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A topological approach to shape optimization

Minimizers of the sum of the elastic compliance and of the weight of a solid structure under specified loadings are investigated in the context of shape optimization. The resulting minimization problem is ill-posed; it must be relaxed to allow for microperforation of the domain. The theory of homogenization permits an explicit computation of the relaxed functional and yields a new class of numerical algorithms which impose no topological restrictions on the admissible designs. Feasible "classical" designs can then be recovered through simple penalization techniques.

1. Introduction

A major issue in shape optimization is the layout. Shape sensitivity analysis is ill-suited to capture the topological complexity of the shape because holes cannot be created through smooth motions of the existing boundary of the material domain. Hole punching may however dramatically improve the performance of a candidate shape.

Therefore, holes of any shape and size should be allowed within the design region, in a manner such that meaningful optimality criteria can be derived. In the setting of compliance minimization for an elastic structure constrained by its weight, the resulting optimization problem is bang-bang (material or void) in the infinite dimensional space of the characteristic functions of the shape. As such, it is generally ill-posed, a well-known connendrum since the work of Pontryaguin.

The class of admissible designs must be enlarged if the existence of a (generalized) optimal design is to be secured: the adequate class is generated by considering fine mixtures of void and material on a microscale.

Of course, many types of microperforations lead to the same local volume fraction of void in a porous material, and the generalized designs will also depend on the resulting effective tensor associated with the specific microgeometry of the holes.

Although the set of all effective tensors resulting from the mixture in fixed volume fraction of two elastic materials is not known – a major stumbling block in the theory of homogenization since its inception – the minimum compliance can be evaluated among the better known subset of sequential laminates. Note that such may not be the case for a different objective functional. In the present context, an additional difficulty is created by the degenerate character of one of the phases, which prohibits straightforward application of the theory of homogenization. A formal remedy consists in filling the holes with a weak material and letting the stiffness of that material tend to zero. Our contribution, exposed in detail in [1], should be viewed as two-fold. Firstly, the passage from the original shape optimization problem to its generalized (relaxed) formulation is carefully analyzed. The adequacy of the hole filling process is partially demonstrated, although many issues are still unresolved. Secondly, a new computational algorithm is proposed, based on our intimate knowledge of the optimal microstructures (multiple layers). It is an alternate direction algorithm, which successively computes the stress field through the solving of a problem of linear elasticity, then the optimal microstructure for that stress field.

References on the different issues addressed in this paper will be found in [1].

2. Original formulation versus relaxed formulation

Consider a bounded domain Ω in $\mathbb{R}^N$ subject to "smooth enough" surface loadings f on its boundary $\partial\Omega$. Part of the domain is occupied by an isotropic linearly elastic material with elasticity

$$A = \lambda I_2 \otimes I_2 + 2\mu I_4, \quad 0 < \lambda + \frac{2\mu}{N}, \mu < +\infty,$$

while the remaining part of Ω is void. Let χ denote the characteristic function of the part Ω_χ of Ω occupied by the elastic material. Define the set $\Sigma(\chi)$ of statically admissible stress field as

$$\Sigma(\chi) = \left\{ \tau \in L^2(\Omega; \mathbb{R}_s^{N^2}) \mid \text{div } \tau = 0 \text{ in } \Omega;\ \tau \cdot n = f \text{ on } \partial\Omega;\ \tau(x) = 0 \text{ a.e. where } \chi(x) = 0 \right\}.$$

A generalized compliance may be defined as

$$c(\chi) := \inf_{\tau \in \Sigma(\chi)} \int_\Omega A^{-1}\tau \cdot \tau dx$$

(Note that the infimum is not necessarily attained.)

The goal of optimal design is to devise the least compliant structure compatible with the loads for a given weight of the structure. Thus, the range of compliances $c(\chi)$ for all characteristic functions χ such that

$$\int_\Omega \chi(x)dx = \Theta, \quad 0 < \Theta \leq |\Omega|,$$

is investigated and the optimal design problem reads as

$$I := \inf \left\{ c(\chi) \ \mid\ \chi \in L^\infty(\Omega; \{0,1\});\ \int_\Omega \chi(x)dx = \Theta \right\}.$$

