QUASI-STATIC EVOLUTION IN BRITTLE FRACTURE: THE CASE OF BOUNDED SOLUTIONS

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Abstract

The main steps of the proof of the existence result for the quasi-static evolution of cracks in brittle materials, obtained in [7] in the vector case and for a general quasiconvex elastic energy, are presented here under the simplifying assumption that the minimizing sequences involved in the problem are uniformly bounded in L^∞.

Keywords: fracture, functions of bounded variation, geometric measure theory, free discontinuity problems, Mumford & Shah functional, quasi-static evolution.

1 Introduction

In recent years, a variational theory of quasi-static crack growth in a brittle solid, first proposed in [10], has been developed on several fronts. Its basic ingredients are few and simple. The main idea – borrowed from D. Mumford & J. Shah's approach to image segmentation [13] and close in spirit to the original idea of A. Griffith in his seminal paper [12] – is that the crack wants to quasi-statically minimize its total energy among all legal competitors. In other words, the crack $\Gamma(t)$ must minimize

$$\mathcal{E}(t,\Gamma) := \mathcal{W}(t,\Gamma) + \mathcal{H}^{N-1}(\Gamma)$$

among all $\Gamma \supset \Gamma(s),\ s < t$, where $\mathcal{W}(t,\Gamma)$ is the potential energy of the sample for the loads that are applied at time t, and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure (i.e., the length, if $N = 2$, and the area, if $N = 3$).

Actually, the evolution must be further constrained by imposition of a condition on the time evolution of the energy $\mathcal{E}(t) := \mathcal{E}(t, \Gamma(t))$, so as to recover the propagation criterion of A. Griffith in the current setting. That constraint is simply that the change in mechanical energy should exactly balance the work of the external loads.

Specifically, let W be the elastic energy density associated to the sample $\Omega \subset \mathbb{R}^N$, ($N = 1$, 2, or 3 in the physically relevant cases). If u is any displacement field, then W is either a function of ∇u (the case of finite elasticity) or of $\varepsilon(u)$, the symmetrized gradient of u (the case of linearized elasticity). We assume the former and briefly comment this choice further below.

Let $g(t)$ be the applied displacement at time t on a part $\partial\Omega_d$ of the boundary $\partial\Omega$ and let $\mathcal{L}(t, v)$ be the work of the surface and body loads at time t for a test displacement field v. We do not attempt to further detail the specific kind of surface or body loads at this point, but merely note that $\mathcal{L}$ *cannot* be a linear map of v at the current stage of the theory, as will be justified later. In particular, the loads cannot be displacement independent.

For a given crack Γ, a closed subset of $\overline{\Omega}$ with $\mathcal{H}^{N-1}(\Gamma) < +\infty$, the displacement field $u(t)$ is a minimizer of the potential energy $\int_\Omega W(\nabla v)dx - \mathcal{L}(t, v)$ among all kinematically admissible displacement fields, that is all v's which may be discontinuous across Γ and satisfy the boundary condition $v = g(t)$ on $\partial\Omega_d \setminus \Gamma$. Note that the crack may in effect debond the sample from the applied displacement. The resulting value of the potential energy is $\mathcal{W}(t, \Gamma)$.

Now, for a given time t, the crack $\Gamma(t)$ must be such that it minimizes the total energy $\mathcal{E}(t, \Gamma)$ among all cracks that contain the prior cracks, that is among all $\Gamma \supset \bigcup_{s<t} \Gamma(s)$. Fracture irreversibility, that is the fact that $\Gamma(t)$ increases with t, is an essential feature of the evolution process; it is implicit in this minimality property.

Finally, the conservation of total energy must be satisfied throughout the evolution. In the current context, this translates into

$$\dot{\mathcal{E}}(t) = \int_{\partial\Omega_d \setminus \Gamma(t)} DW(\nabla u(t))n \cdot \dot{g}(t)\, d\mathcal{H}^{N-1} - \dot{\mathcal{L}}(t, u(t)).$$

In [10], the mechanical significance of the proposed evolution model is investigated at length in the case where W is actually a quadratic function of $\varepsilon(u)$, the setting of linear elasticity, $\mathcal{L} \equiv 0$, so that the only driving mechanism is the boundary condition $g(t)$, and $g(t) \equiv tG$, with G a fixed function. The model then palliates the major defects of the classical theory, most notably its inability to initiate the fracture process and to predict the crack path as well as the time evolution of the crack along that path.

In [3] and [4] various numerical implementations of the time evolution are attempted. The continuous-time evolution is replaced by a finite time step approach. The proposed methods are shown to be both theoretically and numerically sound; the obtained results are striking at times and certainly well beyond the boundaries of classical fracture mechanics computations.

From the mathematical standpoint, the hurdles involved in an adequate handling of the symmetrized gradient case – the case where W is a function of $\varepsilon(u)$ – forbid at present the complete development of the theory for linearized elasticity, notably because the ambient functional space for such a study, that is $SBD(\mathbb{R}^N)$, is only partially understood. The only mathematical study relevant to fracture in the context of linearized elasticity is that of the *two-dimensional setting* with a quadratic W, and under the restrictive assumption that the maximal number of connected components of the potential cracks is *a priori* known [5].

In the case where ∇u is considered, however, the mathematical analysis of the proposed formulation is well under way. The antiplane elasticity setting, that is that where u is scalar-valued, is well understood: in [8], existence is shown under the same restrictions as those just detailed; then, the general quadratic case is solved in [9]. In both settings, the assumption that $\mathcal{L} \equiv 0$ is essential to the analysis, because it permits to obtain L^∞ estimates through the maximum principle. Since the maximum principle is not applicable to the vector-valued case, that restriction becomes moot. In [7], the vector case is analyzed under the only assumption of quasiconvexity of the energy density. The class of loads $\mathcal{L}$ for which the minimization problem at fixed time is meaningful must ensure some reasonable compactness properties of the minimizing sequences, which is why a linear dependence of $\mathcal{L}(t, v)$ on v is not admissible. See [7] for details.

