QUASISTATIC CRACK GROWTH IN FINITE ELASTICITY

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Abstract. In this paper, we prove a new existence result for a variational model of crack growth in brittle materials proposed in [15]. We consider the case of n-dimensional finite elasticity, for an arbitrary $n \geq 1$, with a quasiconvex bulk energy and with prescribed boundary deformations and applied loads, both depending on time.

Keywords: variational models, energy minimization, free-discontinuity problems, quasiconvexity, crack propagation, quasistatic evolution, brittle fracture, Griffith's criterion.

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1. INTRODUCTION

In this paper we present a new existence result for a variational model of quasistatic growth for brittle cracks introduced in [15] and based on Griffith's idea (see [16]) that the crack growth is determined by the competition between the elastic energy of the body and the work needed to produce a new crack, or extend an existing one. The main feature of this model is that the crack path is not prescribed, but is a result of energy balance. In order to obtain our existence theorems in any space dimension and for a general bulk energy, we introduce a mathematical formulation of the problem in a suitable space of functions which may exhibit jump discontinuities on sets of codimension one.

We now describe the model in more detail. The reference configuration is a bounded open set Ω of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, with $\partial_D\Omega \cap \partial_N\Omega = \emptyset$. On the Dirichlet part $\partial_D\Omega$ of the boundary we prescribe the boundary deformation, while on the Neumann part $\partial_N\Omega$ we apply the surface forces. In our formulation, a crack is any rectifiable set Γ contained in $\overline{\Omega}$ and with finite $n-1$ dimensional Hausdorff measure. We assume that the work done to produce the crack Γ can be written as

$$\mathcal{K}(\Gamma) := \int_{\Gamma \setminus \partial_N \Omega} \kappa(x, \nu_\Gamma(x))\, d\mathcal{H}^{n-1}(x)\,,$$

where ν_Γ is a unit normal vector field on Γ and $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure. The function $\kappa(x,\nu)$ depends on the material and satisfies the standard hypotheses which guarantee the lower semicontinuity of $\mathcal{K}$. Since $\kappa(x,\nu)$ depends on the position x and on the orientation ν, we are able to deal with heterogeneous and anisotropic materials (see Subsection 3.2).

We adopt the framework of hyperelasticity and assume that the bulk energy of the uncracked part of the body is given by

$$\mathcal{W}(\nabla u) := \int_{\Omega \setminus \Gamma} W(x, \nabla u(x))\, dx\,,$$

where $u\colon \Omega \setminus \Gamma \to \mathbb{R}^n$ is the unknown deformation of the body, and $W(x,\xi)$ is a given function depending on the material. We only suppose that $W(x,\xi)$ is quasiconvex with respect to ξ and satisfies suitable growth and regularity conditions (see Subsection 3.3). It is convenient to consider the deformation of the uncracked part $\Omega \setminus \Gamma$ of the body as a function u defined almost everywhere on Ω, whose discontinuity set $S(u)$ is contained in

$\varGamma$. An adequate functional setting for these deformations is a suitable subspace of the space $GSBV(\Omega;\mathbb{R}^n)$ introduced in [10] and studied in [1] (see Section 2).

For every time $t \in [0,T]$, the applied load is given by a system of t-dependent body and surface forces. We assume that these forces are conservative and that their work is given by

$$\mathcal{F}(t)(u) := \int_{\Omega\setminus\varGamma} F(t,x,u(x))\,dx\,, \qquad \mathcal{G}(t)(u) := \int_{\partial_S\Omega} G(t,x,u(x))\,d\mathcal{H}^{n-1}(x)\,,$$

where $\partial_S\Omega$ is a subset of $\partial_N\Omega$, and F and G satisfy suitable regularity and growth conditions (see Subsections 3.4 and 3.5 and Section 9). To avoid interactions between cracks and surface forces, we impose that all cracks remain at a positive distance from $\partial_S\Omega$ (see (3.1) and Remark 3.6).

We adopt the following terminology: an admissible configuration is a pair $(u,\varGamma)$, where $\varGamma$ is an admissible crack and u is an admissible deformation with jump set $S(u)$ contained in $\varGamma$. The total energy of $(u,\varGamma)$ at time t is defined by

$$\mathcal{E}(t)(u,\varGamma) := \mathcal{W}(\nabla u) + \mathcal{K}(\varGamma) - \mathcal{F}(t)(u) - \mathcal{G}(t)(u)\,.$$

For every time $t \in [0,T]$ we prescribe a "continuous" boundary displacement $\psi(t)$ on $\partial_D\Omega\setminus\varGamma(t)$, where $\varGamma(t)$ is the unknown crack at time t. We thus assume that $\psi(t)$ is the trace on $\partial_D\Omega$ of a function in a suitable Sobolev space on Ω, so that we cannot a priori impose a "strong discontinuity" at the boundary, like a jump in the prescribed displacement $\psi(t)$. Moreover, we also assume that $t \mapsto \psi(t)$ is sufficiently regular (see Subsection 3.6). The set $AD(\psi(t),\varGamma(t))$ of admissible deformations with crack $\varGamma(t)$ and boundary displacement $\psi(t)$ is then defined as the set of deformations u in a suitable subspace of $GSBV(\Omega;\mathbb{R}^n)$, whose jump set $S(u)$ is contained in $\varGamma(t)$ and whose trace agrees with $\psi(t)$ on $\partial_D\Omega\setminus\varGamma(t)$.

In the spirit of Griffith's original theory, a minimum energy configuration at time t is an admissible configuration $(u(t),\varGamma(t))$, with $u(t) \in AD(\psi(t),\varGamma(t))$, such that

$$\mathcal{E}(t)(u(t),\varGamma(t)) \le \mathcal{E}(t)(u,\varGamma)$$

for every admissible crack $\varGamma$ containing $\varGamma(t)$ and for every $u \in AD(\psi(t),\varGamma)$. In other words, the energy of $(u(t),\varGamma(t))$ can not be reduced by choosing a larger crack and, possibly, a new deformation with the same boundary condition (see Subsection 3.8).

An irreversible quasistatic evolution of minimum energy configurations is a function $t \mapsto (u(t),\varGamma(t))$ which satisfies the following conditions:

(a) static equilibrium: for every $t \in [0,T]$ the pair $(u(t),\varGamma(t))$ is a minimum energy configuration at time t;
(b) irreversibility: $\varGamma(s)$ is contained in $\varGamma(t)$ for $0 \le s < t \le T$;
(c) nondissipativity: the derivative of the internal energy equals the power of the applied forces.

In condition (c) the (loosely named) internal energy is defined by

$$\mathcal{E}^{in}(t)(u(t),\varGamma(t)) := \mathcal{W}(\nabla u(t)) + \mathcal{K}(\varGamma(t))\,,$$

while the power of the external forces is given by

$$\begin{aligned}\int_{\partial_D\Omega\setminus\varGamma(t)} \partial_\xi W(x,\nabla u(t))\nu\,\dot\psi(t)\,d\mathcal{H}^{n-1} + \int_\Omega \partial_z F(t,x,u(t))\dot u(t)\,dx\,+ \\ + \int_{\partial_S\Omega} \partial_z G(t,x,u(t))\dot u(t)\,d\mathcal{H}^{n-1}\,,\end{aligned} \tag{1.1}$$

where ν is the outer unit normal to $\partial\Omega$, $\dot\psi(t)$ and $\dot u(t)$ denote the time derivatives of $\psi(t)$ and $u(t)$, while $\partial_\xi W$, $\partial_z F$, and $\partial_z G$ are the partial derivatives of $W(x,\xi)$, $F(t,x,z)$, and $G(t,x,z)$ with respect to ξ and z. As $\partial_\xi W(x,\nabla u(t))\nu$ is the boundary traction corresponding to the deformation $u(t)$, the first term in (1.1) is the power of the surface force which produces the boundary displacement $\psi(t)$ on $\partial_D\Omega\setminus\Gamma(t)$.

Unfortunately, formula (1.1) makes sense only if $\psi(t)$ and $u(t)$ are sufficiently regular with respect to t (see Remark 3.9), while there are quasistatic evolutions that are discontinuous with respect to t. Therefore we prefer to express the conservation of energy in a weaker form, which makes sense even if u is not regular (see Subsection 3.9).

The main result of this paper is the following existence theorem: if (u_0,Γ_0) is a minimum energy configuration at time $t=0$, then there exists an irreversible quasistatic evolution $t\mapsto(u(t),\Gamma(t))$ with $(u(0),\Gamma(0))=(u_0,\Gamma_0)$ (see Theorem 3.13).

As for the hypothesis of the previous theorem, we remark that for every initial boundary displacement $\psi(0)$ and for every crack Γ there exists a minimum energy configuration (u_0,Γ_0) at time $t=0$ with Γ_0 containing Γ (see Theorem 3.8). For special initial displacements u_0 it is easy to determine the cracks Γ_0 such that (u_0,Γ_0) can be used as initial condition in the existence theorem. For instance, if $\psi(0)$ and u_0 coincide with the identity map u_{id}, then (u_{id},Γ_0) is a minimum energy configuration at time $t=0$ for every crack Γ_0, under very natural assumptions on W, F, and G (see Remark 3.12).

Previous results on this subject have been obtained in [9] in the case $n=2$ for a scalar-valued u and for $W(\xi)=|\xi|^2$, which corresponds to the antiplane case in linear elasticity. In that paper the admissible cracks are assumed to be connected, or with a uniform bound on the number of connected components. This restriction allows to simplify the mathematical formulation of the problem. Indeed, in [9] the cracks are assumed to be closed and, consequently the deformations belong to a suitable Sobolev space.

These results were extended to the case of planar linear elasticity by Chambolle in [7]. In both papers the existence of a solution is obtained by an approximation argument, where the approximating cracks converge in the sense of the Hausdorff metric, while the approximating deformation gradients converge strongly in L^2.

The paper [14] removes the restriction on the connected components of Γ and on the dimension of the space, and introduces a weak formulation in the space $SBV(\Omega)$. The function u is still scalar-valued and this hypothesis is used to obtain some compactness results which need a uniform L^∞-bound that, in the scalar case, can be easily obtained by truncation. It also provides a jump transfer theorem that is instrumental in the present analysis (see Subsection 5.1).

In the present paper, we deal with the vector case, where the deformation u maps a subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$ (or, more generally, into $\mathbb{R}^m$, so as to include the antiplane case when $m=1$). This forces us to introduce a weaker formulation in the larger space $GSBV(\Omega;\mathbb{R}^m)$, where a compactness theorem holds under more general hypotheses. Another new feature of this paper is that we consider the case of finite elasticity, with an arbitrary quasiconvex bulk energy with polynomial growth, and allow for a large class of body and surface forces. In truth however, our formulation is not all encompassing; it does not allow for constant body loads like gravity, or conservative surface loads like pressure.

As in prior works [9], [7], [14], our result is obtained by time discretization. We fix a sequence of subdivisions $(t_k^i)_{0\le i\le k}$ of the interval $[0,T]$, with $0=t_k^0<t_k^1<\cdots<t_k^{k-1}<t_k^k=T$ and $\lim_k\max_i(t_k^i-t_k^{i-1})=0$, and define by induction an approximate solution (u_k^i,Γ_k^i) at time t_k^i: let $(u_k^0,\Gamma_k^0):=(u_0,\Gamma_0)$, and, for $i=1,\dots,k$, let (u_k^i,Γ_k^i) be a solution of the minimum problem

$$\min\{\mathcal{E}(t_k^i)(u,\Gamma):\Gamma_k^{i-1}\subset\Gamma,\ u\in AD(\psi(t_k^i),\Gamma)\},$$

whose existence can be deduced from the $GSBV$ compactness theorem of [2]. For every $t\in[0,T]$ we consider the piecewise constant interpolations

$$\tau_k(t):=t_k^i,\quad u_k(t):=u_k^i,\quad \Gamma_k(t):=\Gamma_k^i,$$

where i is the largest integer such that $t_k^i \le t$. The solution $(u(t), \Gamma(t))$ of the continuous-time evolution problem will be obtained by passing to the limit in the sequence $(u_k(t), \Gamma_k(t))$ as $k \to \infty$.

