$

locally defines $\mathscr{L}$ in the sense that for all smooth trajectories $\mathscr{A}(t)$ satisfying

$$\mathscr{A}(0) = \mathscr{B} , \qquad \mathscr{A}(t) \in \mathscr{L} \text{ for } 0 \leqq t \leqq 1 , \tag{3.63}$$

with end point derivative

$$\mathscr{A}'(0) = (\mathbf{A}', a') \equiv \mathscr{A}' , \tag{3.64}$$

we have

$$\langle \mathscr{N}_i, \mathscr{A}' \rangle \geqq 0 \qquad \forall\ i . \tag{3.65}$$

At points where the boundary $\partial\mathscr{L}$ is smooth we require that the second-order derivative condition (3.57) holds for all $\mathscr{A}'$ satisfying (3.56) and at junctions between smooth boundaries we require that the first-order derivative condition (3.65) holds. It remains to check that these conditions are sufficient to ensure stability of $\mathscr{L}$ under lamination. The first condition (3.57) ensures that for any $\mathbf{n}$ a typical "two-dimensional" cross section of $\mathscr{W}_{\mathbf{n}}(\mathscr{L})$ (i.e., one that is not accidentally aligned with one of the junctions between the piecewise smooth sections $\mathscr{W}_{\mathbf{n}}(\partial\mathscr{L})$) has a piecewise smooth boundary with positive curvature on the smooth sections of the boundary. The second condition implies that the piecewise smooth sections meet with an interior angle of at most π. So these conditions ensure the convexity of the cross section, and hence imply that $\mathscr{W}_{\mathbf{n}}(\mathscr{L})$ must be convex for all $\mathbf{n}$ and consequently that $\mathscr{L}$ must be stable under lamination.

Remark 3.8. Theorem 3.1 and Remark 3.5 can easily be generalized to the case where a is a vector. This is appropriate when, for example, we need to keep track of the volume fractions in multiphase composites. The obvious generalization of (3.57) would hold. Theorem 3.2 below will address more specifically the case where a is a scalar.

If we assume that at a given point $\mathscr{B}$ on $\partial\mathscr{L}$

$$H_c \neq 0\,, \tag{3.66}$$

then a' can be explicitly computed from (3.56) and (3.57) becomes

$$\begin{aligned} &H_{\mathbf{C}}\cdot(\mathbf{A}'\boldsymbol{\Gamma}_{\mathbf{B}}(\mathbf{n})\mathbf{A}') \leqq \\ &\quad -\frac{1}{2}H_{\mathbf{CC}}\cdot\mathbf{A}'\cdot\mathbf{A}' - \frac{H_{cc}}{2H_c^2}(H_{\mathbf{C}}\cdot\mathbf{A}')^2 + \frac{1}{H_c}(H_{\mathbf{C}}\cdot\mathbf{A}')(H_{\mathbf{C}c}\cdot\mathbf{A}')\,, \end{aligned} \tag{3.67}$$

for any $\mathbf{n}$ in S^{d-1}, for any $\mathbf{A}'$ in $L_s(\mathbb{R}^N)$.

Alternatively if at the point $\mathscr{B}$ in question $H_c = 0$ then a' is free to vary and when a' is sufficiently large (3.57) requires $H_{cc} \leqq 0$. Two possibilities arise: The first is when

$$H_c = 0\,, \qquad H_{cc} = 0\,. \tag{3.68}$$

In this event (3.57) implies that for all $\mathbf{A}'$ satisfying

$$H_{\mathbf{C}}\cdot\mathbf{A}' = 0\,, \tag{3.69}$$

we have

$$H_{\mathbf{C}}\cdot(\mathbf{A}'\boldsymbol{\Gamma}_{\mathbf{B}}(\mathbf{n})\mathbf{A}') \leqq -\frac{1}{2}H_{\mathbf{CC}}\cdot\mathbf{A}'\cdot\mathbf{A}'\,, \tag{3.70}$$

$$H_{\mathbf{C}c}\cdot\mathbf{A}' = 0\,. \tag{3.71}$$

The latter implies there exists a constant λ such that

$$H_{\mathbf{C}c} = \lambda H_{\mathbf{C}}\,. \tag{3.72}$$

The second possiblity is that

$$H_c = 0\,, \qquad H_{cc} < 0\,, \tag{3.73}$$

in which case the left-hand side of (3.57) is maximized over a' when

$$a' = -(H_{\mathbf{C}c} \cdot \mathbf{A}')/H_{cc} . \tag{3.74}$$

By substituting this back in (3.57) we deduce that for all $\mathbf{A}'$ satisfying (3.69) we have

$$H_{\mathbf{C}} \cdot (\mathbf{A}'\mathbf{\Gamma_B}(\mathbf{n})\mathbf{A}') \leqq -\frac{1}{2}H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' + \frac{1}{H_{cc}}(H_{\mathbf{C}c} \cdot \mathbf{A}')^2 \tag{3.75}$$

at any point $\mathscr{B}$ where (3.73) applies.