Our goal in this paper is to present the results of [7] in a simpler situation which allows us to remove some *non-trivial* technicalities and to make the main ideas of the proof more transparent. To be definite, we revisit the vector case under the assumption that the minimizing fields are bounded in L^∞ (uniformly in time). We do not attempt to justify this hypothesis (which is automatically satisfied only in the scalar case) and readily agree with any potential criticism. The advantage of this assumption is that we can present our results in a simpler functional framework, using the space $SBV(\mathbb{R}^N;\mathbb{R}^m)$ instead of $GSBV(\mathbb{R}^N;\mathbb{R}^m)$. Another advantage is that we can avoid all coerciveness assumptions on the loads, since the compactness of the minimizing sequences follows now from the L^∞-bound.

Our aim however is not to give an independent proof, but only to describe the main arguments of [7] in the simplest situation. For this reason we present the results only in the case of no applied loads. We still require a few technical lemmas and refer the reader to [7] for their proof.

Throughout, $SBV^p(\mathbb{R}^N;\mathbb{R}^m)$, $1 < p < +\infty$, is the space of functions $v \in SBV(\mathbb{R}^N;\mathbb{R}^m)$ (see [1]) such that $\nabla v \in L^p(\mathbb{R}^N;\mathbb{R}^{mN})$ and $\mathcal{H}^{N-1}(S(v)) < +\infty$, where $S(v)$ denotes the jump set of v. We say that a sequence $u^n \in SBV^p(\mathbb{R}^N;\mathbb{R}^m)$ SBV^p-converges to $u \in SBV^p(\mathbb{R}^N;\mathbb{R}^m)$ if

$$\nabla u^n \rightharpoonup \nabla u \text{ in } L^p(\mathbb{R}^N;\mathbb{R}^{mN}),$$

$$\mathcal{H}^{N-1}(S(u^n)) \text{ is bounded},$$

$$u^n \to u \text{ in } L^p(\mathbb{R}^N;\mathbb{R}^m),$$

$$u^n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}^N;\mathbb{R}^m).$$

We use the same definitions for sequences in $SBV^p(U)$, where U is an open subset of $\mathbb{R}^N$.

Remark 1.1 *If $u^n \overset{SBV^p}{\rightharpoonup} u$, then it is proved in [1] that, for any open set U,*

$$\mathcal{H}^{N-1}(S(u) \cap U) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(S(u^n) \cap U).$$

Further, it is easily seen (see, e.g., [7], [9]) that, for any Borel set E with $\mathcal{H}^{N-1}(E) < \infty$,

$$\mathcal{H}^{N-1}(S(u) \setminus E) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(S(u^n) \setminus E).$$

2 Setting of the problem and statement of the results

As mentioned in the introductory section, our analysis is restricted to the case where we act on the body only through prescribed displacements on a part $\partial\Omega_d := \partial\Omega \setminus \partial\Omega_f$ of the boundary.

The energy density W is a nonnegative quasiconvex C^1 function on $\mathbb{R}^{mN}$ which further satisfies

$$(1/C)|\xi|^p - C \le W(\xi) \le C|\xi|^p + C, \quad \xi \in \mathbb{R}^{mN}, \tag{2.1}$$

for some constants $C \ge 1$ and $1 < p < \infty$. Note that the assumptions on W immediately imply that (see, e.g., [6])

$$|DW(\xi)| \le C(1 + |\xi|^{p-1}), \tag{2.2}$$

for some (possibly different) constant $C \ge 1$.

The domain Ω under consideration is assumed throughout to be bounded and Lipschitz, and the function g, which appears in the boundary condition on $\partial\Omega_d$, is assumed to be defined on all of $\mathbb{R}^N$; actually, it is taken to be in $W^{1,1}_{loc}([0,\infty); W^{1,p}(\mathbb{R}^N;\mathbb{R}^m))$. In particular, g belongs to $C^0([0,\infty); W^{1,p}(\mathbb{R}^N;\mathbb{R}^m))$ and its time derivative $\dot g$ belongs to $L^1_{loc}([0,\infty); W^{1,p}(\mathbb{R}^N;\mathbb{R}^m))$.

The traction-free part $\partial\Omega_f$ of the boundary $\partial\Omega$ is assumed to be closed.

We will denote throughout inclusion, up to a set of $\mathcal{H}^{N-1}$-measure 0, by $\tilde\subset$, and set equality, up to a set of $\mathcal{H}^{N-1}$-measure 0, by $\cong$. A crack is a subset Γ of $\overline{\Omega}$ with $\mathcal{H}^{N-1}(\Gamma) < +\infty$.

The condition that the deformation field u, physically defined only on Ω, has a jump set contained in $\Gamma(t)$ and agrees with $g(t)$ on $\partial\Omega_d \setminus \Gamma(t)$ in the sense of traces, will be expressed in an equivalent way by defining u on all of $\mathbb{R}^N$ and by requiring that $u \in SBV^p(\mathbb{R}^N;\mathbb{R}^m)$, $u = g(t)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$, and $S(u) \,\tilde\subset\, \Gamma(t) \cup \partial\Omega_f$. Note that, under these hypotheses $W(\nabla u) \in L^1(\Omega)$ and $DW(\nabla u) \in L^{p'}(\Omega;\mathbb{R}^{mN})$, with $p' := p/(p-1)$.

We will consider an initial crack Γ_0 and an initial deformation $u_0 \in SBV^p(\mathbb{R}^N;\mathbb{R}^m)$ with $u_0 = g(0)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$ and $S(u_0) \,\tilde\subset\, \Gamma_0 \cup \partial\Omega_f$. We assume also that the Griffith equilibrium condition is satisfied by the initial configuration Γ_0, u_0, i.e., u_0 minimizes

$$\int_\Omega W(\nabla v)\, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma_0 \cup \partial\Omega_f))$$

among all v in $SBV^p(\mathbb{R}^N;\mathbb{R}^m)$ with $v = g(0)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$;

In the remainder of this paper, we intend to prove the following result, under an additional uniform boundedness assumption detailed in **(H)** below.