To this aim we introduce a new notion of convergence of sets, called σ^p-convergence, related to the notion of jump sets of SBV functions. We study the main properties of this convergence and prove, in particular, a compactness theorem (see Subsections 4.1 and 4.2). Using these results we show that there exists a subsequence, still denoted $\Gamma_k(t)$, such that $\Gamma_k(t) \cup \partial_N \Omega$ σ^p-converges to a crack $\Gamma(t)$ for every $t \in [0,T]$. Since this subsequence does not depend on t, the cracks $\Gamma(t)$ are easily shown to satisfy the irreversibility condition (b).

We now fix $t \in [0,T]$ and, using the $GSBV$ compactness theorem, we extract a further subsequence of $u_k(t)$, depending on t, which converges to some function $u(t)$. The very definition of σ^p-convergence implies that the jump set $S(u(t))$ is contained in $\Gamma(t)$, so that $u(t) \in AD(\psi(t), \Gamma(t))$. Then we show that $(u(t), \Gamma(t))$ is a minimum energy configuration at time t (condition (a)), using the fact that $(u_k(t), \Gamma_k(t))$ satisfies the same property at time $\tau_k(t)$. A crucial tool in the proof of this stability result for minimizers is the jump transfer theorem, established in [14] in the case of SBV functions, and extended here to the $GSBV$ setting (see Subsections 5.1 and 5.2).

It remains to prove condition (c) on the conservation of energy, in the weak form given in Subsection 3.9. To this aim we introduce the functions

$$\begin{aligned} \theta_k(t) := \langle \partial\mathcal{W}(\nabla u_k(t)), \nabla\dot\psi(t)\rangle - \langle \partial\mathcal{F}(\tau_k(t))(u_k(t)), \dot\psi(t)\rangle - \\ - \dot{\mathcal{F}}(t)(u_k(t)) - \langle \partial\mathcal{G}(\tau_k(t))(u_k(t)), \dot\psi(t)\rangle - \dot{\mathcal{G}}(t)(u_k(t)), \end{aligned}$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing in suitable L^p spaces, $\partial\mathcal{F}$ and $\partial\mathcal{G}$ are the differentials of the functionals $\mathcal{F}$ and $\mathcal{G}$ in these function spaces, while $\dot{\mathcal{F}}$ and $\dot{\mathcal{G}}$ are the time derivatives of $\mathcal{F}$ and $\mathcal{G}$. By using the minimality property which defines (u_k^i, Γ_k^i), we prove the fundamental estimate

$$\mathcal{E}(\tau_k(t))(u_k(t), \Gamma_k(t)) \le \mathcal{E}(0)(u_0, \Gamma_0) + \int_0^{\tau_k(t)} \theta_k(s)\, ds + R_k,$$

with $R_k \to 0$ as $k \to \infty$ (see Section 6).

The main difficulty is to pass to the limit in the first term in the definition of $\theta_k(t)$, since $\nabla u_k(t)$ converges only weakly. We overcome this difficulty by proving a technical result (Lemma 4.11), which shows that convergence of the energies implies convergence of the stresses (see Subsection 4.3). Since $\mathcal{E}$ is lower semicontinuous, we can pass to the limit in the previous estimate, obtaining the energy inequality

$$\mathcal{E}(t)(u(t), \Gamma(t)) \le \mathcal{E}(0)(u_0, \Gamma_0) + \int_0^t \theta(s)\, ds,$$

where

$$\begin{aligned} \theta(t) := \langle \partial\mathcal{W}(\nabla u(t)), \nabla\dot\psi(t)\rangle - \langle \partial\mathcal{F}(t)(u(t)), \dot\psi(t)\rangle - \\ - \dot{\mathcal{F}}(t)(u(t)) - \langle \partial\mathcal{G}(t)(u(t)), \dot\psi(t)\rangle - \dot{\mathcal{G}}(t)(u(t)). \end{aligned}$$

Recalling the weak formulation of condition (c) given in Subsection 3.9, it remains to show that

$$\mathcal{E}(t)(u(t), \Gamma(t)) \ge \mathcal{E}(0)(u_0, \Gamma_0) + \int_0^t \theta(s)\, ds.$$

To prove this inequality we consider a sequence of subdivisions $(s_k^i)_{0\le i\le i_k}$ of the interval $[0,t]$, with $0 = s_k^0 < s_k^1 < \cdots < s_k^{i_k-1} < s_k^{i_k} = t$ and $\lim_k \max_i (s_k^i - s_k^{i-1}) = 0$, and compare $\mathcal{E}(s_k^{i-1})(u(s_k^{i-1}), \Gamma(s_k^{i-1}))$ with $\mathcal{E}(s_k^i)(u(s_k^i), \Gamma(s_k^i))$, thanks to the minimality property of $(u(s_k^{i-1}), \Gamma(s_k^{i-1}))$ given by condition (a). In this way we obtain an estimate of the form

$$\mathcal{E}(t)(u(t), \Gamma(t)) \ge \mathcal{E}(0)(u_0, \Gamma_0) + \Theta_k(t),$$

where $\Theta_k(t)$ is an intricate expression that can be written in terms of Riemann sums of the functions $\theta(s)$, $\nabla\dot{\psi}(s)$, $\dot{\psi}(s)$, $\dot{\mathcal{F}}(s)(u(s))$, and $\dot{\mathcal{G}}(s)(u(s))$ (see Section 7). Although these functions are only Lebesgue integrable, $\Theta_k(t)$ converges to the integral of θ on $[0,t]$ for a suitable choice of the subdivisions (s_k^i) (see Subsections 4.4 and 5.3).

Finally we prove a result which can be used to justify the numerical approximation of the quasistatic evolution based on time discretization. Even if the deformation $u(t)$ is not uniquely determined by the crack $\Gamma(t)$, for every $t \in [0,T]$ the elastic energies and the crack energies of the discrete-time problems converge to the corresponding energies for the continuous-time problems (see Section 8).

2. SPACES OF FUNCTIONS WITH BOUNDED VARIATION

Throughout the paper $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure in $\mathbb{R}^n$ and the $n-1$ dimensional Hausdorff measure, respectively. Unless otherwise specified, the expression *almost everywhere* (abbreviated as *a.e.*) always refers to $\mathcal{L}^n$. If $1 \le r \le \infty$ and E is a set, we use the notation $\|\cdot\|_r$ or $\|\cdot\|_{r,E}$ for the L^r norm on E with respect to $\mathcal{L}^n$ or $\mathcal{H}^{n-1}$ (or to some other measure as dictated by the context).

Given two sets A, B in $\mathbb{R}^n$ we write $A \tilde{\subset} B$ if $\mathcal{H}^{n-1}(A\backslash B) = 0$ and we write $A \cong B$ if $\mathcal{H}^{n-1}(A\triangle B) = 0$, where $A\triangle B := (A\backslash B)\cup(B\backslash A)$ denotes the symmetric difference of A and B.

We say that a set $\Gamma \subset \mathbb{R}^n$ is *rectifiable* if there exists a sequence Γ_i of C^1-manifolds of dimension $n-1$ such that $\Gamma \cong \bigcup_i \Gamma_i$ (these sets are called $(\mathcal{H}^{n-1}, n-1)$ rectifiable in [13]). A *unit normal vector field* ν on Γ is an $\mathcal{H}^{n-1}$-measurable function $\nu\colon \Gamma \to \mathbb{R}^n$, with $|\nu(x)| = 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, such that $\nu(x)$ is normal to Γ_i for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_i$ and for every i. It is well-known that every rectifiable set has a unit normal vector field (indeed, infinitely many, since there is no continuity assumption) and that the definition does not depend on the decomposition of Γ (see, e.g., [4, Remark 2.87]).

Let U be a bounded open set in $\mathbb{R}^n$ and let $u\colon U \to \mathbb{R}^m$ be a measurable function. Given $x \in U$ we say that $\tilde{u}(x) \in \mathbb{R}^m$ is the *approximate limit* of u at x, and write $\tilde{u}(x) = \operatorname*{ap\,lim}_{y\to x} u(y)$, if for every $\varepsilon > 0$ we have

$$\lim_{\rho\to 0+} \rho^{-n}\mathcal{L}^n(\{y \in B_\rho(x) : |u(y) - \tilde{u}(x)| > \varepsilon\}) = 0\,, \tag{2.1}$$

where $B_\rho(x)$ is the open ball with centre x and radius ρ. We define the *jump set* $S(u)$ of u as the set of points $x \in U$ where the approximate limit of u does not exist. Given $x \in U$ such that $\tilde{u}(x)$ exists, we say that the $m\times n$ matrix $\nabla u(x)$ is the *approximate differential* of u at x if

$$\operatorname*{ap\,lim}_{y\to x} \frac{u(y) - \tilde{u}(x) - \nabla u(x)(y-x)}{|y-x|} = 0\,.$$

The space $BV(U;\mathbb{R}^m)$ of *functions of bounded variation* is defined as the set of all $u \in L^1(U;\mathbb{R}^m)$ such that the distributional gradient Du is a bounded Radon measure on U with values in the space $\mathbb{M}^{m\times n}$ of $m\times n$ matrices. If $u \in BV(U;\mathbb{R}^m)$, we can consider the Lebesgue decomposition $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous with respect to $\mathcal{L}^n$ and $D^s u$ is singular with respect to $\mathcal{L}^n$. In this case the approximate differential $\nabla u(x)$ exists for a.e. $x \in U$ and the function ∇u belongs to $L^1(U;\mathbb{M}^{m\times n})$ and coincides a.e. with the density of $D^a u$ with respect to $\mathcal{L}^n$ (Calderón-Zygmund Theorem, see, e.g., [4, Theorem 3.83]). Note that $S(u)$ coincides with the complement of the set of Lebesgue points for u, up to a set of $\mathcal{H}^{n-1}$-measure 0.

The space $SBV(U;\mathbb{R}^m)$ of *special functions of bounded variation* is defined as the set of all $u \in BV(U;\mathbb{R}^m)$ such that $D^s u$ is concentrated on $S(u)$, i.e., $|D^s u|(U\backslash S(u)) = 0$. As usual, $SBV_{loc}(U;\mathbb{R}^m)$ denotes the space of functions which belong to $SBV(U';\mathbb{R}^m)$ for every open set $U' \subset\subset U$.

Let us fix an exponent p, with $1 < p < +\infty$. The space $SBV^p(U;\mathbb{R}^m)$ is defined as the set of functions $u \in SBV(U;\mathbb{R}^m)$ with $\nabla u \in L^p(U;\mathbb{M}^{m\times n})$ and $\mathcal{H}^{n-1}(S(u)) < +\infty$.

Definition 2.1. A sequence u_k converges to u weakly in $SBV^p(U;\mathbb{R}^m)$ if and only if u_k, $u \in SBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$, $u_k \to u$ a.e. in U, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(U;\mathbb{M}^{m\times n})$, and $\|u_k\|_\infty$ and $\mathcal{H}^{n-1}(S(u_k))$ are bounded uniformly with respect to k.

If $u \in W^{1,p}(U;\mathbb{R}^m)$, then $u \in SBV^p(U;\mathbb{R}^m)$ and $S(u) \cong \emptyset$. The following compactness theorem is proved in [1] (see also [4, Section 4.2]).

Theorem 2.2. *Let u_k be a sequence in $SBV^p(U;\mathbb{R}^m)$ such that $\|u_k\|_\infty$, $\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S(u_k))$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly in $SBV^p(U;\mathbb{R}^m)$.*

This result is not enough for the study of the fracture problem in dimension n, because we have no a priori bound on the L^∞ norm of the solutions. To overcome this difficulty we have to use the wider space $GSBV(U;\mathbb{R}^m)$ of *generalized special functions of bounded variation*, defined as the set of all functions $u\colon U \to \mathbb{R}^m$ such that $\varphi(u) \in SBV_{loc}(U,\mathbb{R}^m)$ for every $\varphi \in C^1(\mathbb{R}^m;\mathbb{R}^m)$ with $\mathrm{supp}(\nabla\varphi) \subset\subset \mathbb{R}^m$. It is easy to see that $SBV(U;\mathbb{R}^m) \subset GSBV(U;\mathbb{R}^m)$ and $GSBV(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m) = SBV_{loc}(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$. If $u \in GSBV(U;\mathbb{R}^m)$, then the approximate differential $\nabla u(x)$ exists for a.e. $x \in U$ (see [3, Propositions 1.3 and 1.4]).