The constraint on χ is dealt with through the introduction of a Lagrange multiplier l. It should be emphasized that the existence of such a multiplier is still an open problem in the present setting. In any case, the previous formulation becomes

$$I(\ell) := \inf_{\chi \in L^\infty(\Omega;\{0,1\})} \left\{ c(\chi) + \ell \int_\Omega \chi(x)dx \right\},$$

or equivalently,

$$I(\ell) = \inf_{\tau \in \Sigma(\Omega)} \int_\Omega f_\ell(\tau)dx,$$

where

$$\Sigma(\Omega) := \left\{ \tau \in L^2(\Omega; \mathbb{R}_s^{N^2}) \ \mid\ \text{div } \tau = 0 \text{ in } \Omega;\ \ \tau \cdot n = f \text{ on } \partial\Omega \right\},$$

and

$$f_\ell(\tau) := \begin{cases} 0 & \text{if } \tau = 0 \\ A^{-1}\tau \cdot \tau + \ell & \text{if } \tau \neq 0. \end{cases}$$

We prove in particular that

$$I(\ell) := \inf_{\tau \in \Sigma(\Omega)} \int_\Omega f_\ell(\tau)dx = I^*(\ell) := \min_{\tau \in \Sigma(\Omega)} \int_\Omega F_\ell(\tau)dx,$$

where,

$$F_\ell(\tau) = \min_{0 \leq \theta \leq 1} \{F(\tau,\theta) + \ell\theta\}$$

and

$$F(\tau,\theta) = \min_{A^* \in L_\theta^0} A^{*-1}\tau \cdot \tau, \tag{1}$$

where L_θ^0 is the set of all tensors A^* obtained by rank-N sequential "lamination" (in dimension N) of A with void in the eigendirections of τ (see Theorem 3.1 of [1]). The tensor A^{*-1} is given by

$$A^{*-1} = A^{-1} + \frac{1-\theta}{\theta} \left(\sum_{i=1}^{p} m_i f_A^c(e_i) \right)^{-1},$$

where $f_A^c(e_i)$ is a fourth order tensor defined, for any symmetric matrix ξ, by the quadratic form

$$f_A^c(e_i)\xi \cdot \xi = A\xi \cdot \xi - \frac{1}{\mu}|A\xi e_i|^2 + \frac{\mu+\lambda}{\mu(2\mu+\lambda)}((A\xi)e_i \cdot e_i)^2.$$

In the above expression, the e_i's denote the normalized principal directions of τ, the proportions m_i satisfy $0 \leq m_i \leq 1$ and $\sum_i m_i = 1$. Note that the only remaining parameters are the m_i's. In the two or three dimensional case, those can be explicitly determined as well as the (unique) density θ, which realizes the minimum in (1) (see Propositions 4.7, 4.8 and (69) in [1]).

3. A numerical algorithm for 2 and 3-d shape optimization

The algorithm relies on two key ideas. The first one is to consider the relaxed problem $I^*(\ell)$ as a minimization problem not only for the stress, but also for the structural parameters, the density θ, and the microstructure A^*. The second key idea is not to try to minimize directly in the triplet of variables (τ, θ, A^*), but rather to adopt an iterative approach and minimize separately and successively in the design variables (θ, A^*) and in the field variable τ. The algorithm is structured as follows:

1. Initialization of the design parameters (θ_0, A_0^*) (for example, taking $\theta_0 = 1$ and $A_0^* = A$ everywhere in the domain).
2. Iteration until convergence:
   (a) Computation of τ_n through a linear elasticity problem with $(\theta_{n-1}, A_{n-1}^*)$ as design variables.
   (b) Updating of the design variables (θ_n, A_n^*) by using the stress τ_n in the explicit optimality formulae for θ and for the m_i's.

The resulting generalized design is characterized by a density of material. It includes large regions of composite material which are undesirable since the primary goal is to find a real shape. This drawback is avoided through a post-processing technique that *penalizes* composite regions. The goal is to deduce a quasi-optimal shape from the optimal densities. In loose terms, the solution of the relaxed problem is projected onto the set of classical solutions of the original problem, in the hope that the value of the objective functional will not increase too much in the process.

The strategy is as follows. Upon convergence to an optimal density, we run a few more iterations of our algorithm where we *force* the density to take values close to 0 or 1. This changes the optimal density and produces a quasi-optimal shape. Numerical implementation of this simple penalization technique seems to produce satisfactory designs.

4. Numerical results: the optimal throne!

We present below a typical 3-D numerical computation. The workspace Ω is the truncated box shown on the figures. Her Optimal Highness applies a uniform pressure on the seat and back of the throne, which is held fixed to the floor on its whole base. The figures show the penalized shapes for a volume fraction of 11%, seen from different angles.

5. References

1 Allaire G., Bonnetier E., Francfort G., Jouve F.: Shape optimization by the homogenization method. *to appear.*

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