These results are summarized in the following:

THEOREM 3.2. *In the context of Theorem* 3.1 *the conditions* (3.56) *and* (3.57) *hold if and only if one of the three sets of conditions are met*:

(i) $H_c \neq 0$ *and*

$$H_{\mathbf{C}} \cdot (\mathbf{A}'\mathbf{\Gamma_B}(\mathbf{n})\mathbf{A}') \leqq -\frac{1}{2}H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' - \frac{H_{cc}}{2H_c^2}(H_{\mathbf{C}} \cdot \mathbf{A}')^2 + \frac{1}{H_c}(H_{\mathbf{C}} \cdot \mathbf{A}')(H_{\mathbf{C}c} \cdot \mathbf{A}') , \tag{3.76}$$

for all $\mathbf{n} \in S^{d-1}$ *and all* $\mathbf{A}' \in L_s(\mathbb{R}^N)$;

(ii) $H_c = 0$ *and* $H_{cc} < 0$ *and*

$$H_{\mathbf{C}} \cdot (\mathbf{A}'\mathbf{\Gamma_B}(\mathbf{n})\mathbf{A}') \leqq -\frac{1}{2}H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' + \frac{1}{H_{cc}}(H_{\mathbf{C}c} \cdot \mathbf{A}')^2 , \tag{3.77}$$

for all $\mathbf{n} \in S^{d-1}$ *and all* $\mathbf{A}'$ *satisfying*

$$H_{\mathbf{C}} \cdot \mathbf{A}' = 0 ; \tag{3.78}$$

(iii) $H_c = H_{cc} = 0$ *and* $H_{\mathbf{C}c}$ *parallel to* $H_{\mathbf{C}}$ *and*

$$H_{\mathbf{C}} \cdot (\mathbf{A}'\mathbf{\Gamma_B}(\mathbf{n})\mathbf{A}') \leqq -\frac{1}{2}H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' , \tag{3.79}$$

for all $\mathbf{n} \in S^{d-1}$ *and all* $\mathbf{A}'$ *satisfying*

$$H_{\mathbf{C}} \cdot \mathbf{A}' = 0 . \tag{3.80}$$

Theorem (3.1) also has an important corollary, when it is applied to sets $\mathscr{L}$ of the form (3.8) for which we necessarily have $a' = 0$ for all admissible trajectories.

COROLLARY 3.1. *A compact set L of tensors with smooth boundary is stable under lamination if and only if whenever H is a locally smooth parameterization of ∂L in a neighborhood V such that*

$$
\begin{array}{llll}
H(\mathbf{C}) & \geqq 0 & \text{iff} & \mathbf{C} \in V \cap L\,, \\
H(\mathbf{C}) & = 0 & \text{iff} & \mathbf{C} \in V \cap \partial L \equiv V'\,, \\
\partial H/\partial \mathbf{C} & \neq 0 & \text{when} & \mathbf{C} \in V'\,,
\end{array}
\tag{3.81}
$$

then at any point $\mathbf{B}$ *of* V' *and for any* $\mathbf{A}'$ *in* $L_s(\mathbb{R}^N)$ *satisfying*

$$
H_{\mathbf{C}} \cdot \mathbf{A}' = 0\,,
\tag{3.82}
$$

we have

$$
H_{\mathbf{C}} \cdot (\mathbf{A}'\mathbf{\Gamma_B(n)}\mathbf{A}') + \frac{1}{2} H_{\mathbf{CC}} \cdot \mathbf{A}' \cdot \mathbf{A}' \leqq 0\,,
\tag{3.83}
$$

where $H_{\mathbf{C}}$ *and* $H_{\mathbf{CC}}$ *respectively stand for* $\frac{\partial H}{\partial \mathbf{C}}(\mathbf{B})$ *and* $\frac{\partial^2 H}{\partial \mathbf{C} \partial \mathbf{C}}(\mathbf{B})$.