We define $GSBV^p(U;\mathbb{R}^m)$ as the set of functions $u \in GSBV(U;\mathbb{R}^m)$ such that $\nabla u \in L^p(U;\mathbb{M}^{m\times n})$ and $\mathcal{H}^{n-1}(S(u)) < +\infty$. If $u \in GSBV^p(U;\mathbb{R}^m)$ and $\varphi \in C^1(\mathbb{R}^m;\mathbb{R}^m)$ with $\mathrm{supp}(\nabla\varphi) \subset\subset \mathbb{R}^m$, then the function $v := \varphi(u)$ belongs to $SBV_{loc}(U;\mathbb{R}^m)$ and $S(v) \tilde{\subset} S(u)$. As $\nabla v = \nabla\varphi(u)\nabla u$ a.e. in U, we have $\nabla v \in L^p(U;\mathbb{M}^{m\times n})$. Since by [4, Section 3.9]

$$|Dv|(U) \le \int_U |\nabla v(x)|\,dx + 2\|v\|_\infty \mathcal{H}^{n-1}(S(v))\,,$$

we deduce that $v \in BV(U;\mathbb{R}^m)$, and recalling that $v \in SBV_{loc}(U;\mathbb{R}^m)$ we conclude that $v \in SBV^p(U;\mathbb{R}^m)$. The previous discussion shows that $SBV^p(U;\mathbb{R}^m) \subset GSBV^p(U;\mathbb{R}^m)$ and $GSBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m) = SBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$. As usual we set $SBV^p(U) := SBV^p(U;\mathbb{R})$ and $GSBV^p(U) := GSBV^p(U;\mathbb{R})$.

The following proposition proves some basic properties of the space $GSBV^p(U;\mathbb{R}^m)$. Note that the same properties do not hold for $GSBV(U;\mathbb{R}^m)$ (see [4, Remark 4.27]).

Proposition 2.3. *$GSBV^p(U;\mathbb{R}^m)$ is a vector space. A function $u := (u^1,\dots,u^m)\colon U \to \mathbb{R}^m$ belongs to $GSBV^p(U;\mathbb{R}^m)$ if and only if each component u^i belongs to $GSBV^p(U)$.*

Proof. Let u, $v \in GSBV^p(U;\mathbb{R}^m)$ and let $\varphi \in C^1(\mathbb{R}^m;\mathbb{R}^m)$ with $\mathrm{supp}(\nabla\varphi) \subset\subset \mathbb{R}^m$. We have to prove that the function $w := \varphi(u+v)$ belongs to $SBV_{loc}(U;\mathbb{R}^m)$. To this aim we consider a function $\varphi_1 \in C^1_c(\mathbb{R}^m;\mathbb{R}^m)$ such that $\varphi_1(z) = z$ for $|z| \le 1$, and we define $\varphi_k(z) := k\varphi_1(z/k)$. Then $\varphi_k \in C^1_c(\mathbb{R}^m;\mathbb{R}^m)$, $\varphi_k(z) = z$ for $|z| \le k$, and $|\nabla\varphi_k| \le C$ for some constant C independent of k. Let $w_k := \varphi(\varphi_k(u) + \varphi_k(v))$. Since $\varphi_k(u)$ and $\varphi_k(v)$ belong to $SBV^p(U;\mathbb{R}^m)$, the functions w_k belong to $SBV^p(U;\mathbb{R}^m)$. As

$$\nabla w_k = \nabla\varphi(\varphi_k(u) + \varphi_k(v))[\nabla\varphi_k(u)\nabla u + \nabla\varphi_k(v)\nabla v]\,,$$

the sequence ∇w_k is bounded in $L^p(U;\mathbb{M}^{m\times n})$. Moreover $S(w_k) \tilde{\subset} S(u) \cup S(v)$ and $\|w_k\|_\infty \le \|\varphi\|_\infty < +\infty$. Since w_k converges to w a.e. in U, from Theorem 2.2 we deduce that $w \in SBV^p(U;\mathbb{R}^m)$.

Let $u := (u^1,\dots,u^m)$ be a function in $GSBV^p(U;\mathbb{R}^m)$, let $i = 1,\dots,m$, and let $\psi \in C^1(\mathbb{R})$ with $\mathrm{supp}\,\psi' \subset\subset \mathbb{R}$. In order to prove that $u^i \in GSBV^p(U)$ it is enough to show that $\psi(u^i)$ belongs to $SBV(U)$. Let ψ_k be a sequence in $C^1_c(\mathbb{R}^m)$ such that $\|\psi_k\|_\infty \le 1$, $\|\nabla\psi_k\|_\infty \le 1$, and $\psi_k(z) = 1$ for $|z| \le k$, and let $v_k := \psi(u^i)\psi_k(u)$. Then $v_k \in SBV^p(U;\mathbb{R}^m)$, $\|v_k\|_\infty \le \|\psi\|_\infty < +\infty$, $S(v_k) \tilde{\subset} S(u)$, and

$$\nabla v_k = \psi'(u^i)\psi_k(u)\nabla u^i + \psi(u^i)\nabla\psi_k(u)\nabla u\,,$$

so that ∇v_k is bounded in $L^p(U;\mathbb{R}^m)$. Since $v_k \to \psi(u^i)$ a.e. in U, from Theorem 2.2 we deduce that $\psi(u^i) \in SBV^p(U)$.

Conversely, if all components u^i of u belong to $GSBV^p(U)$, then it is easy to see that $u^i e_i$ belongs to $GSBV^p(U;\mathbb{R}^m)$, e_i being the i^{th} vector of the canonical basis of $\mathbb{R}^m$, so that $u \in GSBV^p(U;\mathbb{R}^m)$ by the vector space property. □

From Proposition 2.3 and [4, Theorems 4.34 and 4.40] we obtain that $S(u)$ is rectifiable for every $u \in GSBV^p(U;\mathbb{R}^m)$, and, if ν_u is a unit normal vector field on $S(u)$, then for $\mathcal{H}^{n-1}$-a.e. $x \in S(u)$ there exist two distinct vectors $u^+(x)$, $u^-(x) \in \mathbb{R}^m$ such that

$$u^+(x) = \operatorname*{ap\,lim}_{y\to x,\, y\in H^+(x)} u(y)\,, \qquad u^-(x) = \operatorname*{ap\,lim}_{y\to x,\, y\in H^-(x)} u(y)\,, \tag{2.2}$$

where $H^+(x) := \{y \in U : (y-x)\cdot\nu_u(x) > 0\}$ and $H^-(x) := \{y \in U : (y-x)\cdot\nu_u(x) < 0\}$.

We introduce now the notion of trace on the boundary in the $GSBV^p$ setting.

Proposition 2.4. *If U has a Lipschitz boundary and $u \in GSBV^p(U;\mathbb{R}^m)$, then there exists a function $\tilde u\colon \partial U \to \mathbb{R}^m$ such that*

$$\operatorname*{ap\,lim}_{y\to x,\, y\in U} u(y) = \tilde u(x) \tag{2.3}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \partial U$.

The $\mathcal{H}^{n-1}$-a.e. defined function $\tilde u\colon \partial U \to \mathbb{R}^m$ is called the trace of u on ∂U and in the rest of the paper will be denoted simply by u.

Proof of Proposition 2.4. Let us fix a bounded open set U_0 containing $\overline{U}$. Given $u \in GSBV^p(U;\mathbb{R}^m)$, let u_0 be the function defined by $u_0 := u$ on U and $u_0 := 0$ on $U_0 \setminus U$. For every $\varphi \in C^1(\mathbb{R}^m;\mathbb{R}^m)$, with $\operatorname{supp}(\nabla\varphi) \subset\subset \mathbb{R}^m$, we have $\varphi(u) \in SBV^p(U;\mathbb{R}^m)$. Therefore $\varphi(u_0) \in SBV^p(U_0;\mathbb{R}^m)$ and we conclude that $u_0 \in GSBV^p(U_0;\mathbb{R}^m)$. By the definition of $S(u_0)$, for every $x \in \partial U \setminus S(u_0)$ we have

$$\operatorname*{ap\,lim}_{y\to x} u_0(y) = \tilde u_0(x)\,,$$

which implies (2.3) with $\tilde u(x) := \tilde u_0(x)$. We can choose a unit normal vector field ν_{u_0} on $S(u_0)$ which coincides with the outward unit normal to ∂U $\mathcal{H}^{n-1}$-a.e. on $S(u_0) \cap \partial U$ (see, e.g., [4, Proposition 2.85]). Since in this case

$$\lim_{\rho\to 0+} \rho^{-n}\mathcal{L}^n(B_\rho(x) \cap (H^-(x) \triangle U)) = 0$$

for $\mathcal{H}^{n-1}$-a.e. $x \in S(u_0) \cap \partial U$, the second equality in (2.2) gives (2.3) with $\tilde u(x) := u_0^-(x)$. □

For every $q \geq 1$ we set $GSBV^p_q(U;\mathbb{R}^m) := GSBV^p(U;\mathbb{R}^m) \cap L^q(U;\mathbb{R}^m)$.

Lemma 2.5. *Assume that U has a Lipschitz boundary and that $u \in GSBV^p_q(U;\mathbb{R}^m)$ for some $q \geq 1$. If $S(u) \cong \emptyset$, then u belongs to $W^{1,p}(U;\mathbb{R}^m) \cap L^q(U;\mathbb{R}^m)$.*

Proof. Let $\varphi \in C^1_c(\mathbb{R}^m;\mathbb{R}^m)$ be a function such that $\varphi(z) = z$ for $|z| \leq 1$, and let $\varphi_k(z) := k\varphi(z/k)$. Then $\varphi_k \in C^1_c(\mathbb{R}^m;\mathbb{R}^m)$, $\varphi_k(z) = z$ for $|z| \leq k$, and $|\nabla\varphi_k| \leq C$ for some constant C independent of k. Under our assumptions on u, the functions $v_k := \varphi_k(u)$ belong to $SBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$ and $S(v_k) \cong \emptyset$. This implies that $v_k \in W^{1,p}(U,\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$. Since U has a Lipschitz boundary there exists a constant $\gamma > 0$, depending only on p, q, and U, such that $\|v_k\|_p \leq \gamma(\|\nabla v_k\|_p + \|v_k\|_q)$ for every k. As $|\varphi_k(z)| \leq C|z|$ for every $z \in \mathbb{R}^m$ and every k, we have $\|v_k\|_q \leq C\|u\|_q$. Since $\nabla v_k = \nabla\varphi_k(u)\nabla u$, we have also $\|\nabla v_k\|_p \leq C\|\nabla u\|_p$. Therefore v_k is bounded in $W^{1,p}(U;\mathbb{R}^m)$. As v_k converges to u pointwise a.e. on U, we conclude that $u \in W^{1,p}(U;\mathbb{R}^m)$. □

In the spirit of Definition 2.1, we introduce the following notion of convergence.

Definition 2.6. A sequence u_k converges to u weakly in $GSBV^p(U;\mathbb{R}^m)$ if and only if u_k, u belong to $GSBV^p(U;\mathbb{R}^m)$, $u_k \to u$ a.e. in U, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(U;\mathbb{M}^{m\times n})$, and $\mathcal{H}^{n-1}(S(u_k))$ is bounded uniformly with respect to k.

It is immediate that weak convergence in $SBV^p(U;\mathbb{R}^m)$ implies weak convergence in $GSBV^p(U;\mathbb{R}^m)$. The following compactness theorem for $GSBV^p(U;\mathbb{R}^m)$ is proved in [2, Theorem 2.2] (see also [4, Section 4.5]).