Theorem 2.1 *There exists a family of time dependent cracks $\Gamma(t) \subset \overline{\Omega}$, $t \ge 0$, and a field $u\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^m$ such that*

- $u(0) = u_0$ *a.e. and* $\Gamma(0) \cong \Gamma_0$;
- $u(t) \in SBV^p(\mathbb{R}^N;\mathbb{R}^m)$ *for every* $t \ge 0$, *so that* $DW(\nabla u(t)) \in L^{p'}(\Omega;\mathbb{R}^{mN})$;
- $\Gamma(t)$ *increases with* t *and* $\mathcal{H}^{N-1}(\Gamma(t)) < +\infty$ *for every* $t \ge 0$;
- $S(u(t)) \,\tilde\subset\, \Gamma(t) \cup \partial\Omega_f$ *and* $u(t) = g(t)$ *a.e. on* $\mathbb{R}^N \setminus \overline{\Omega}$ *for every* $t \ge 0$;

- *for every $t \geq 0$ the deformation $u(t)$ minimizes*

$$\int_\Omega W(\nabla v)\, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t) \cup \partial\Omega_f))$$

among all v in $SBV^p(\mathbb{R}^N; \mathbb{R}^m)$ with $v = g(t)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$;

- *the total energy*

$$\mathcal{E}(t) := \int_\Omega W(\nabla u(t))\, dx + \mathcal{H}^{N-1}(\Gamma(t) \setminus \partial\Omega_f)$$

is absolutely continuous, $DW(\nabla u) \cdot \nabla \dot{g} \in L^1_{loc}([0, \infty); L^1(\mathbb{R}^N))$, and

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_\Omega DW(\nabla u(s)) \cdot \nabla \dot{g}(s)\, dx\, ds \tag{2.3}$$

for every $t > 0$.

The strategy for proving the result is close to that developed in [9] for the case of quadratic energy densities. As mentioned in the introduction, we will appeal, without repeating the proofs, to various results in [7], since the present paper favors simplicity over completeness.

There is no loss of generality in restricting the study to a time interval $[0, T]$. We then choose a countable dense set I_∞ in $[0, T]$ (containing 0), and, for each $n \in \mathbb{N}$, a subset $I_n = \{t_0^n = 0 < t_1^n < \cdots < t_n^n\}$, such that $\{I_n\}$ form an increasing sequence of nested sets whose union is I_∞. We set $\Delta_n := \sup_{k \in \{1,\dots,n\}}(t_k^n - t_{k-1}^n)$. Note that $\Delta_n \searrow 0$.

We set $\Gamma_0^n := \Gamma_0$ and $u_0^n := u_0$. Suppose that u_j^n is defined for $j = 0, 1, \dots, k-1$, and let

$$\Gamma_{k-1}^n := \Gamma_0 \cup \bigcup_{0 \leq j \leq k-1} S(u_j^n).$$

At time t_k^n, $k \geq 1$, we define u_k^n to be a minimizer for

$$\int_\Omega W(\nabla v)\, dx + \mathcal{H}_c^{N-1}(S(v) \setminus \Gamma_{k-1}^n) \tag{2.4}$$

in $\{v \in SBV^p(\mathbb{R}^N; \mathbb{R}^m) : v \equiv g_k^n := g(t_k^n)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}\}$. In (2.4) and onward, $\mathcal{H}_c^{N-1} := \mathcal{H}^{N-1} \lfloor \partial\Omega_f^c$, where $\partial\Omega_f^c := \mathbb{R}^N \setminus \partial\Omega_f$. In other words, $\mathcal{H}_c^{N-1}(E) = \mathcal{H}^{N-1}(E \setminus \partial\Omega_f)$ for every Borel set $E \subset \mathbb{R}^N$.

As mentioned in the introductory section, we a priori impose that

(H) *each problem (2.4) has a minimizing sequence which is bounded in $L^\infty(\mathbb{R}^N; \mathbb{R}^m)$, with a bound independent of n and k.*

By a truncation argument it is easy to see that **(H)** is automatically satisfied in the scalar case $m = 1$, when $g \in L^\infty([0, T] \times \mathbb{R}^N) \cap W^{1,1}([0, T]; W^{1,p}(\mathbb{R}^N))$.

In view of the bound from below on W and of assumption **(H)**, the existence of a minimizer for (2.4) is a straightforward iterated application of the SBV-compactness theorem (see, e.g., [1], [9]).

We then define

$$\Gamma^n(t) := \Gamma_k^n, \quad u^n(t) := u_k^n, \quad g^n(t) := g_k^n, \quad \text{in } [t_k^n, t_{k+1}^n),$$

and note that, for each $t \in I_\infty$, $g(t) = g_n(t)$, if n is large enough.

Remark 2.2 *For every* $t \in [0,T]$, $S(u^n(t)) \,\tilde{\subset}\, \Gamma^n(t) \cup \partial\Omega_f$ *and* $u^n(t) = g^n(t)$ *a.e. on* $\mathbb{R}^N \backslash \overline{\Omega}$. *Moreover,* $u^n(t)$ *minimizes*

$$\int_\Omega W(\nabla v)\, dx + \mathcal{H}_c^{N-1}(S(v) \setminus \Gamma^n(t))$$

on $\{v \in SBV^p(\mathbb{R}^N;\mathbb{R}^m) : v = g^n(t) \text{ a.e. on } \mathbb{R}^N \backslash \overline{\Omega}\}$. *In particular, with terminology from [9],* $u^n(t)$ *is a minimizer for its own jump set.*

The construction of $\Gamma^n(t)$ and $u^n(t)$ can be viewed as a discrete time approximation of the solution to the continuous time problem. Indeed, we will also establish the following result.

Theorem 2.3 *Consider a subsequence of* $\{n\}$, *independent of* t *and still labeled* $\{n\}$, *such that for every* $t \in [0,T]$ $\Gamma^n(t)$ σ^p*-converges to* $\Gamma(t)$ *according to Definition 3.1 below. Let* $u(t)$ *be a deformation field such that the pair* $\Gamma(t), u(t)$ *satisfies all the conclusions of Theorem 2.1. Then for every* $t \in [0,T]$,

$$\int_\Omega W(\nabla u^n(t))\, dx \to \int_\Omega W(\nabla u(t))\, dx,$$

while

$$\mathcal{H}_c^{N-1}(\Gamma^n(t)) \to \mathcal{H}_c^{N-1}(\Gamma(t)).$$

Note that the existence of a pair $\Gamma(t), u(t)$ is guaranteed through the proof of Theorem 2.1. This latter result, a generalization in the present context of a result of [11], demonstrates that the discrete time approximation provides a reasonable estimate of both the bulk energy and the length of the crack as the discretization step becomes small.

3 Proofs

As mentioned in the introduction, the proof presented here is a special case of a more general result obtained in [7]. In particular, we use below the set convergenge introduced in [7], Section 4, under the name of σ^p-convergence, which we now define.