4. Two Examples of Sets Stable under Lamination

In our first example we focus on the conductivity problem ($N = d$) and investigate a specific class of subsets of $\mathcal{M} \times [0, 1]$ of the form

$$
\begin{aligned}
\mathcal{L} = \{(\mathbf{A}, a) \\
\mid \mathbf{A} \in \mathcal{M}, a \in [0,1]\,, \mathbf{A} \geqq \mathbf{A}_0 \ \text{ and } \ \mathrm{Tr}[\mathbf{A}_0(\mathbf{A} - \mathbf{A}_0)^{-1}] + p_1 \leqq p_2/a\}\,,
\end{aligned}
\tag{4.1}
$$

where $\mathbf{A}_0 \in \mathcal{M}$ and p_1 and p_2 are fixed positive constants. The motivation for studying such subsets stems from the form of the well-known trace bounds for the conductivity of two-phase mixtures: see Remark 4.1. Our attention will be exclusively focused on the part of the boundary $\partial\mathcal{L}$ parametrized by

$$
H(\mathbf{C}, c) = -\,\mathrm{Tr}[\mathbf{A}_0(\mathbf{C} - \mathbf{A}_0)^{-1}] - p_1 + p_2/c = 0\,.
\tag{4.2}
$$

Since H_c is clearly non-zero, stability under lamination is guaranteed provided the condition (3.76) of Theorem 3.2 is satisfied. In view of (4.2) this condition written at the point $(\mathbf{B}, b)$ reduces to

$$
\begin{aligned}
&(\mathbf{B} - \mathbf{A}_0)^{-1}\mathbf{A}_0(\mathbf{B} - \mathbf{A}_0)^{-1} \cdot \mathbf{A}'\mathbf{\Gamma_B(n)}\mathbf{A}' \leqq \\
&(\mathbf{B} - \mathbf{A}_0)^{-1}\mathbf{A}_0(\mathbf{B} - \mathbf{A}_0)^{-1} \cdot \mathbf{A}'(\mathbf{B} - \mathbf{A}_0)^{-1}\mathbf{A}' \\
&-(b/p_2)[(\mathbf{B} - \mathbf{A}_0)^{-1}\mathbf{A}_0(\mathbf{B} - \mathbf{A}_0)^{-1} \cdot \mathbf{A}']^2
\end{aligned}
\tag{4.3}
$$

which upon setting

$$
\mathbf{A}'' = \mathbf{A}_0^{1/2}(\mathbf{B} - \mathbf{A}_0)^{-1}\mathbf{A}'(\mathbf{B} - \mathbf{A}_0)^{-1}\mathbf{A}_0^{1/2} \in L_s(\mathbb{R}^d)\,,
\tag{4.4}
$$

and recalling the formula (3.11) for $\mathbf{\Gamma_B(n)}$, becomes

$$\mathbf{A}'' \cdot \mathbf{A}_0^{-1/2}\left[(\mathbf{B}-\mathbf{A}_0)-(\mathbf{B}-\mathbf{A}_0)\frac{\mathbf{n}\otimes\mathbf{n}}{\mathbf{B}\cdot\mathbf{n}\cdot\mathbf{n}}(\mathbf{B}-\mathbf{A}_0)\right]\mathbf{A}_0^{-1/2}\mathbf{A}'' \geqq (b/p_2)(\mathrm{Tr}\mathbf{A}'')^2 , \tag{4.5}$$

and this must hold for all $\mathbf{n} \in S^{d-1}$ and $\mathbf{A}'' \in L_s(\mathbb{R}^d)$. Now any relation of the form

$$\mathbf{A}'' \cdot \mathbf{X}\mathbf{A}'' \geqq k(\mathrm{Tr}\mathbf{A}'')^2 , \qquad \mathbf{X} \in L_s(\mathbb{R}^d) , \quad \mathbf{X} > 0 , \quad k > 0 , \tag{4.6}$$

is satisfied for all $\mathbf{A}'' \in L_s(\mathbb{R}^d)$ if and only if

$$1/k \geqq \mathrm{Tr}\mathbf{X}^{-1} . \tag{4.7}$$

In our setting $k = b/p_2$ and

$$\mathbf{X} = \mathbf{A}_0^{-1/2}\left[(\mathbf{B}-\mathbf{A}_0)-(\mathbf{B}-\mathbf{A}_0)\frac{\mathbf{n}\otimes\mathbf{n}}{\mathbf{B}\cdot\mathbf{n}\cdot\mathbf{n}}(\mathbf{B}-\mathbf{A}_0)\right]\mathbf{A}_0^{-1/2} \tag{4.8}$$