Theorem 2.7. *Let u_k be a sequence in $GSBV^p(U;\mathbb{R}^m)$ such that $\|u_k\|_1$, $\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S(u_k))$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly in $GSBV^p(U;\mathbb{R}^m)$.*

We recall that a function $W\colon \mathbb{M}^{m\times n} \to \mathbb{R}$ is said to be *quasiconvex* if

$$\int_U W(\xi + \nabla\varphi(x))\,dx \ge W(\xi)\mathcal{L}^n(U)$$

for every $\xi \in \mathbb{M}^{m\times n}$ and every $\varphi \in C^1_c(U;\mathbb{R}^m)$. The following theorem collects the lower semicontinuity results with respect to weak convergence in $GSBV^p(U;\mathbb{R}^m)$ that we shall use in the rest of the paper.

Theorem 2.8. *Let $W\colon U\times\mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$W(x,\cdot) \text{ is quasiconvex on } \mathbb{M}^{m\times n} \text{ for every } x\in U\,, \tag{2.4}$$

$$a_0|\xi|^p - b_0(x) \le W(x,\xi) \le a_1|\xi|^p + b_1(x) \quad \text{for every } (x,\xi)\in U\times\mathbb{M}^{m\times n} \tag{2.5}$$

for some constants $a_0 > 0$, $a_1 > 0$, and some nonnegative functions b_0, $b_1 \in L^1(U)$. Let $\kappa\colon U\times\mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function such that

$$\kappa(x,\cdot) \text{ is a norm on } \mathbb{R}^n \text{ for every } x\in U\,, \tag{2.6}$$

$$\kappa_1|\nu| \le \kappa(x,\nu) \le \kappa_2|\nu| \quad \text{for every } (x,\nu)\in U\times\mathbb{R}^n \tag{2.7}$$

for some constants $\kappa_1 > 0$ and $\kappa_2 > 0$. If u_k converges to u weakly in $GSBV^p(U;\mathbb{R}^m)$, then for every $\mathcal{H}^{n-1}$-measurable set E, with $\mathcal{H}^{n-1}(E) < +\infty$, we have

$$\int_U W(x,\nabla u(x))\,dx \le \liminf_{k\to\infty} \int_U W(x,\nabla u_k(x))\,dx\,, \tag{2.8}$$

$$\int_{S(u)\setminus E} \kappa(x,\nu_u(x))\,d\mathcal{H}^{n-1}(x) \le \liminf_{k\to\infty} \int_{S(u_k)\setminus E} \kappa(x,\nu_{u_k}(x))\,d\mathcal{H}^{n-1}(x)\,, \tag{2.9}$$

where ν_{u_k} and ν_u are unit normal vector fields on $S(u_k)$ and $S(u)$, respectively.

Proof. If u_k converges to u weakly in $SBV^p(U;\mathbb{R}^m)$, inequality (2.8) is proved in [3] (see also [4, Theorem 5.29]). The general case is proved in [20].

If $E = \emptyset$, the proof of (2.9) can be found in [2, Theorem 3.7] when κ does not depend on x. The extension to the case of a general κ can be obtained by standard localization techniques. When $E \ne \emptyset$ is compact it is enough to replace U by $U\setminus E$. To prove (2.9) in the general case let $\varepsilon > 0$ and let $K \subset E$ be a compact set such that $\mathcal{H}^{n-1}(E\setminus K) < \varepsilon$. Since $S(u)\setminus E \subset S(u)\setminus K$ and $S(u_k)\setminus K \subset (S(u_k)\setminus E)\cup(E\setminus K)$, we have

$$\int_{S(u)\setminus E} \kappa(x,\nu_u(x))\,d\mathcal{H}^{n-1}(x) \le \int_{S(u)\setminus K} \kappa(x,\nu_u(x))\,d\mathcal{H}^{n-1}(x) \le$$

$$\le \liminf_{k\to\infty} \int_{S(u_k)\setminus K} \kappa(x,\nu_{u_k}(x))\,d\mathcal{H}^{n-1}(x) \le$$

$$\le \liminf_{k\to\infty} \int_{S(u_k)\setminus E} \kappa(x,\nu_{u_k}(x))\,d\mathcal{H}^{n-1}(x) + \kappa_2\mathcal{H}^{n-1}(E\setminus K) \le$$

$$\le \liminf_{k\to\infty} \int_{S(u_k)\setminus E} \kappa(x,\nu_{u_k}(x))\,d\mathcal{H}^{n-1}(x) + \kappa_2\varepsilon\,.$$

As $\varepsilon \to 0$ we obtain (2.9). ☐

Remark 2.9. Let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E) < +\infty$. Theorem 2.8 implies that, if u_k converges to u weakly in $GSBV^p(U;\mathbb{R}^m)$ and $S(u_k) \tilde{\subset} E$ for every k, then $S(u) \tilde{\subset} E$.

Remark 2.10. Let Γ be a rectifiable subset of U with $\mathcal{H}^{n-1}(\Gamma) < +\infty$ and let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E) < +\infty$. Since $(S(u)\cup\Gamma)\backslash E = (S(u)\backslash(\Gamma\cup E))\cup(\Gamma\backslash E)$ and $(S(u_k)\cup\Gamma)\backslash E = (S(u_k)\backslash(\Gamma\cup E))\cup(\Gamma\backslash E)$, Theorem 2.8 implies that, if u_k converges to u weakly in $GSBV^p(U;\mathbb{R}^m)$, then

$$\int_{(S(u)\cup\Gamma)\backslash E} \kappa(x,\nu(x))\, d\mathcal{H}^{n-1}(x) \leq \liminf_{k\to\infty} \int_{(S(u_k)\cup\Gamma)\backslash E} \kappa(x,\nu_k(x))\, d\mathcal{H}^{n-1}(x)\,,$$

where ν and ν_k are unit normal vector fields on $S(u)\cup\Gamma$ and $S(u_k)\cup\Gamma$, respectively.

3. FORMULATION OF THE PROBLEM

3.1. **The reference configuration.** Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, and let Ω_B be an open subset of Ω with Lipschitz boundary. The set $\overline{\Omega}$ represents the *reference configuration* of an elastic body with cracks, while $\overline{\Omega}_B$ represents its *brittle part*, in the sense that every crack in the reference configuration will be contained in $\overline{\Omega}_B$.

We fix a closed subset $\partial_N\Omega$ of $\partial\Omega$, called the *Neumann part* of the boundary, on which we will prescribe the boundary forces. On the *Dirichlet part* of the boundary $\partial_D\Omega := \partial\Omega\backslash\partial_N\Omega$ we will prescribe the boundary deformation, that will be attained only in the part of $\partial_D\Omega$ which is not contained in the crack. We fix also a closed subset $\partial_S\Omega$ of $\partial_N\Omega$, which will contain the support of all boundary forces applied to the body. We assume that

$$\overline{\Omega}_B \cap \partial_S\Omega = \emptyset\,. \tag{3.1}$$

The reason for such a condition will be explained later (see Remark 3.6).

3.2. **The cracks.** A *crack* is represented in the reference configuration by a rectifiable set $\Gamma \tilde{\subset} \overline{\Omega}_B$ with $\mathcal{H}^{n-1}(\Gamma) < +\infty$. The collection of *admissible cracks* is given by

$$\mathcal{R}(\overline{\Omega}_B) := \{\Gamma : \Gamma \text{ is rectifiable, } \Gamma \tilde{\subset} \overline{\Omega}_B\,,\ \mathcal{H}^{n-1}(\Gamma) < +\infty\}\,. \tag{3.2}$$

The set $\Gamma\cap\partial_D\Omega$ is interpreted as the part of $\partial_D\Omega$ where the prescribed boundary deformation is not attained. On the contrary $\Gamma\cap\partial_N\Omega$ will not produce any effect, since $\overline{\Omega}_B\cap\partial_N\Omega$ is traction free by (3.1).

According to Griffith's theory, we assume that the *energy spent to produce the crack* $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$ is given by

$$\mathcal{K}(\Gamma) := \int_{\Gamma\backslash\partial_N\Omega} \kappa(x,\nu_\Gamma(x))\, d\mathcal{H}^{n-1}(x)\,, \tag{3.3}$$

where ν_Γ is a unit normal vector field on Γ and $\kappa\colon \overline{\Omega}_B\times\mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous function, which takes into account the *toughness* of the material in different locations and in different directions. Note that, since with our definition $\mathcal{K}(\Gamma) = \mathcal{K}(\Gamma\backslash\partial_N\Omega)$, there is no energy associated with the part of the crack that lies on $\partial_N\Omega$. Nevertheless for mathematical convenience (see Section 5) it is sometimes useful to consider also cracks Γ with $\Gamma\cap\partial_N\Omega \neq \emptyset$. As in Theorem 2.8, we assume that

$$\kappa(x,\cdot) \text{ is a norm on } \mathbb{R}^n \text{ for every } x \in \overline{\Omega}_B\,, \tag{3.4}$$

$$\kappa_1|\nu| \leq \kappa(x,\nu) \leq \kappa_2|\nu| \text{ for every } (x,\nu) \in \overline{\Omega}_B\times\mathbb{R}^n\,, \tag{3.5}$$

for some constants $\kappa_1 > 0$ and $\kappa_2 > 0$.

To simplify the exposition of auxiliary results, we extend κ to $\mathbb{R}^n\times\mathbb{R}^n$ by setting $\kappa(x,\nu) := \kappa_2|\nu|$ if $x \notin \overline{\Omega}_B$, and we define $\mathcal{K}(\Gamma)$ by (3.3) for every rectifiable subset Γ of $\mathbb{R}^n$. By (3.5) we have

$$\kappa_1\mathcal{H}^{n-1}(\Gamma\backslash\partial_N\Omega) \leq \mathcal{K}(\Gamma) \leq \kappa_2\mathcal{H}^{n-1}(\Gamma\backslash\partial_N\Omega) \tag{3.6}$$

for every rectifiable subset Γ of $\mathbb{R}^n$.

3.3. **The body deformations and their bulk energy.** Given an admissible crack Γ, an *admissible deformation* with crack Γ will be any function $u \in GSBV(\Omega;\mathbb{R}^m)$ with $S(u) \mathrel{\tilde{\subset}} \Gamma$. This implies that u has a representative $\tilde{u}$ which coincides with u a.e. on Ω, is defined at $\mathcal{H}^{n-1}$-a.e. point of $\Omega\backslash\Gamma$, and is approximately continuous $\mathcal{H}^{n-1}$-a.e. on $\Omega\backslash\Gamma$, in the sense that

$$\tilde{u}(x) = \operatorname*{ap\,lim}_{y\to x,\, y\notin\Gamma} \tilde{u}(y)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega\backslash\Gamma$. Note that when $m = n$ we are in the classical case of finite elasticity on $\Omega\backslash\Gamma$, and when $m = 1$ we are in the antiplane setting.

We assume that the uncracked part of the body is hyperelastic and that its *bulk energy* relative to the deformation $u \in GSBV(\Omega;\mathbb{R}^m)$ can be written as

$$\int_\Omega W(x, \nabla u(x))\, dx\,,$$

where $W\colon \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ is a given Carathéodory function such that $\xi \mapsto W(x,\xi)$ is quasiconvex and C^1 on $\mathbb{M}^{m\times n}$ for every $x \in \Omega$. As in Theorem 2.8, we assume that there exist three constants $p > 1$, $a_0^W > 0$, $a_1^W > 0$, and two nonnegative functions b_0^W, $b_1^W \in L^1(\Omega)$, such that

$$a_0^W|\xi|^p - b_0^W(x) \le W(x,\xi) \le a_1^W|\xi|^p + b_1^W(x) \tag{3.7}$$

for every $(x,\xi) \in \Omega{\times}\mathbb{M}^{m\times n}$. Since $\xi \mapsto W(x,\xi)$ is rank-one convex on $\mathbb{M}^{m\times n}$ for every $x \in \Omega$ (see, e.g., [8]), we deduce from the previous inequalities an estimate for the partial gradient $\partial_\xi W\colon \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ of W with respect to ξ (see, e.g., [8]). Specifically, there exist a constant $a_2^W > 0$ and a nonnegative function $b_2^W \in L^{p'}(\Omega)$, with $p' = p/(p{-}1)$, such that

$$|\partial_\xi W(x,\xi)| \le a_2^W|\xi|^{p-1} + b_2^W(x) \tag{3.8}$$

for every $(x,\xi) \in \Omega{\times}\mathbb{M}^{m\times n}$.