Definition 3.1 *We say that* Γ^n σ^p*-converges to* Γ *if* Γ^n, $\Gamma \subset \mathbb{R}^N$, $\mathcal{H}^{N-1}(\Gamma^n)$ *is bounded uniformly with respect to* n, *and the following conditions are satisfied:*

(a) *if* u^j *converges weakly to* u *in* $SBV^p(\mathbb{R}^N)$ *and* $S(u^j) \,\tilde{\subset}\, \Gamma^{n_j}$ *for some sequence* $n_j \nearrow \infty$, *then* $S(u) \,\tilde{\subset}\, \Gamma$;

(b) *there exist a function* $u \in SBV^p(\mathbb{R}^N)$ *and a sequence* $u^n \overset{SBV^p}{\rightharpoonup} u$ *such that* $S(u) \,\tilde{=}\, \Gamma$ *and* $S(u^n) \,\tilde{\subset}\, \Gamma^n$ *for every* n.

The following compactness result proved in Theorem 4.8 in [7] is central to our argument. Note that, in the quadratic case, one does not need to appeal to the notion of σ^p-convergence: see [9]. Although this is true of the convex case as well, the general quasiconvex case seems to necessitate that notion.

Theorem 3.2 *Let $t \mapsto \Gamma^n(t)$ be a sequence of increasing set functions defined on an interval I with values contained in a bounded set $B \subset \mathbb{R}^N$, i.e.,*

$$\Gamma^n(s) \tilde{\subset} \Gamma^n(t) \subset B \quad \textit{for every } s,\, t \in I \textit{ with } s < t.$$

Assume that the measures $\mathcal{H}^{N-1}(\Gamma^n(t))$ are bounded uniformly with respect to n and t. Then there exist a subsequence Γ^{n_j} and an increasing set function $t \mapsto \Gamma(t)$ on I such that

$$\Gamma^{n_j}(t) \quad \sigma^p\textit{-converges to } \Gamma(t) \tag{3.1}$$

for every $t \in I$. Furthermore, $\mathcal{H}^{N-1}(\Gamma(t))$ is bounded uniformly with respect to t.

In all that follows, we will not relabel converging subsequences of a given sequence, unless confusion might ensue.

3.1 The discrete formulation

We first derive the necessary *a priori* estimates. For some constant $C > 0$ the following holds true:

$$\|\nabla u^n(t)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})} \leq C, \tag{3.2}$$

$$\mathcal{H}^{N-1}(\Gamma^n(t)) = \mathcal{H}^{N-1}\Big(\Gamma_0 \cup \bigcup_{\tau \leq t} S(u^n(\tau))\Big) \leq C, \tag{3.3}$$

$$\|u^n(t)\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^m)} \leq C. \tag{3.4}$$

Indeed, at time t_k^n, take g_k^n as test function for the minimality of u_k^n in (2.4). We obtain

$$\int_\Omega W(\nabla u_k^n)\, dx + \mathcal{H}_c^{N-1}(S(u_k^n) \setminus \Gamma_{k-1}^n) \leq \int_\Omega W(\nabla g_k^n)\, dx,$$

which implies, since $u_k^n \equiv g_k^n$ a.e. on $\mathbb{R}^N \backslash \overline{\Omega}$ and by virtue of the p-growth of W, that

$$\|\nabla u^n(t)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})} \leq C\|\nabla g^n(t)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})} + C$$

for some constant $C > 0$. This proves (3.2), since $\|\nabla g^n(t)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})}$ is bounded uniformly with respect to t and n.

Now, at time t_{k+1}^n, take $u_k^n + g_{k+1}^n - g_k^n$ as a test function for the minimality of u_{k+1}^n in (2.4). Since $\Phi \mapsto \int_\Omega W(\Phi)\, dx$ is a C^1-map from $L^p(\mathbb{R}^{mN})$ into $\mathbb{R}$ with differential $\Psi \mapsto \int_\Omega DW(\Phi) \cdot \Psi\, dx$, we obtain, for some $\theta_k^n \in [0, 1]$,

$$\begin{aligned}
&\int_\Omega W(\nabla u_{k+1}^n)\, dx \;+\; \mathcal{H}_c^{N-1}(S(u_{k+1}^n) \setminus \Gamma_k^n) \\
&\qquad \leq \int_\Omega W(\nabla(u_k^n + g_{k+1}^n - g_k^n))\, dx \\
&\qquad = \int_\Omega W(\nabla u_k^n)\, dx + \int_\Omega DW(\nabla(u_k^n + \theta_k^n(g_{k+1}^n - g_k^n))) \cdot \nabla(g_{k+1}^n - g_k^n)\, dx \\
&\qquad = \int_\Omega W(\nabla u_k^n)\, dx + \int_{t_k^n}^{t_{k+1}^n} \int_\Omega DW(\nabla u^n(s) + \Psi^n(s)) \cdot \nabla \dot{g}(s)\, dx\, ds,
\end{aligned}$$

where $\Psi^n \in L^\infty((0,T); L^p(\mathbb{R}^N;\mathbb{R}^{mN}))$ is defined as

$$\Psi^n(s) := \theta^n_k \int_{t^n_k}^{t^n_{k+1}} \nabla\dot g(\sigma) d\sigma, \quad s \in [t^n_k, t^n_{k+1}).$$

Because $\Delta_n \searrow 0$ and $g(t)$ is absolutely continuous with values in $W^{1,p}(\mathbb{R}^N;\mathbb{R}^m)$,

$$\|\Psi^n(t)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})} \to 0, \text{ uniformly in } t \in [0,T]. \tag{3.5}$$

Summing up the previous inequality for $k = 0, \ldots, i-1$, we obtain

$$\begin{aligned}\int_\Omega W(\nabla u^n_i)\, dx \;+\; \mathcal{H}^{N-1}_c(\Gamma^n_i) \\ \le \int_\Omega W(\nabla u_0)\, dx + \mathcal{H}^{N-1}_c(\Gamma_0) \\ + \int_0^{t^n_i} \int_\Omega DW(\nabla u^n(s) + \Psi^n(s)) \cdot \nabla \dot g(s)\, dx\, ds.\end{aligned} \tag{3.6}$$

The already established *a priori* bound (3.2), (3.5), the growth estimate (2.2) on DW and Hölder's inequality yield for every $t \in [0,T]$

$$\mathcal{H}^{N-1}_c(\Gamma^n(t)) \le \int_\Omega W(\nabla u(0))\, dx + \mathcal{H}^{N-1}_c(\Gamma_0) + C\int_0^t \|\nabla \dot g(s)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})} ds\,, \tag{3.7}$$

which is bounded in view of the assumed regularity of g. Since $\mathcal{H}^{N-1}(\partial\Omega_f)$ is finite, we conclude that

$$\mathcal{H}^{N-1}(\Gamma^n(t)) \le C.$$

This proves (3.3).