is easily verified to be positive definite when $\mathbf{B} > \mathbf{A}_0 > 0$, and has inverse

$$\mathbf{X}^{-1} = \mathbf{A}_0^{1/2}\left[(\mathbf{B}-\mathbf{A}_0)^{-1} + \frac{\mathbf{n}\otimes\mathbf{n}}{\mathbf{A}_0\cdot\mathbf{n}\cdot\mathbf{n}}\right]\mathbf{A}_0^{1/2} . \tag{4.9}$$

So the condition (4.7) is

$$p_2/b \geqq \mathrm{Tr}(\mathbf{A}_0(\mathbf{B}-\mathbf{A}_0)^{-1}) + 1 , \qquad \mathbf{n} \in S^{d-1} . \tag{4.10}$$

Now, because $\mathscr{B}$ is on the boundary, we have $H(\mathbf{B}, b) = 0$ and in view of (4.2) the condition (4.10) reduces to

$$p_1 \geqq 1 . \tag{4.11}$$

Remark 4.1. When the set $\mathscr{U}$ of initial tensor-scalar pairs consists of only two points

$$\mathscr{U} = \{(\mathbf{A}_0, 0), (\mathbf{A}_1, 1) \mid \mathbf{A}_0, \mathbf{A}_1 \in \mathscr{M}, \ \ \mathbf{A}_1 > \mathbf{A}_0\} , \tag{4.12}$$

then the set $G\mathscr{U}$ has been completely characterized for conductivity (see [11], [17], [23], and [25]) and is given by

$$\begin{aligned} G\mathscr{U} = \{(\mathbf{A}^*, a) \in \mathscr{M} \times [0,1] \mid \ & \mathbf{A}^* \leqq a\mathbf{A}_1 + (1-a)\mathbf{A}_0 , \\ & \mathrm{Tr}[\mathbf{A}_0(\mathbf{A}^* - \mathbf{A}_0)^{-1}] + 1 \leqq (1/a)(\mathrm{Tr}[\mathbf{A}_0(\mathbf{A}_1-\mathbf{A}_0)^{-1}] + 1) , \ \text{and} \\ & \mathrm{Tr}[\mathbf{A}_1(\mathbf{A}_1 - \mathbf{A}^*)^{-1}] - 1 \leqq [1/(1-a)](\mathrm{Tr}[\mathbf{A}_1(\mathbf{A}_1-\mathbf{A}_0)^{-1}] - 1)\} . \end{aligned} \tag{4.13}$$

Part of the boundary $\partial G\mathscr{U}$ is parametrized by an equation of the type (4.2) with

$$p_1 = 1 , \qquad p_2 = \mathrm{Tr}[\mathbf{A}_0(\mathbf{A}_1 - \mathbf{A}_0)^{-1}] + 1 , \tag{4.14}$$

and the condition (4.11) is obviously satisfied, as it should be. The fact that (4.11) is satisfied as an equality is consistent with the existence of trajectories of lamination that remain on the boundary of $G\mathcal{U}$. If this were not the case then it would be impossible for any point on the boundary of $G\mathcal{U}$, aside from $\mathcal{U}$, to be attained through a lamination process beginning with $\mathcal{U}$.

In our second example we consider a set L of the form

$$L = \{\mathbf{A} \in \mathcal{M} \mid \operatorname{Tr}[\mathbf{A}_0(\mathbf{A} + \mathbf{A}_0)^{-1}] + p_1 \leqq 0\}, \tag{4.15}$$

where $\mathbf{A}_0 \in \mathcal{M}$ and p_1 is a fixed constant. The motivation this time derives from the form of the bounds for the conductivity of a polycrystal; see Remark 4.2. We investigate the boundary ∂L parametrized by

$$H(\mathbf{C}) = - \operatorname{Tr}[\mathbf{A}_0(\mathbf{C} + \mathbf{A}_0)^{-1}] - p_1 . \tag{4.16}$$