Note that in the case $m = n$ the boundedness assumption (3.7) prohibits the introduction of the "classical" constraint that $W(\xi) \to \infty$ as $\det \xi \to 0$.

To shorten the notation we introduce the function $\mathcal{W}\colon L^p(\Omega;\mathbb{M}^{m\times n}) \to \mathbb{R}$ defined by

$$\mathcal{W}(\Phi) := \int_\Omega W(x,\Phi(x))\, dx \tag{3.9}$$

for every $\Phi \in L^p(\Omega;\mathbb{M}^{m\times n})$. By (3.7) and (3.8) the functional $\mathcal{W}$ is of class C^1 on $L^p(\Omega;\mathbb{M}^{m\times n})$ and its differential $\partial\mathcal{W}\colon L^p(\Omega;\mathbb{M}^{m\times n}) \to L^{p'}(\Omega;\mathbb{M}^{m\times n})$ is given by

$$\langle \partial\mathcal{W}(\Phi), \Psi\rangle = \int_\Omega \partial_\xi W(x,\Phi(x))\Psi(x)\, dx\,, \tag{3.10}$$

for every Φ, $\Psi \in L^p(\Omega;\mathbb{M}^{m\times n})$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ and $L^p(\Omega;\mathbb{M}^{m\times n})$.

By (3.7) and (3.8) there exist six constants $\alpha_0^{\mathcal{W}} > 0$, $\alpha_1^{\mathcal{W}} > 0$, $\alpha_2^{\mathcal{W}} > 0$, $\beta_0^{\mathcal{W}} \ge 0$, $\beta_1^{\mathcal{W}} \ge 0$, $\beta_2^{\mathcal{W}} \ge 0$ such that

$$\alpha_0^{\mathcal{W}}\|\Phi\|_p^p - \beta_0^{\mathcal{W}} \le \mathcal{W}(\Phi) \le \alpha_1^{\mathcal{W}}\|\Phi\|_p^p + \beta_1^{\mathcal{W}}\,, \tag{3.11}$$

$$|\langle \partial\mathcal{W}(\Phi), \Psi\rangle| \le (\alpha_2^{\mathcal{W}}\|\Phi\|_p^{p-1} + \beta_2^{\mathcal{W}})\|\Psi\|_p\,, \tag{3.12}$$

for every Φ, $\Psi \in L^p(\Omega;\mathbb{M}^{m\times n})$.

If $u \in GSBV(\Omega;\mathbb{R}^m)$ is an admissible deformation for some crack Γ and u has finite bulk energy, then u belongs to $GSBV^p(\Omega;\mathbb{R}^m)$ by (3.7) and its bulk energy is given by $\mathcal{W}(\nabla u)$.

3.4. **The body forces.** We assume that at each time $t \in [0,T]$ the applied *body forces* depend on the deformation u and are conservative. This means that there exists a function $F\colon [0,T]\times\Omega\times\mathbb{R}^m \to \mathbb{R}$ such that the density of the applied body forces per unit volume in the reference configuration is given by $\partial_z F(t,x,u(x))$, where $\partial_z F(t,x,z)$ denotes the partial gradient of F with respect to z. We always assume that for every $t \in [0,T]$ the function $(x,z) \mapsto F(t,x,z)$ is $\mathcal{L}^n$-measurable in x and C^1 in z.

Rather than imposing further regularity conditions on F, we prefer to impose appropriate conditions on the associated work, corresponding to the deformation u, given by

$$\mathcal{F}(t)(u) := \int_\Omega F(t,x,u(x))\,dx\,. \tag{3.13}$$

This is because only $\mathcal{F}(t)$ enters in the expression for the energy.

First of all we assume that there exists $q > 1$ such that for every $t \in [0,T]$ the function $\mathcal{F}(t)$ is of class C^1 on $L^q(\Omega;\mathbb{R}^m)$, with differential $\partial\mathcal{F}(t)\colon L^q(\Omega;\mathbb{R}^m) \to L^{q'}(\Omega;\mathbb{R}^m)$ given by

$$\langle \partial\mathcal{F}(t)(u), v\rangle = \int_\Omega \partial_z F(t,x,u(x))\,v(x)\,dx \tag{3.14}$$

for every u, $v \in L^q(\Omega;\mathbb{R}^m)$, where $\langle\cdot,\cdot\rangle$ is the duality pairing between $L^{q'}(\Omega;\mathbb{R}^m)$ and $L^q(\Omega;\mathbb{R}^m)$, and $q' := q/(q-1)$. We assume also that

$$\mathcal{F}(t)(u) \geq \limsup_{k\to\infty} \mathcal{F}(t)(u_k) \tag{3.15}$$

whenever u_k, $u \in L^q(\Omega;\mathbb{R}^m)$ and $u_k \to u$ a.e. on Ω. Notice that this inequality follows from Fatou's Lemma and from the continuity hypothesis on F, provided suitable upper bounds are satisfied.

About the regularity in t we assume that there exist a constant $\dot q \in [1,q)$ and, for a.e. $t \in [0,T]$, a function $\dot{\mathcal{F}}(t)\colon L^{\dot q}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ of class C^1, with differential $\partial\dot{\mathcal{F}}(t)\colon L^{\dot q}(\Omega;\mathbb{R}^m) \to L^{\dot q'}(\Omega;\mathbb{R}^m)$, $\dot q' := \dot q/(\dot q - 1)$, such that

$$\mathcal{F}(t)(u) = \mathcal{F}(0)(u) + \int_0^t \dot{\mathcal{F}}(s)(u)\,ds\,, \tag{3.16}$$

$$\langle \partial\mathcal{F}(t)(u), v\rangle = \langle \partial\mathcal{F}(0)(u), v\rangle + \int_0^t \langle \partial\dot{\mathcal{F}}(s)(u), v\rangle\,ds \tag{3.17}$$

for every u, $v \in L^q(\Omega;\mathbb{R}^m)$ and for every $t \in [0,T]$. In order for (3.16) and (3.17) to make sense, we assume in addition that $t \mapsto \dot{\mathcal{F}}(t)(u)$ and $t \mapsto \langle \partial\dot{\mathcal{F}}(t)(u), v\rangle$ are integrable on $[0,T]$ for every u, $v \in L^q(\Omega;\mathbb{R}^m)$. Of course, under these assumptions the functions $t \mapsto \mathcal{F}(t)(u)$ and $t \mapsto \langle \partial\mathcal{F}(t)(u), v\rangle$ are absolutely continuous on $[0,T]$ for every u, $v \in L^q(\Omega;\mathbb{R}^m)$, and their time derivatives coincide a.e. with $t \mapsto \dot{\mathcal{F}}(t)(u)$ and $t \mapsto \langle \partial\dot{\mathcal{F}}(t)(u), v\rangle$, which justifies the notation.

We also require some growth conditions on $\mathcal{F}(t)$, $\partial\mathcal{F}(t)$, $\dot{\mathcal{F}}(t)$, and $\partial\dot{\mathcal{F}}(t)$ to pass to the limit in our approximating sequences. To be explicit we assume that there exist six constants $\alpha_0^{\mathcal{F}} > 0$, $\alpha_1^{\mathcal{F}} > 0$, $\alpha_2^{\mathcal{F}} > 0$, $\beta_0^{\mathcal{F}} \geq 0$, $\beta_1^{\mathcal{F}} \geq 0$, $\beta_2^{\mathcal{F}} \geq 0$ and four nonnegative functions $\alpha_3^{\mathcal{F}}$, $\alpha_4^{\mathcal{F}}$, $\beta_3^{\mathcal{F}}$, $\beta_4^{\mathcal{F}} \in L^1([0,T])$ such that

$$\alpha_0^{\mathcal{F}}\|u\|_q^q - \beta_0^{\mathcal{F}} \leq -\mathcal{F}(t)(u) \leq \alpha_1^{\mathcal{F}}\|u\|_q^q + \beta_1^{\mathcal{F}}\,, \tag{3.18}$$

$$|\langle \partial\mathcal{F}(t)(u), v\rangle| \leq (\alpha_2^{\mathcal{F}}\|u\|_q^{q-1} + \beta_2^{\mathcal{F}})\|v\|_q\,, \tag{3.19}$$

$$|\dot{\mathcal{F}}(t)(u)| \leq \alpha_3^{\mathcal{F}}(t)\|u\|_{\dot q}^{\dot q} + \beta_3^{\mathcal{F}}(t)\,, \tag{3.20}$$

$$|\langle \partial\dot{\mathcal{F}}(t)(u), v\rangle| \leq (\alpha_4^{\mathcal{F}}(t)\|u\|_{\dot q}^{\dot q-1} + \beta_4^{\mathcal{F}}(t))\|v\|_{\dot q} \tag{3.21}$$

for a.e. $t \in [0,T]$ and for every u, $v \in L^q(\Omega;\mathbb{R}^m)$. Note that by continuity (3.18) and (3.19) hold for every $t \in [0,T]$.

Remark 3.1. The conditions $q > 1$ and $\alpha_0^F > 0$ in (3.18) play a crucial role in our results. From a mathematical viewpoint, they ensure that every sequence of deformations with bounded total energy is bounded in $L^q(\Omega;\mathbb{R}^m)$. From a mechanical viewpoint, these conditions ensure that, even if the cracks divide the body into several components, no part of the body is sent to infinity by the applied forces. Unfortunately, they exclude the case of a constant body force, which corresponds to a potential F which is linear with respect to z.

Remark 3.2. Let us explain the roles of the different exponents q and $\dot q$ in (3.18)–(3.21). By (3.18) we obtain the exponent q in (3.51), so that, when we apply the $GSBV$ compactness theorem to a sequence u_k of functions with bounded energy, we obtain a subsequence which converges pointwise a.e. on Ω and is bounded in $L^q(\Omega;\mathbb{R}^m)$, but, in general, does not converge strongly in $L^q(\Omega;\mathbb{R}^m)$. In some estimates we need to pass to the limit in sequences like $\dot{\mathcal{F}}(t)(u_k)$, and this is made possible by (3.20), since $\dot q < q$ and, therefore, u_k converges strongly in $L^{\dot q}(\Omega;\mathbb{R}^m)$.

Remark 3.3. All the conditions for $\mathcal{F}(t)$ and $\dot{\mathcal{F}}(t)$ listed above are satisfied whenever it is assumed that

$$\text{for every } (t,z)\in[0,T]\times\mathbb{R}^m \text{ the function } x\mapsto F(t,x,z) \text{ is integrable on } \Omega, \tag{3.22}$$

$$\text{for every } x\in\Omega \text{ the function } (t,z)\mapsto F(t,x,z) \text{ belongs to } C^2([0,T]\times\mathbb{R}^m), \tag{3.23}$$

and that there exist seven constants $q > \dot q \geq 1$, $a_0^F > 0$, $a_1^F > 0$, $a_2^F > 0$, $a_3^F \geq 0$, $a_4^F \geq 0$ and five nonnegative functions b_0^F, $b_1^F \in C^0([0,T];L^1(\Omega))$, $b_2^F \in C^0([0,T];L^{q'}(\Omega))$, $b_3^F \in L^1(\Omega)$, and $b_4^F \in L^{\dot q'}(\Omega)$ such that

$$a_0^F|z|^q - b_0^F(t,x) \leq -F(t,x,z) \leq a_1^F|z|^q + b_1^F(t,x)\,, \tag{3.24}$$

$$|\partial_z F(t,x,z)| \leq a_2^F|z|^{q-1} + b_2^F(t,x)\,, \tag{3.25}$$

$$|\partial_t F(t,x,z)| \leq a_3^F|z|^{\dot q} + b_3^F(x)\,, \tag{3.26}$$

$$|\partial_z\partial_t F(t,x,z)| \leq a_4^F|z|^{\dot q-1} + b_4^F(x) \tag{3.27}$$

for every $(t,x,z)\in[0,T]\times\Omega\times\mathbb{R}^m$, where ∂_t denotes the partial derivative with respect to t. In this case (3.16) and (3.17) are satisfied with

$$\dot{\mathcal{F}}(t)(u) := \int_\Omega \partial_t F(t,x,u(x))\,dx\,, \tag{3.28}$$

since for every u, $v\in L^q(\Omega)$ the functions $t\mapsto\dot{\mathcal{F}}(t)(u)$ and $t\mapsto\langle\partial\dot{\mathcal{F}}(t)(u),v\rangle$ are the continuous time derivatives of the functions $t\mapsto\mathcal{F}(t)(u)$ and $t\mapsto\langle\partial\mathcal{F}(t)(u),v\rangle$. Weaker hypotheses on F will be considered in Section 9.