The third bound (3.4) is an immediate consequence of assumption **(H)**.

According to Theorem 3.2 applied to $\Gamma^n(t)$ and thanks to (3.3), there exists a subsequence of $\{n\}$, still labeled $\{n\}$, and an increasing set function $\Gamma(t) \subset \mathbb{R}^N$ such that

$$\Gamma^n(t) \stackrel{\sigma^p}{\to} \Gamma(t) \tag{3.8}$$

for every $t \in [0,T]$. Also, since $\Gamma^n(t) \subset \overline{\Omega}$, we obtain that $\Gamma(t) \mathrel{\tilde\subset} \overline{\Omega}$. This follows from the definition of σ^p-convergence and from Remark 1.1 (applied with $U = \mathbb{R}^N \setminus \overline{\Omega}$).

We now set for a.e. $t \in [0,T]$

$$\begin{aligned}\theta^n(t) &:= \int_\Omega DW(\nabla u^n(t)) \cdot \nabla \dot g(t)\, dx, \\ \theta(t) &:= \limsup_{n\to\infty} \theta^n(t).\end{aligned} \tag{3.9}$$

In view of the growth assumption on DW, the $L^1((0,T); L^p(\mathbb{R}^N;\mathbb{R}^{mN}))$-regularity of $\nabla\dot g$, and the uniform bound (3.2) on $\|\nabla u^n(t)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})}$, Fatou's lemma immediately implies that $\theta \in L^1(0,T)$ and that

$$\limsup_{n\to\infty} \int_0^t \theta^n(s)\, ds \le \int_0^t \theta(s)\, ds. \tag{3.10}$$

Furthermore, we are at liberty to extract, for a.e. $t \in [0, T]$, a t-dependent subsequence of θ^n, denoted by θ^{n_t}, such that

$$\theta(t) = \lim_{n_t \to \infty} \theta^{n_t}(t) = \lim_{n_t \to \infty} \int_\Omega DW(\nabla u^{n_t}(t)) \cdot \nabla \dot{g}(t)\, dx. \tag{3.11}$$

On the other hand, thanks to estimates (3.2)–(3.4), we are in a position to apply AMBROSIO's SBV-compactness theorem (see, e.g., [1]) to $\{u^{n_t}(t)\}$, for any $t \in [0, T]$, and to conclude the existence of $u(t) \in SBV^p(\mathbb{R}^N; \mathbb{R}^m)$ such that, for a yet another t-dependent subsequence of u^{n_t}, still denoted by u^{n_t},

$$\begin{aligned} \nabla u^{n_t}(t) &\rightharpoonup \nabla u(t) \text{ in } L^p(\mathbb{R}^N; \mathbb{R}^{mN}), \\ u^{n_t}(t) &\to u(t) \text{ in } L^p(\mathbb{R}^N, \mathbb{R}^m), \\ u^{n_t}(t) &\overset{*}{\rightharpoonup} u(t) \text{ in } L^\infty(\mathbb{R}^N; \mathbb{R}^m). \end{aligned}$$

In view of the boundedness of $\mathcal{H}^{N-1}(S(u^{n_t}(t))$, we conclude, following the terminology of the introduction, that $u^{n_t}(t)$ SBV^p-converges to $u(t)$.

Further, (3.2)–(3.4) and the lower semi-continuous character of the $\mathcal{H}^{N-1}$-measure with respect to σ^p-convergence (an immediate consequence of item (b) in Definition 3.1) imply the existence of a constant C such that,

$$\|\nabla u(t)\|_{L^p(\mathbb{R}^N; \mathbb{R}^{mN})} \leq C, \tag{3.12}$$

$$\mathcal{H}^{N-1}(\Gamma(t)) \leq \liminf_{n_t \to \infty} \mathcal{H}^{N-1}(\Gamma^{n_t}(t)) \leq C, \tag{3.13}$$

$$\|u(t)\|_{L^\infty(\mathbb{R}^N; \mathbb{R}^m)} \leq C. \tag{3.14}$$

We now investigate the minimality properties of $u(t)$. This is the object of the following lemma, which is an easy consequence of the jump transfer theorem in [9] (see Theorems 2.1, 2.8 of that reference) and of the properties of σ^p-convergence.

Lemma 3.3 *For every $t \in [0, T]$ we have $S(u(t)) \tilde{\subset} \Gamma(t) \cup \partial\Omega_f$ and $u(t) = g(t)$ a.e. on $\mathbb{R}^N \backslash \overline{\Omega}$. Moreover, $u(t)$ minimizes*

$$\int_\Omega W(\nabla v)\, dx + \mathcal{H}_c^{N-1}(S(v) \setminus \Gamma(t)) \tag{3.15}$$

on $\{v \in SBV^p(\mathbb{R}^N; \mathbb{R}^m) : v = g(t) \text{ a.e. on } \mathbb{R}^N \backslash \overline{\Omega}\}$.

Further, for every $t \in [0, T]$,

$$\int_\Omega W(\nabla u^{n_t}(t))\, dx \to \int_\Omega W(\nabla u(t))\, dx, \tag{3.16}$$

and

$$DW(\nabla u^{n_t}(t)) \rightharpoonup DW(\nabla u(t)), \text{ weakly in } L^{p'}(\mathbb{R}^N; \mathbb{R}^{mN}). \tag{3.17}$$

Finally, θ defined in (3.9) lies in $L^1(0, T)$ and

$$\theta(t) = \int_\Omega DW(\nabla u(t)) \cdot \nabla \dot{g}(t)\, dx \tag{3.18}$$

for almost every $t \in [0, T]$.