At the expense of replacing $\mathbf{B} - \mathbf{A}_0$ by $\mathbf{B} + \mathbf{A}_0$ wherever it appears in the previous calculation, and applying Corollary 3.1 in place of Theorem 3.2 we obtain

$$\mathbf{A}'' \cdot \mathbf{X}\mathbf{A}'' \geqq 0 \tag{4.17}$$

for all $\mathbf{A}'' \in L_s(\mathbb{R}^d)$ satisfying $\operatorname{Tr}\mathbf{A}'' = 0$, where

$$\mathbf{X} = \mathbf{A}_0^{-1/2}\left[(\mathbf{B} + \mathbf{A}_0) - (\mathbf{B} + \mathbf{A}_0)\frac{\mathbf{n} \otimes \mathbf{n}}{\mathbf{B} \cdot \mathbf{n} \cdot \mathbf{n}}(\mathbf{B} + \mathbf{A}_0)\right]\mathbf{A}_0^{-1/2} . \tag{4.18}$$

Note that because $\mathbf{A}_0$ and $\mathbf{B}$ are both positive definite $\mathbf{X}$ is non-singular (but it has both positive and negative eigenvalues). The inverse of $\mathbf{X}$ is

$$\mathbf{X}^{-1} = \mathbf{A}_0^{1/2}\left[(\mathbf{B} + \mathbf{A}_0)^{-1} - \frac{\mathbf{n} \otimes \mathbf{n}}{\mathbf{A}_0 \cdot \mathbf{n} \cdot \mathbf{n}}\right]\mathbf{A}_0^{1/2} , \tag{4.19}$$

and

$$\operatorname{Tr}\mathbf{X}^{-1} = -(p_1 + 1) . \tag{4.20}$$

Assuming that $p_1 \neq -1$, we obtain through a standard procedure in linear algebra that (4.17) is equivalent to requiring that for all $\mathbf{G} \in L_s(\mathbb{R}^d)$

$$\mathbf{G} \cdot \mathbf{X}^{-1}\mathbf{G} - \frac{1}{\operatorname{Tr}\mathbf{X}^{-1}}\left[\operatorname{Tr}(\mathbf{X}^{-1}\mathbf{G})\right]^2 \geqq 0 , \tag{4.21}$$

which in view of (4.19) and (4.20) becomes

$$\mathbf{G} \cdot (\mathbf{S} - \mathbf{m} \otimes \mathbf{m})\mathbf{G} + \frac{1}{p_1 + 1}\{\operatorname{Tr}[(\mathbf{S} - \mathbf{m} \otimes \mathbf{m})\mathbf{G}]\}^2 \geqq 0 , \tag{4.22}$$

where

$$\mathbf{S} = \mathbf{A}_0^{1/2}(\mathbf{B} + \mathbf{A}_0)^{-1}\mathbf{A}_0^{1/2} , \tag{4.23}$$

$$\mathbf{m} = \mathbf{A}_0^{1/2}\mathbf{n}/|\mathbf{A}_0^{1/2}\mathbf{n}| \,, \tag{4.24}$$

and this must hold for all $\mathbf{m} \in S^{d-1}$. Note that $\mathbf{S}$ has eigenvalues strictly between 0 and 1. Taking $\mathbf{G} = \mathbf{m} \otimes \mathbf{m}$ in equality (4.22) constrains p_1 to be strictly between -1 and 0.

Let us introduce $\mathbf{Y} \in L_s(\mathbb{R}^d)$ and $\tilde{\mathbf{Y}} \in L_s(L_s(\mathbb{R}^d))$ defined by

$$\mathbf{Y} = \mathbf{S} - \mathbf{m} \otimes \mathbf{m} \,, \qquad \tilde{\mathbf{Y}}\mathbf{G} = \mathbf{Y}\mathbf{G} \,, \tag{4.25}$$

where $\mathbf{G} \in L_s(\mathbb{R}^d)$. Then (4.22) reads as

$$\tilde{\mathbf{Y}} + \frac{1}{p_1 + 1}\mathbf{Y} \otimes \mathbf{Y} \geqq 0 \,. \tag{4.26}$$

For (4.26) to hold it is sufficient to check that the eigenvalues of the left-hand side are non-negative. Let λ be such an eigenvalue and $\mathbf{G}_\lambda$ an associated eigenvector. Then $\mathbf{G}_\lambda$ satisfies

$$\mathbf{Y}\mathbf{G}_\lambda + \frac{1}{p_1 + 1}\mathbf{Y}\mathrm{Tr}(\mathbf{Y}\mathbf{G}_\lambda) = \lambda \mathbf{G}_\lambda \,. \tag{4.27}$$