3.5. **The surface forces.** We assume that at each time $t\in[0,T]$ the *surface forces* applied on $\partial_S\Omega$ depend on the deformation u, are conservative, and can be expressed by means of a potential function $G\colon[0,T]\times\partial_S\Omega\times\mathbb{R}^n\to\mathbb{R}$. More precisely, we assume that for a given deformation u the density of the applied surface forces per unit area in the reference configuration is given by $\partial_z G(t,x,u(x))$, where $\partial_z G(t,x,z)$ denotes the partial gradient of G with respect to z. We assume also that for every $t\in[0,T]$ the function $(x,z)\mapsto G(t,x,z)$ is $\mathcal{H}^{n-1}$-measurable in x and C^1 in z.

Remark 3.4. These assumptions are natural when the surface forces applied to the deformed body depend on a conservative field acting on a charge distribution which is deformed with the body, i.e., the charge density per unit area in the reference configuration does not depend on the deformation. Indeed, in this case the change in the area elements between the reference and the deformed configuration is compensated by the corresponding change in the charge densities, so that the surface force applied to the deformed body has a density per unit area in the reference configuration which depends on the position x and on the deformation $u(x)$, but not on the deformation gradient $\nabla u(x)$.

Unfortunately, pressure forces do not satisfy this assumption, so they cannot be treated directly in the framework of this paper.

As we did for body forces, we will impose appropriate conditions on the associated work, corresponding to the deformation u, which in this case is given by

$$\mathcal{G}(t)(u) := \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^{n-1}(x)\,. \tag{3.29}$$

First of all we assume that there exists $r>1$ such that for every $t\in[0,T]$ the function $\mathcal{G}(t)$ is of class C^1 on $L^r(\partial_S\Omega;\mathbb{R}^m)$, with differential $\partial\mathcal{G}(t)\colon L^r(\partial_S\Omega;\mathbb{R}^m)\to L^{r'}(\partial_S\Omega;\mathbb{R}^m)$ given by

$$\langle \partial\mathcal{G}(t)(u), v\rangle = \int_{\partial_S\Omega} \partial_z G(t,x,u(x))\, v(x)\, d\mathcal{H}^{n-1}(x) \tag{3.30}$$

for every $u,\ v\in L^r(\partial_S\Omega;\mathbb{R}^m)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^{r'}(\partial_S\Omega;\mathbb{R}^m)$ and $L^r(\partial_S\Omega;\mathbb{R}^m)$, and $r':=r/(r-1)$. If the exponent p which appears in (3.7) is less than the dimension n of Ω, we suppose that $p\le r<p(n-1)/(n-p)$. If $p\ge n$, we just suppose that $p\le r$.

Let us fix an open set $\Omega_S\subset\Omega\backslash\overline{\Omega}_B$ with Lipschitz boundary and such that $\partial_S\Omega\subset\partial\Omega_S$. Under our hypothesis on p and r the trace operator from $W^{1,p}(\Omega_S;\mathbb{R}^m)$ into $L^r(\partial_S\Omega;\mathbb{R}^m)$ is compact (see, e.g., [22]). Therefore there exists a constant $\gamma_S>0$ such that

$$\|u\|_{r,\partial_S\Omega} \le \gamma_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{p,\Omega_S}) \tag{3.31}$$

for every $u\in W^{1,p}(\Omega_S;\mathbb{R}^m)$. Here and in the rest of the paper we use the same symbol to denote a function defined on (a set containing) Ω_S and its trace on $\partial_S\Omega$. In particular, if $u\in GSBV(\Omega;\mathbb{R}^m)$ is a deformation with $S(u)\,\tilde{\subset}\,\overline{\Omega}_B$, $\mathcal{W}(\nabla u)<+\infty$, and $\mathcal{F}(t)(u)<+\infty$ for some $t\in[0,T]$, then by (3.11) and (3.18) $u\in GSBV^p_q(\Omega;\mathbb{R}^m):=GSBV^p(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$ and $S(u)\cap\Omega_S\cong\emptyset$, so that by Lemma 2.5 $u\in W^{1,p}(\Omega_S;\mathbb{R}^m)\cap L^q(\Omega_S;\mathbb{R}^m)$. Therefore (the trace of) u belongs to $L^r(\partial_S\Omega;\mathbb{R}^m)$ and $\mathcal{G}(t)(u)$ is well defined.

As for the regularity in t we assume that for a.e. $t\in[0,T]$ there exists a function $\dot{\mathcal{G}}(t)\colon L^r(\partial_S\Omega;\mathbb{R}^m)\to\mathbb{R}$ of class C^1, with differential $\partial\dot{\mathcal{G}}(t)\colon L^r(\partial_S\Omega;\mathbb{R}^m)\to L^{r'}(\partial_S\Omega;\mathbb{R}^m)$, such that

$$\mathcal{G}(t)(u) = \mathcal{G}(0)(u) + \int_0^t \dot{\mathcal{G}}(s)(u)\, ds\,, \tag{3.32}$$

$$\langle\partial\mathcal{G}(t)(u), v\rangle = \langle\partial\mathcal{G}(0)(u), v\rangle + \int_0^t \langle\partial\dot{\mathcal{G}}(s)(u), v\rangle\, ds \tag{3.33}$$

for every $u,\ v\in L^r(\partial_S\Omega;\mathbb{R}^m)$ and for every $t\in[0,T]$. In order for (3.32) and (3.33) to make sense, we also assume that $t\mapsto\dot{\mathcal{G}}(t)(u)$ and $t\mapsto\langle\partial\dot{\mathcal{G}}(t)(u),v\rangle$ are integrable on $[0,T]$ for every $u,\ v\in L^r(\partial_S\Omega;\mathbb{R}^m)$. This implies that the functions $t\mapsto\mathcal{G}(t)(u)$ and $t\mapsto\langle\partial\mathcal{G}(t)(u),v\rangle$ are absolutely continuous on $[0,T]$ for every $u,\ v\in L^r(\partial_S\Omega;\mathbb{R}^m)$.

As we did for the body forces, we require some growth conditions on $\mathcal{G}(t)$, $\partial\mathcal{G}(t)$, $\dot{\mathcal{G}}(t)$, and $\partial\dot{\mathcal{G}}(t)$. To be explicit we assume that there exist six nonnegative constants $\alpha_0^{\mathcal{G}}$, $\alpha_1^{\mathcal{G}}$, $\alpha_2^{\mathcal{G}}$, $\beta_0^{\mathcal{G}}$, $\beta_1^{\mathcal{G}}$, $\beta_2^{\mathcal{G}}$ and four nonnegative functions $\alpha_3^{\mathcal{G}}$, $\alpha_4^{\mathcal{G}}$, $\beta_3^{\mathcal{G}}$, $\beta_4^{\mathcal{G}}\in L^1([0,T])$ such that

$$-\alpha_0^{\mathcal{G}}\|u\|_{r,\partial_S\Omega} - \beta_0^{\mathcal{G}} \le -\mathcal{G}(t)(u) \le \alpha_1^{\mathcal{G}}\|u\|^r_{r,\partial_S\Omega} + \beta_1^{\mathcal{G}}\,, \tag{3.34}$$

$$|\langle\partial\mathcal{G}(t)(u),v\rangle| \le (\alpha_2^{\mathcal{G}}\|u\|^{r-1}_{r,\partial_S\Omega} + \beta_2^{\mathcal{G}})\|v\|_{r,\partial_S\Omega}\,, \tag{3.35}$$

$$|\dot{\mathcal{G}}(t)(u)| \le \alpha_3^{\mathcal{G}}(t)\|u\|^r_{r,\partial_S\Omega} + \beta_3^{\mathcal{G}}(t) \tag{3.36}$$

$$|\langle\partial\dot{\mathcal{G}}(t)(u),v\rangle| \le (\alpha_4^{\mathcal{G}}(t)\|u\|^{r-1}_{r,\partial_S\Omega} + \beta_4^{\mathcal{G}}(t))\|v\|_{r,\partial_S\Omega} \tag{3.37}$$

for a.e. $t\in[0,T]$ and for every $u,\ v\in L^r(\partial_S\Omega;\mathbb{R}^m)$. By continuity (3.34) and (3.35) hold for every $t\in[0,T]$. Note that in the first inequality of (3.34) the term $\|u\|_{r,\partial_S\Omega}$ appears with exponent 1. This is because in (3.51) we want a constant $\alpha_0^{\mathcal{E}}>0$.

Remark 3.5. All the conditions for $\mathcal{G}(t)$ and $\dot{\mathcal{G}}(t)$ listed above are satisfied whenever it is assumed that

$$\text{for every } (t,z)\in[0,T]\times\mathbb{R}^m \text{ the function } x\mapsto G(t,x,z) \text{ is } \mathcal{H}^{n-1}\text{-integrable on } \partial_S\Omega, \tag{3.38}$$

$$\text{for every } x\in\partial_S\Omega \text{ the function } (t,z)\mapsto G(t,x,z) \text{ belongs to } C^2([0,T]\times\mathbb{R}^m), \tag{3.39}$$

and that there exist four constants $a_1^G\ge 0$, $a_2^G\ge 0$, $a_3^G\ge 0$, $a_4^G\ge 0$ and six nonnegative functions a_0^G, $b_2^G\in C^0([0,T];L^{r'}(\partial_S\Omega))$, b_0^G, $b_1^G\in C^0([0,T];L^1(\partial_S\Omega))$, $b_3^G\in L^1(\partial_S\Omega)$, and $b_4^G\in L^{r'}(\partial_S\Omega)$ such that

$$-a_0^G(t,x)|z|-b_0^G(t,x)\le -G(t,x,z)\le a_1^G|z|^r+b_1^G(t,x)\,, \tag{3.40}$$

$$|\partial_z G(t,x,z)|\le a_2^G|z|^{r-1}+b_2^G(t,x)\,, \tag{3.41}$$

$$|\partial_t G(t,x,z)|\le a_3^G|z|^r+b_3^G(x)\,, \tag{3.42}$$

$$|\partial_z\partial_t G(t,x,z)|\le a_4^G|z|^{r-1}+b_4^G(x) \tag{3.43}$$

for every $(t,x,z)\in[0,T]\times\partial_S\Omega\times\mathbb{R}^m$, where ∂_t denotes the partial derivative with respect to t. As in Remark 3.3 we can prove that (3.32) and (3.33) are satisfied with

$$\dot{\mathcal{G}}(t)(u):=\int_{\partial_S\Omega}\partial_t G(t,x,u(x))\,d\mathcal{H}^{n-1}(x)\,. \tag{3.44}$$

Weaker hypotheses on G will be considered in Section 9.