Proof. Since $u^n(t) = g^n(t)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$ we have $u(t) = g(t)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$. That $S(u(t)) \tilde{\subset} \Gamma(t)$ is an immediate consequence of item (a) in Definition 3.1, since $S(u^n(t)) \subset \Gamma^n(t)$. By item (b) in the same definition, there exists $v \in SBV^p(\mathbb{R}^N; \mathbb{R}^m)$ with $S(v) \tilde{=} \Gamma(t)$ and a sequence $v^n \in SBV^p(\mathbb{R}^N; \mathbb{R}^m)$ with $S(v^n) \tilde{\subset} \Gamma^n(t)$ such that $v^n \overset{SBV^p}{\rightharpoonup} v$. We now apply the jump transfer theorem (Theorem 2.1 in [9]) and conclude to the existence, for an arbitrary element $w \in SBV^p(\mathbb{R}^N; \mathbb{R}^m)$ with $w = g(t)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$, of a sequence $w^n \in SBV^p(\mathbb{R}^N; \mathbb{R}^m)$ such that

$$\begin{gathered} w^n \equiv w = g(t) \text{ a.e. on } \mathbb{R}^N \setminus \overline{\Omega}, \\ w^n \to w \text{ in } L^1(\mathbb{R}^N; \mathbb{R}^m), \\ \nabla w^n \to \nabla w \text{ in } L^p(\mathbb{R}^N; \mathbb{R}^{mN}), \\ \mathcal{H}^{N-1}\Big((S(w^n) \setminus S(v^n)) \setminus (S(w) \setminus \Gamma(t))\Big) \longrightarrow 0. \end{gathered} \tag{3.19}$$

Because $S(v^n) \subset \Gamma^n(t)$, the last inequality above *a fortiori* implies that

$$\mathcal{H}^{N-1}\Big((S(w^n) \setminus \Gamma^n(t)) \setminus (S(w) \setminus \Gamma(t))\Big) \longrightarrow 0. \tag{3.20}$$

Now, in view of Remark 2.2, $u^{n_t}(t)$ minimizes

$$\int_\Omega W(\nabla v)\, dx + \mathcal{H}_c^{N-1}(S(v) \setminus \Gamma^{n_t}(t))$$

on $\{v \in SBV^p(\mathbb{R}^N; \mathbb{R}^m) : v = g^{n_t}(t) \text{ a.e. on } \mathbb{R}^N \setminus \overline{\Omega}\}$, so that

$$\begin{aligned} &\int_\Omega W(\nabla u^{n_t}(t))\, dx \\ &\qquad \le \int_\Omega W(\nabla w^{n_t}(t) + \nabla g^{n_t}(t) - \nabla g(t))\, dx + \mathcal{H}_c^{N-1}(S(w^{n_t}(t)) \setminus \Gamma^{n_t}(t)). \end{aligned} \tag{3.21}$$

Since W is quasiconvex with p-growth, $u^{n_t}(t)$ SBV^p-converges to $u(t)$, and the sequence $\mathcal{H}^{N-1}(S(u^{n_t}(t))$ is uniformly bounded, Theorem 5.29 in [2] implies that

$$\int_\Omega W(\nabla u(t))\, dx \le \liminf_{n_t \to \infty} \int_\Omega W(\nabla u^{n_t}(t))\, dx. \tag{3.22}$$

Recalling (3.19)–(3.22), and the fact that $\nabla g^{n_t}(t) \overset{L^p(\mathbb{R}^N;\mathbb{R}^{mN})}{\longrightarrow} \nabla g(t)$, we conclude that

$$\int_\Omega W(\nabla u(t))\, dx \le \int_\Omega W(\nabla w)\, dx + \mathcal{H}_c^{N-1}(S(w) \setminus \Gamma(t))$$

and obtain the minimality result.

To prove (3.16), we apply the jump transfer theorem once again, this time to $u(t)$, thus obtaining a sequence $w_{n_t} \in SBV^p(\mathbb{R}^N; \mathbb{R}^m)$ with $w_{n_t} \equiv g^{n_t}(t)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$ and such that

$$\begin{gathered} \nabla w_{n_t} \to \nabla u(t) \text{ in } L^p(\mathbb{R}^N; \mathbb{R}^{mN}), \\ \mathcal{H}_c^{N-1}(S(w_{n_t}) \setminus S(u^{n_t}(t))) \to 0. \end{gathered}$$

Since $u^{n_t}(t)$ is in particular a minimizer for its own jump set,

$$\int_\Omega W(\nabla u^{n_t}(t))\, dx \le \int_\Omega W(\nabla w_{n_t})\, dx + \mathcal{H}_c^{N-1}(S(w_{n_t}) \setminus S(u^{n_t}(t))).$$

Thus

$$\limsup_{n_t\to\infty}\int_\Omega W(\nabla u^{n_t}(t))\,dx \le \int_\Omega W(\nabla u(t))\,dx,$$

which, together with (3.22), yields the desired result.

To prove (3.17), we appeal to Lemma 4.11 in [7], which states in essence that SBV^p-convergence of $u^{n_t}(t)$ to $u(t)$, together with convergence (3.16) of the energy, implies weak convergence of the stresses.

Finally, (3.18) is an immediate consequence of (3.11) and (3.17). $\square$

We now derive an elementary estimate on the total energy at time t, that is

$$\mathcal{E}(t) := \int_\Omega W(\nabla u(t))\,dx + \mathcal{H}_c^{N-1}(\Gamma(t)). \tag{3.23}$$

We also define the corresponding total energy for $u^n(t)$, namely

$$\mathcal{E}^n(t) := \int_\Omega W(\nabla u^n(t))\,dx + \mathcal{H}_c^{N-1}(\Gamma^n(t)). \tag{3.24}$$

The following lemma then holds.

Lemma 3.4 *For any $t \in [0,T]$,*

$$\mathcal{E}(t) \le \mathcal{E}(0) + \int_0^t \int_\Omega DW(\nabla u(s))\cdot\nabla\dot g(s)\,dx\,ds. \tag{3.25}$$

Proof. We recall (3.6), namely

$$\begin{aligned}
\mathcal{E}^{n_t}(t) &= \int_\Omega W(\nabla u^{n_t}(t))\,dx + \mathcal{H}_c^{N-1}(\Gamma^{n_t}(t)) \\
&\le \int_\Omega W(\nabla u^{n_t}(0))\,dx + \mathcal{H}_c^{N-1}(\Gamma_0) \\
&\qquad + \int_0^t\int_\Omega DW(\nabla u^{n_t}(s) + \Psi^{n_t}(s))\cdot\nabla\dot g(s)\,dx\,ds.
\end{aligned}$$

and pass to the limit in n_t. The SBV^p-convergence of $u^{n_t}(t)$ to $u(t)$, together with Theorem 5.29 in [2] permits us to pass to the lim-inf in the first term of the left side of the above inequality while we appeal to (3.13) for the surface term.