Observing through transposition of (4.27) that $\mathbf{Y}$ and $\mathbf{G}_\lambda$ commute and denoting by y_i and g_i the (possibly repeated) eigenvalues of $\mathbf{Y}$ and $\mathbf{G}_\lambda$ yields

$$y_i g_i + y_i \mathrm{Tr}(\mathbf{Y}\mathbf{G}_\lambda)/(p_1 + 1) = \lambda g_i \,, \qquad 1 \leqq i \leqq d \,. \tag{4.28}$$

Further by virtue of (4.16) and the fact that $\mathbf{B}$ belongs to ∂L, we have

$$\mathrm{Tr}\mathbf{Y} = \sum_{i=1}^{d} y_i = -(1 + p_1) \,. \tag{4.29}$$

In addition since the eigenvalues of $\mathbf{S}$ are positive all the y_i are positive except one which we take to be y_d. If $\mathrm{Tr}\mathbf{Y}\mathbf{G}_\lambda = 0$ then $\mathbf{G}_\lambda$ must have at least two non-zero eigenvalues, and (4.28) implies that the two corresponding eigenvalues of $\mathbf{Y}$ are equal to λ which implies that λ must be positive. If $\mathrm{Tr}\mathbf{Y}\mathbf{G}_\lambda \neq 0$ then (4.28) implies that $\lambda \neq y_i$ for all i. We solve (4.28) for g_i and obtain the following relation upon computing $\sum_{i=1}^{d} y_i g_i$,

$$f(\lambda) \equiv \sum_{i=1}^{d} \frac{y_i^2}{y_i - \lambda} = -(p_1 + 1) \,. \tag{4.30}$$

Note that in view of (4.29) we have

$$f(-\infty) = 0 \,, \qquad f(0) = -(p_1 + 1) \,. \tag{4.31}$$

Furthermore $f(\lambda)$ is strictly monotonic increasing over the intervals $\lambda \in (-\infty, y_d)$ and $\lambda \in (y_d, 0)$ while $f(y_d^-) = +\infty$ and $f(y_d^+) = -\infty$. Thus (4.30) cannot be satisfied when λ is strictly negative, and $p_1 \in (-1, 0)$. This establishes that all eigenvalues of the left-hand side of (4.26) are non-negative which in turn proves (4.21) and hence the stability of the set L when p_1 is between -1, and 0.

Remark 4.2. When the set U of initial tensors consists of

$$U = \{\mathbf{R}^T\mathbf{B}_0\mathbf{R} \mid \mathbf{B}_0 \in \mathcal{M},\ \ \mathbf{R} \in SO_d\}\,, \tag{4.32}$$

then the set GU, although not yet fully characterized, has part of its boundary parametrized by (see [2], [24], and [26])

$$\mathrm{Tr}[\theta(\theta\mathbf{I} + \mathbf{B})^{-1}] = 1 \qquad (\leqq 1 \ \text{ when } \ \mathbf{B} \in GU)\,, \tag{4.33}$$

where θ is the positive root of the equation

$$\mathrm{Tr}[\theta(\theta\mathbf{I} + \mathbf{B}_0)^{-1}] = 1\,. \tag{4.34}$$

Letting p_1 tend to -1 in the previous analysis shows that the set

$$L = \{\mathbf{B} \in \mathcal{M} \mid \mathrm{Tr}[\theta(\theta\mathbf{I} + \mathbf{B})^{-1}] \leqq 1\} \tag{4.35}$$

is stable under lamination.

Remark 4.3. It would appear in our two examples that only part of the boundary has been dealt with, and thus that global stability under lamination remains to be established. Elementary considerations would however ensure stability under lamination and under homogenization of the sets of the form $\{\mathbf{A} \in \mathcal{M} \mid \mathbf{A} \leqq \mathbf{C}_0$ (respectively $\mathbf{A} \geqq \mathbf{C}_0)\}$ with $\mathbf{C}_0 \in \mathcal{M}$. Because the intersection of sets stable under lamination is itself stable under lamination it follows that $\mathcal{L}$ is stable when $p_1 \geqq 1$ in (4.1) and L is stable when $p_1 \in [-1, 0)$ in (4.15). Equivalently we have also proved stability under lamination of the sets (4.13) in Remark 4.1 and (4.35) in Remark 4.2.

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