Remark 3.6. In this model $\overline{\Omega}_B$ represents the reference configuration of the brittle part of the material, while $\Omega\setminus\overline{\Omega}_B$ can be considered as the reference configuration of an elastic unbreakable part attached to it through the interface $\Omega\cap\partial\Omega_B$. Since W is not assumed to be continuous with respect to x, it may happen that the bulk energy density is discontinuous across $\Omega\cap\partial\Omega_B$, so that we can interpret $\overline{\Omega}_B$ and $\Omega\setminus\overline{\Omega}_B$ as representing two bodies with different material properties. In other words, in this model the surface forces can act on the brittle body Ω_B only through the layer of unbreakable material $\Omega\setminus\overline{\Omega}_B$. At the moment it is not known what happens if the thickness of this layer tends to 0.

3.6. **The prescribed boundary deformations.** In this paper we do not consider the case of imposed "discontinuous" boundary deformations, but only boundary deformations that are traces on $\partial_D\Omega$ of functions $\psi\in W^{1,p}(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$, so that there is always a configuration with finite energy without cracks which satisfies the boundary conditions. The choice of the exponents is determined by (3.11) and (3.18).

The set $AD(\psi,\Gamma)$ of *admissible deformations in* Ω *with finite energy*, corresponding to a crack $\Gamma\in\mathcal{R}(\overline{\Omega}_B)$ and to a boundary deformation $\psi\in W^{1,p}(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$, is defined by

$$AD(\psi,\Gamma):=\{u\in GSBV_q^p(\Omega;\mathbb{R}^m): S(u)\,\tilde{\subset}\,\Gamma,\ u=\psi\ \ \mathcal{H}^{n-1}\text{-a.e. on } \partial_D\Omega\setminus\Gamma\}\,, \tag{3.45}$$

where the last equality in the previous formula refers to the traces of u and ψ on $\partial\Omega$ introduced in Proposition 2.4.

Note that if Γ is closed, then $AD(\psi,\Gamma)$ coincides with the space of all functions $u\in L^q(\Omega;\mathbb{R}^m)$ whose distributional gradient on $\Omega\setminus\Gamma$ belongs to $L^p(\Omega\setminus\Gamma;\mathbb{M}^{m\times n})$ and which agree with ψ on $\partial_D\Omega\setminus\Gamma$ in the standard sense of Sobolev spaces (to prove this fact we can use Lemma 2.5 and [11, Lemma 2.3]). This space is frequently used in the variational approach to nonlinear elasticity. Our "non-conventional" definition of $AD(\psi,\Gamma)$ stems from the potential failure of the crack Γ to remain closed in our existence theorem.

We assume that the boundary deformation $\psi(t)$ depends on time and that the function $t\mapsto\psi(t)$ is absolutely continuous from $[0,T]$ into $W^{1,p}(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$ (endowed with the sum of the norms), so that the time derivative $t\mapsto\dot\psi(t)$ belongs to the space $L^1([0,T];W^{1,p}(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m))$ and its spatial gradient $t\mapsto\nabla\dot\psi(t)$ belongs to the space $L^1([0,T];L^p(\Omega;\mathbb{M}^{m\times n}))$.

3.7. **The admissible configurations and their total energy.** An *admissible configuration* is a pair $(u, \varGamma)$, where $\varGamma \in \mathcal{R}(\overline{\Omega}_B)$ is an admissible crack and $u \in GSBV^p_q(\Omega;\mathbb{R}^m)$ is a deformation with finite energy and with $S(u) \,\tilde{\subset}\, \varGamma$.

For every $t \in [0,T]$, $\varGamma \in \mathcal{R}(\overline{\Omega}_B)$, and $u \in AD(\psi(t), \varGamma)$, the *total energy* of the admissible configuration $(u, \varGamma)$ at time t is given by

$$\mathcal{E}(t)(u,\varGamma) := \mathcal{E}^{el}(t)(u) + \mathcal{K}(\varGamma)\,, \tag{3.46}$$

where for every $u \in GSBV^p_q(\Omega;\mathbb{R}^m)$ the *elastic energy* is defined by

$$\mathcal{E}^{el}(t)(u) := \mathcal{W}(\nabla u) - \mathcal{F}(t)(u) - \mathcal{G}(t)(u)\,. \tag{3.47}$$

Note that $u \in W^{1,p}(\Omega_S;\mathbb{R}^m)$ by Lemma 2.5, so that $u \in L^r(\partial_S\Omega, \mathbb{R}^m)$ and $\mathcal{G}(t)(u)$ is well defined. We will sometimes write

$$\mathcal{E}(t)(u,\varGamma) := \mathcal{E}^{in}(u,\varGamma) - \mathcal{L}(t)(u)\,, \tag{3.48}$$

where

$$\mathcal{E}^{in}(u,\varGamma) := \mathcal{W}(\nabla u) + \mathcal{K}(\varGamma) \tag{3.49}$$

is the *internal energy*, while

$$\mathcal{L}(t)(u) := \mathcal{F}(t)(u) + \mathcal{G}(t)(u) \tag{3.50}$$

is the *work done by the applied loads*.

There exist four constants $\alpha_0^{\mathcal{E}} > 0$, $\alpha_1^{\mathcal{E}} > 0$, $\beta_0^{\mathcal{E}} \ge 0$, and $\beta_1^{\mathcal{E}} \ge 0$ such that

$$\mathcal{E}^{el}(t)(u) \ge \alpha_0^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q) - \beta_0^{\mathcal{E}}\,, \tag{3.51}$$

$$\mathcal{E}^{el}(t)(u) \le \alpha_1^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q + \|u\|^r_{r,\partial_S\Omega}) + \beta_1^{\mathcal{E}} \tag{3.52}$$

for every $t \in [0,T]$ and for every $u \in GSBV^p_q(\Omega;\mathbb{R}^m)$.

To prove this fact let us fix t and u. By Lemma 2.5 we have $u \in W^{1,p}(\Omega_S, \mathbb{R}^m)$, and by (3.11), (3.18), and (3.34) we have

$$\mathcal{E}^{el}(t)(u) \ge \alpha_0^{\mathcal{W}}\|\nabla u\|_p^p - \beta_0^{\mathcal{W}} + \alpha_0^{\mathcal{F}}\|u\|_q^q - \beta_0^{\mathcal{F}} - \alpha_0^{\mathcal{G}}\|u\|_{r,\partial_S\Omega} - \beta_0^{\mathcal{G}}\,. \tag{3.53}$$

Since Ω_S has a Lipschitz boundary, there exists a constant $k_S > 0$, depending only on p, q, and Ω_S, such that

$$\|u\|_{p,\Omega_S} \le k_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{q,\Omega_S}) \tag{3.54}$$

for every $u \in W^{1,p}(\Omega_S;\mathbb{R}^m) \cap L^q(\Omega_S;\mathbb{R}^m)$.

Using Young's inequality, it follows from (3.31) and (3.54) that there exists a constant $\lambda \ge 0$, depending only on p, r, q, $\alpha_0^{\mathcal{W}}$, $\alpha_0^{\mathcal{F}}$, $\alpha_0^{\mathcal{G}}$, and Ω_S, such that

$$\alpha_0^{\mathcal{G}}\|u\|_{r,\partial_S\Omega} \le \frac{\alpha_0^{\mathcal{W}}}{2}\|\nabla u\|^p_{p,\Omega_S} + \frac{\alpha_0^{\mathcal{F}}}{2}\|u\|^q_{q,\Omega_S} + \lambda \tag{3.55}$$

for every $u \in W^{1,p}(\Omega_S;\mathbb{R}^m) \cap L^q(\Omega_S;\mathbb{R}^m)$. Therefore (3.51) follows from (3.53), with $\alpha_0^{\mathcal{E}} := \min\{\frac{1}{2}\alpha_0^{\mathcal{W}}, \frac{1}{2}\alpha_0^{\mathcal{F}}\}$, and $\beta_0^{\mathcal{E}} := \beta_0^{\mathcal{W}} + \beta_0^{\mathcal{F}} + \beta_0^{\mathcal{G}} + \lambda$.

By (3.11), (3.18), and (3.34) we also have

$$\mathcal{E}^{el}(t)(u) \le \alpha_1^{\mathcal{W}}\|\nabla u\|_p^p + \beta_1^{\mathcal{W}} + \alpha_1^{\mathcal{F}}\|u\|_q^q + \beta_1^{\mathcal{F}} + \alpha_1^{\mathcal{G}}\|u\|^r_{r,\partial_S\Omega} + \beta_1^{\mathcal{G}}\,,$$

which gives (3.52) with $\alpha_1^{\mathcal{E}} := \max\{\alpha_1^{\mathcal{W}}, \alpha_1^{\mathcal{F}}, \alpha_1^{\mathcal{G}}\}$ and $\beta_1^{\mathcal{E}} := \beta_1^{\mathcal{W}} + \beta_1^{\mathcal{F}} + \beta_1^{\mathcal{G}}$.

3.8. **Minimum energy configurations.** For a given time $t \in [0,T]$ and a given crack $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B)$, a deformation u which corresponds to static equilibrium is a critical point of the functional $\mathcal{E}^{el}(t)$ on the set $AD(\psi(t),\Gamma(t))$ defined in (3.45). Among such critical points, the minimum points of the problem

$$\min_{u\in AD(\psi(t),\Gamma(t))} \mathcal{E}^{el}(t)(u) \tag{3.56}$$

play an important role and can be regarded as the most stable equilibria. We will call them *minimum energy deformations* at time t with crack $\Gamma(t)$. The existence of these minimizers is guaranteed by the following theorem, that will be proved in Section 5.

Theorem 3.7. *For every $t \in [0,T]$ and every $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B)$ the minimum problem (3.56) has a solution.*

Let $u(t)$ be a minimum energy deformation at time t with crack $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B)$. For every $v \in AD(0,\Gamma(t))$ and every $\varepsilon \in \mathbb{R}$ the function $u(t)+\varepsilon v$ belongs to $AD(\psi(t),\Gamma(t))$. Therefore $\mathcal{E}^{el}(t)(u(t)) \le \mathcal{E}^{el}(t)(u(t)+\varepsilon v)$ for every $\varepsilon \in \mathbb{R}$. By taking the derivative with respect to ε at $\varepsilon = 0$, we obtain the weak formulation of the *Euler equation*

$$\langle \partial\mathcal{W}(\nabla u(t)), \nabla v\rangle = \langle \partial\mathcal{F}(t)(u(t)), v\rangle + \langle \partial\mathcal{G}(t)(u(t)), v\rangle \tag{3.57}$$

for every $v \in AD(0,\Gamma(t))$. The critical points of the functional $\mathcal{E}^{el}(t)$ on $AD(\psi(t),\Gamma(t))$ are, by definition, the solutions $u(t) \in AD(\psi(t),\Gamma(t))$ of (3.57), which turns out to be the equation of equilibrium with prescribed crack $\Gamma(t)$, and coincides with the classical equilibrium equation considered in nonlinear elasticity when $\Gamma(t)$ is closed.

If $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B)$ and $u(t) \in AD(\psi(t),\Gamma(t))$ is a solution of (3.57), we introduce the linear functional $g(t)$ on $GSBV^p_q(\Omega;\mathbb{R}^m)$ defined by

$$\langle g(t), v\rangle := \langle \partial\mathcal{W}(\nabla u(t)), \nabla v\rangle - \langle \partial\mathcal{F}(t)(u(t)), v\rangle - \langle \partial\mathcal{G}(t)(u(t)), v\rangle \tag{3.58}$$

for every $v \in GSBV^p_q(\Omega;\mathbb{R}^m)$. By the Euler equation we have $\langle g(t), v_1\rangle = \langle g(t), v_2\rangle$ for every $v_1, v_2 \in GSBV^p_q(\Omega;\mathbb{R}^m)$ with $S(v_1) \,\tilde{\subset}\, \Gamma(t)$, $S(v_2) \,\tilde{\subset}\, \Gamma(t)$, and $v_1 = v_2$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega\backslash\Gamma(t)$, since in this case $v_1 - v_2 \in AD(0,\Gamma(t))$. In other words $\langle g(t), v\rangle$ depends only on the trace of v on $\partial_D\Omega\backslash\Gamma(t)$, provided $S(v) \,\tilde{\subset}\, \Gamma(t)$.