Finally, in view of (3.5), a simple argument based on the uniform continuity of DW on compact sets, together with the already established uniform bound on $\nabla u^n(t)$ in $L^p(\mathbb{R}^N;\mathbb{R}^{mN})$ (cf. (3.2)) permits to drop $\Psi^{n_t}(s)$ in the remaining term; see Lemma 4.9 in [7] for a complete proof in a more general setting. Specifically, for a.e. $s \in [0,t]$, Lemma 4.9 in [7] yields

$$\int_\Omega DW(\nabla u^{n_t}(s) + \Psi^{n_t}(s))\cdot\nabla\dot g(s)\,dx - \int_\Omega DW(\nabla u^{n_t}(s))\cdot\nabla\dot g(s)\,dx \overset{n\to\infty}{\longrightarrow} 0.$$

The growth property of DW, together with the uniform $L^p(\mathbb{R}^N;\mathbb{R}^{mN})$-bound on ∇u^{n_t} and (3.5), imply that

$$\int_0^t \Big(\int_\Omega DW(\nabla u^{n_t}(s) + \Psi^{n_t}(s))\cdot\nabla\dot g(s)\,dx - \int_\Omega DW(\nabla u^{n_t}(s))\cdot\nabla\dot g(s)\,dx\Big)\,ds \overset{n\to\infty}{\longrightarrow} 0,$$

We obtain

$$\mathcal{E}(t) \le \mathcal{E}(0) + \liminf_{n_t\to\infty} \int_0^t \Big(\int_\Omega DW(\nabla u^{n_t}(s)) \cdot \nabla \dot g(s)\, dx \Big)\, ds.$$

In view of (3.10) and (3.18), the last term in the above inequality is bounded from above by the announced expression. □

It now remains to establish that inequality (3.25) is actually an equality. This is the object of the following lemma.

Lemma 3.5 *We have*

$$\mathcal{E}(t) \ge \mathcal{E}(0) + \int_0^t \int_\Omega DW(\nabla u(s)) \cdot \nabla \dot g(s)\, dx\, ds. \tag{3.26}$$

Proof. We take $v \equiv u(t) + g(s) - g(t)$ as a competitor for $u(s)$ in the minimum problem for (3.15), and get, since $S(u(t)) \,\tilde{\subset}\, \Gamma(t)$,

$$\begin{aligned}\mathcal{E}(s) \;&\le\; \int_\Omega W(\nabla v)\, dx + \mathcal{H}^{N-1}_c(S(u(t)) \setminus \Gamma(s)) + \mathcal{H}^{N-1}_c(\Gamma(s)) \\ &\le\; \int_\Omega W(\nabla v)\, dx) + \mathcal{H}^{N-1}_c(\Gamma(t)),\end{aligned}$$

so that, for some $\rho(s,t) \in [0,1]$,

$$\begin{aligned}\mathcal{E}(t) - \mathcal{E}(s) \;&\ge\; \int_\Omega (W(\nabla u(t)) - W(\nabla v))\, dx \\ &=\; \int_\Omega \Big[DW\Big(\nabla u(t) + \rho(s,t) \int_s^t \nabla \dot g(\tau)\, d\tau\Big) \cdot \int_s^t \nabla \dot g(\tau)\, d\tau \Big]\, dx.\end{aligned} \tag{3.27}$$

Consider a partition $0 := s^n_0 \le s^n_1 \le \cdots \le s^n_{k(n)} = t$ such that

$$\lim_{n\to\infty} \max_{1\le i\le k(n)} (s^n_i - s^n_{i-1}) = 0\,, \tag{3.28}$$

define

$$u_n(s) := u(s^n_{i+1}) \qquad \text{and} \qquad X_n(s) := \rho(s^n_i, s^n_{i+1}) \int_{s^n_i}^{s^n_{i+1}} \nabla \dot g(\tau)\, d\tau\,,$$

for $s \in (s^n_i, s^n_{i+1}]$, and note that, since $g \in W^{1,1}((0,t); W^{1,p}(\mathbb{R}^N;\mathbb{R}^m))$,

$$\|X_n(s)\|_{L^p(\mathbb{R}^N;\mathbb{R}^{mN})} \to 0, \text{ uniformly on } [0,t]. \tag{3.29}$$

We apply (3.27) for $s = s^n_i$ and $t = s^n_{i+1}$, and sum the result for $i = 0, \dots, k(n)-1$; we obtain

$$\mathcal{E}(t) - \mathcal{E}(0) \ge \int_0^t \int_\Omega DW(\nabla u_n(s) + X_n(s)) \cdot \nabla \dot g(s)\, dx\, ds.$$

Recalling (3.29), we immediately infer, using an argument similar to that which allowed to drop the term in Ψ^n in (3.6) for the proof of Lemma 3.4 (see once again Lemma 4.9 in [7]), that, for a.e. $s \in [0,t]$,

$$\int_\Omega DW(\nabla u_n(s) + X_n(s)) \cdot \nabla \dot g(s)\, dx - \int_\Omega DW(\nabla u_n(s)) \cdot \nabla \dot g(s)\, dx \overset{n\to\infty}{\longrightarrow} 0.$$

The growth property of DW, together with the uniform $L^p(\mathbb{R}^N;\mathbb{R}^{mN})$-bound on ∇u_n and (3.29), imply that

$$\int_0^t \Big(\int_\Omega DW(\nabla u_n(s) + X_n(s)) \cdot \nabla \dot g(s)\, dx - \int_\Omega DW(\nabla u_n(s)) \cdot \nabla \dot g(s)\, dx \Big)\, ds \overset{n\to\infty}{\longrightarrow} 0,$$

so that

$$\mathcal{E}(t) - \mathcal{E}(0) \geq \limsup_{n\to\infty} \int_0^t \int_\Omega DW(\nabla u_n(s)) \cdot \nabla \dot g(s)\, dx\, ds. \tag{3.30}$$

To complete the proof, we need to appeal to the following result in measure theory (see Lemma 4.12 in [7]).

Lemma 3.6 *Let X be a Banach space and $f \in L^1((0,t);X)$. Then, there exists a sequence of subdivisions $0 = s_0^n \leq s_1^n \leq \cdots \leq s_{k(n)}^n = t$, satisfying (3.28), such that*

$$\lim_{n\to\infty} \sum_{i=1}^{k(n)} \Big\| (s_i^n - s_{i-1}^n) f(s_i^n) - \int_{s_{i-1}^n}^{s_i^n} f(t)\, dt \Big\|_X = 0.$$

We apply this lemma to

$$f := (\nabla \dot g, \theta) \in L^1((0,t); L^p(\Omega;\mathbb{R}^{mN}) \times \mathbb{R}),$$

which allows to find a sequence of subdivisions $0 = s_0^n \leq s_1^n \leq \cdots \leq s_{k(n)}^n = t$, so that first $\nabla \dot g(s)$ is replaced by

$$G_n(s) := \nabla \dot g(s_i^n),\ s_{i-1}^n < s \leq s_i^n$$

in (3.30), and also so that

$$\int_0^t \int_\Omega DW(\nabla u_n(s)) \cdot G_n(s)\, dx\, ds \longrightarrow \int_0^t \theta(s)\, ds.$$

In view of the expression (3.18) for $\theta(s)$, we get the desired result. □

The proof of Theorem 2.1 is complete.

Proof of Theorem 2.3. First of all we observe that if the pair $\Gamma(t)$, $u(t)$ satisfies all the conclusions of Theorem 2.1, then the value of the integral

$$\int_\Omega W(\nabla u(t))\, dx \tag{3.31}$$

does not depend on the choice of $u(t)$. Indeed, (3.31) minimizes $\int_\Omega W(\nabla v)\, dx$ among all $v \in SBV^p(\mathbb{R}^N;\mathbb{R}^m)$ with $v = g(t)$ a.e. on $\mathbb{R}^N \setminus \overline{\Omega}$ and $S(v) \,\tilde{\subset}\, \Gamma(t) \cup \partial\Omega_f$.

Recalling the definition (3.24) of the total energy $\mathcal{E}^n(t)$ and using an argument identical to that used in the proof of Lemma 3.4, we obtain, with the function $u(t)$ constructed in the proof of Theorem 2.1,

$$\begin{aligned} \limsup_{n\to\infty} \mathcal{E}^n(t) - \mathcal{E}(0) &\leq \limsup_{n\to\infty} \int_0^t \theta^n(s)\, ds \leq \int_0^t \theta(s)\, ds \\ &= \int_0^t \int_\Omega DW(\nabla u(s)) \cdot \nabla \dot g(s)\, dx\, ds, \end{aligned} \tag{3.32}$$

where we have appealed to (3.10) in deriving the last inequality and invoked the expression (3.18) for $\theta(t)$. Hence, *a fortiori*,

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega W(\nabla u^n(t))\,dx + \liminf_{n\to\infty} \mathcal{H}_c^{N-1}(\Gamma^n(t)) - \mathcal{E}(0) \\ \le \int_0^t \int_\Omega DW(\nabla u(s)) \cdot \nabla \dot g(s)\,dx\,ds = \mathcal{E}(t) - \mathcal{E}(0),\end{aligned}$$

where the last equality follows from (2.3). In view of (3.13) we conclude that

$$\limsup_{n\to\infty} \int_\Omega W(\nabla u^n(t))\,dx \le \int_\Omega W(\nabla u(t))\,dx. \tag{3.33}$$

Now, consider a t-dependent sequence $\{n_t\}$ such that

$$\liminf_{n\to\infty} \int_\Omega W(\nabla u^n(t))\,dx = \lim_{n_t\to\infty} \int_\Omega W(\nabla u^{n_t}(t))\,dx. \tag{3.34}$$

The sequence $\{u^{n_t}(t)\}$ may be assumed to SBV^p-converge to some $\overline{u}(t) \in SBV^p(\mathbb{R}^N;\mathbb{R}^m)$, and, as in the proof of Lemma 3.3, σ^p-convergence, together with the jump transfer theorem imply that, just like $u(t)$, $\overline{u}(t)$ minimizes

$$\int_\Omega W(\nabla v)\,dx + \mathcal{H}_c^{N-1}(S(v) \setminus \Gamma(t))$$

among all v in $\{v \in SBV^p(\mathbb{R}^N;\mathbb{R}^m) : v = g(t) \text{ a.e. on } \mathbb{R}^N\backslash\overline{\Omega}\}$, and that $S(\overline{u}(t)) \mathrel{\tilde{\subset}} \Gamma(t)$. Thus,

$$\int_\Omega W(\nabla \overline{u}(t))\,dx = \int_\Omega W(\nabla u(t))\,dx.$$

But, by sequential weak lower semi-continuity,

$$\int_\Omega W(\nabla \overline{u}(t))\,dx \le \liminf_{n\to\infty} \int_\Omega W(\nabla u^n(t))\,dx,$$

hence

$$\int_\Omega W(\nabla u(t))\,dx \le \liminf_{n\to\infty} \int_\Omega W(\nabla u^n(t))\,dx, \tag{3.35}$$

which, together with (3.33) yields

$$\lim_{n\to\infty} \int_\Omega W(\nabla u^n(t))\,dx = \int_\Omega W(\nabla u(t))\,dx. \tag{3.36}$$

Finally, recalling (3.32), (2.3), (3.13) and (3.35), we have

$$\begin{aligned}\int_\Omega W(\nabla u(t))\,dx \;+\; \mathcal{H}_c^{N-1}(\Gamma(t)) \le \liminf_{n\to\infty} \mathcal{E}^n(t) \\ \le \limsup_{n\to\infty} \mathcal{E}^n(t) \le \int_\Omega W(\nabla u(t))\,dx + \mathcal{H}_c^{N-1}(\Gamma(t)),\end{aligned}$$

so that,

$$\lim_{n\to\infty} \mathcal{E}^n(t) = \int_\Omega W(\nabla u(t))\,dx + \mathcal{H}_c^{N-1}(\Gamma(t))$$

and, in view of (3.36),

$$\mathcal{H}_c^{N-1}(\Gamma(t)) = \lim_{n\to\infty} \mathcal{H}_c^{N-1}(\Gamma^n(t)).$$

The proof of Theorem 2.3 is complete. □

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