Under suitable regularity assumptions we have

$$\langle g(t), v\rangle = \int_{\partial_D\Omega\backslash\Gamma(t)} \partial_\xi W(\nabla u(t))\nu\, v\, d\mathcal{H}^{n-1}, \tag{3.59}$$

where ν is the outer unit normal to $\partial\Omega$, so that $g(t)$ can be identified with the function $\partial_\xi W(\nabla u(t))\nu$ defined on $\partial_D\Omega\backslash\Gamma(t)$, which represents the density per unit area of the surface force acting on $\partial_D\Omega\backslash\Gamma(t)$ at time t.

Returning to the general case considered at the beginning, the expression $\langle g(t), v\rangle$ can always be interpreted as the work done by the surface forces acting on $\partial_D\Omega\backslash\Gamma(t)$ at time t under the deformation v.

In the spirit of Griffith's theory, an *equilibrium configuration* at time $t \in [0,T]$ is an admissible configuration $(u(t),\Gamma(t))$ which is a "critical point", in a sense that has not yet been made mathematically precise, of the functional $\mathcal{E}(t)(u,\Gamma)$ on the set of configurations (u,Γ) with $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, $\Gamma(t) \,\tilde{\subset}\, \Gamma$, and $u \in AD(\psi(t),\Gamma)$. Following [15], we will consider only *minimum energy configurations at time* t, which are defined as the admissible configurations $(u(t),\Gamma(t))$, with $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B)$ and $u(t) \in AD(\psi(t),\Gamma(t))$, such that

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(u,\Gamma)$$

for every $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, with $\Gamma(t) \,\tilde{\subset}\, \Gamma$, and every $u \in AD(\psi(t),\Gamma)$. These are regarded as the most stable equilibrium configurations.

The following theorem, which will be proved in Section 5, ensures that for every $t \in [0,T]$ and for every $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B)$ there exists at least a minimum energy configuration $(u(t),\Gamma(t))$ such that $\Gamma_0 \,\tilde{\subset}\, \Gamma(t)$.

Theorem 3.8. *Let* $t \in [0,T]$ *and let* $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B)$. *Then the minimum problem*

$$\min\{\mathcal{E}(t)(u,\Gamma) : \Gamma \in \mathcal{R}(\overline{\Omega}_B),\ \Gamma_0 \tilde{\subset} \Gamma,\ u \in AD(\psi(t),\Gamma)\} \tag{3.60}$$

has a solution.

3.9. **Quasistatic evolution.** An *irreversible quasistatic evolution of minimum energy configurations* is a function $t \mapsto (u(t),\Gamma(t))$ which satisfies the following conditions:

(a) *static equilibrium*: for every $t \in [0,T]$ the pair $(u(t),\Gamma(t))$ is a minimum energy configuration at time t, i.e., $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B)$, $u(t) \in AD(\psi(t),\Gamma(t))$, and

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(v,\Gamma)$$

for every $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, with $\Gamma(t) \tilde{\subset} \Gamma$, and every $v \in AD(\psi(t),\Gamma)$;

(b) *irreversibility*: $\Gamma(s) \tilde{\subset} \Gamma(t)$ for $0 \le s < t \le T$;

(c) *nondissipativity*: the function $t \mapsto E(t) := \mathcal{E}(t)(u(t),\Gamma(t))$ is absolutely continuous on $[0,T]$ and its time derivative $\dot{E}(t)$ satisfies

$$\dot{E}(t) = \langle g(t),\dot{\psi}(t)\rangle - \dot{\mathcal{F}}(t)(u(t)) - \dot{\mathcal{G}}(t)(u(t)) \tag{3.61}$$

for a.e. $t \in [0,T]$.

Remark 3.9. To explain why condition (c) can be interpreted as the conservation of energy in this model, let us consider the very special case where $\Gamma(t) \cong \Gamma_0$ for every $t \in [0,T]$, with Γ_0 a closed set, and the function $t \mapsto u(t)$ is absolutely continuous from $[0,T]$ into $W^{1,p}(\Omega\backslash\Gamma_0;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$ (endowed with the sum of the norms). Then the time derivative $t \mapsto \dot{u}(t)$ belongs to the space $L^1([0,T];W^{1,p}(\Omega\backslash\Gamma_0;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m))$. Since $u(t)-\psi(t) = 0$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega\backslash\Gamma_0$ for every $t \in [0,T]$, we obtain that $\dot{u}(t) - \dot{\psi}(t) = 0$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega\backslash\Gamma_0$ for a.e. $t \in [0,T]$, so that $\dot{u}(t) - \dot{\psi}(t) \in AD(0,\Gamma_0)$ for a.e. $t \in [0,T]$. From the Euler equation (3.57) and from (3.58) we obtain $\langle g(t),\dot{u}(t)\rangle = \langle g(t),\dot{\psi}(t)\rangle$, so that (3.61) yields

$$\dot{E}(t) = \langle g(t),\dot{u}(t)\rangle - \dot{\mathcal{F}}(t)(u(t)) - \dot{\mathcal{G}}(t)(u(t)) \tag{3.62}$$

for a.e. $t \in [0,T]$. On the other hand, we have

$$\tfrac{d}{dt}\mathcal{F}(t)(u(t)) = \langle \partial\mathcal{F}(t)(u(t)),\dot{u}(t)\rangle + \dot{\mathcal{F}}(t)(u(t)), \tag{3.63}$$

$$\tfrac{d}{dt}\mathcal{G}(t)(u(t)) = \langle \partial\mathcal{G}(t)(u(t)),\dot{u}(t)\rangle + \dot{\mathcal{G}}(t)(u(t)) \tag{3.64}$$

for a.e. $t \in [0,T]$. This follows from our qualitative hypotheses on $\mathcal{F}$, $\dot{\mathcal{F}}$, $\mathcal{G}$, $\dot{\mathcal{G}}$, together with the estimates given by (3.16), (3.17), (3.20), (3.21), (3.32), (3.33), (3.36), and (3.37).

Let

$$E^{in}(t) := \mathcal{E}^{in}(u(t),\Gamma_0) = \mathcal{E}(t)(u(t),\Gamma_0) + \mathcal{F}(t)(u(t)) + \mathcal{G}(t)(u(t))$$

be the interior energy of the solution at time t. By (3.62), (3.63), and (3.64) we have

$$\dot{E}^{in}(t) = \langle g(t),\dot{u}(t)\rangle + \langle \partial\mathcal{F}(t)(u(t)),\dot{u}(t)\rangle + \langle \partial\mathcal{G}(t)(u(t)),\dot{u}(t)\rangle,$$

where the right hand side is the power of the exterior forces applied to the body at time t, including the surface forces acting on $\partial_D\Omega\backslash\Gamma_0$ (see (3.59)). Similar results can be obtained under weaker regularity conditions on $u(t)$ and $\Gamma(t)$.

Remark 3.10. If $t \mapsto (u(t),\Gamma(t))$ is an irreversible quasistatic evolution of minimum energy configurations, so is $t \mapsto (u(t),\Gamma(t)\backslash\partial_N\Omega)$.

Remark 3.11. In the definition of quasistatic evolution we make no measurability assumption on the function $t \mapsto u(t)$. However, the nondissipativity condition (c) implies that the function $t \mapsto \langle g(t),\dot{\psi}(t)\rangle - \dot{\mathcal{F}}(t)(u(t)) - \dot{\mathcal{G}}(t)(u(t))$ is measurable and belongs to $L^1([0,T])$ being the a.e. derivative of an absolutely continuous function.

Given an initial admissible configuration (u_0, Γ_0), we look for an irreversible quasistatic evolution such that $(u(0), \Gamma(0)) = (u_0, \Gamma_0)$. From the definition it follows that $(u(0), \Gamma(0))$ is a minimum energy configuration at time 0. Therefore a necessary condition for the solvability of the initial value problem is that (u_0, Γ_0) is a minimum energy configuration at time 0, i.e., $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B)$, $u_0 \in AD(\psi(0), \Gamma_0)$, and

$$\mathcal{E}(0)(u_0, \Gamma_0) \le \mathcal{E}(0)(u, \Gamma) \tag{3.65}$$

for every $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, with $\Gamma_0 \,\tilde{\subset}\, \Gamma$, and every $u \in AD(\psi(0), \Gamma)$.

The following remark shows that, if no forces are applied at time 0, then there are minimum energy configurations at time 0 with an arbitrary crack Γ_0, provided that some very weak additional conditions are satisfied.

Remark 3.12. Let $m = n$ and let $u_{id}(x) := x$ be the identical deformation, so that $\nabla u_{id}(x)$ is the identity matrix in $\mathbb{M}^{n\times n}$. If $\psi(0) = u_{id}$ and

$$\begin{gathered} W(x,\xi) \ge W(x, \nabla u_{id}(x)), \\ F(0,x,z) \le 0 = F(0,x,u_{id}(x)), \\ G(0,x,z) \le 0 = G(0,x,u_{id}(x)) \end{gathered}$$

for every $x \in \Omega$, $z \in \mathbb{R}^n$, $\xi \in \mathbb{M}^{n\times n}$, then no force is applied to the body at time 0 and the reference configuration u_{id} belongs to $AD(\psi(0), \emptyset)$ and is stress free. Moreover for every admissible crack $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$ the configuration (u_{id}, Γ) is a minimum energy configuration at time 0.

The main result of this paper is the following theorem, which will be proved in Section 7.

Theorem 3.13. *Let (u_0, Γ_0) be a minimum energy configuration at time 0, i.e., assume $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B)$, $u_0 \in AD(\psi(0), \Gamma_0)$, and (3.65). Then there exists an irreversible quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with $(u(0), \Gamma(0)) = (u_0, \Gamma_0)$.*

4. A FEW TOOLS

In this section we develop a few tools which will be very useful in the proof of Theorem 3.13. Subsections 4.1 and 4.2 introduce a weak notion of set convergence, which plays the role of Hausdorff convergence when no restriction is placed on either dimensionality or connectedness. Hausdorff convergence, which was instrumental in [9] and [7], can not be used in the present setting for two reasons: first, $\mathcal{H}^{n-1}$ is not lower semicontinuous with respect to Hausdorff convergence; then, Hausdorff convergence does not imply convergence of the associated minimum energy deformations.

Subsection 4.3 establishes the weak convergence of the stresses associated to the minimum energy deformations, while Subsection 4.4 deals with a technical result concerning the approximation of Bochner integrals with Riemann sums.

As in Section 2, let U be a bounded open set in $\mathbb{R}^n$ and let $1 < p < +\infty$.

4.1. **A convergence of sets.** We introduce a notion of convergence of sets based on the weak convergence in $SBV^p(U)$.

Definition 4.1. We say that Γ_k σ^p-converges to Γ in U if $\Gamma_k, \Gamma \subset U$, $\mathcal{H}^{n-1}(\Gamma_k)$ is bounded uniformly with respect to k, and the following conditions are satisfied:

(a) if u_j converges weakly to u in $SBV^p(U)$ and $S(u_j) \,\tilde{\subset}\, \Gamma_{k_j}$ for some sequence $k_j \to \infty$, then $S(u) \,\tilde{\subset}\, \Gamma$;
(b) there exist a function $u \in SBV^p(U)$ and a sequence u_k converging to u weakly in $SBV^p(U)$ such that $S(u) \,\tilde{=}\, \Gamma$ and $S(u_k) \,\tilde{\subset}\, \Gamma_k$ for every k.

Remark 4.2. The rectifiability of the σ^p-limit of any sequence of sets follows from the rectifiability of $S(u)$ for any $u \in SBV^p(U)$.

It is clear from the definition that, if a sequence σ^p-converges, then every subsequence σ^p-converges to the same limit. If Γ_k and Γ'_k σ^p-converge to Γ and Γ', respectively, then

$$\Gamma_k \tilde{\subset} \Gamma'_k \text{ for every } k \quad \Longrightarrow \quad \Gamma \tilde{\subset} \Gamma' .$$

Let us consider now the special case where $\Gamma_k = \Gamma_0$ for every k, with $\mathcal{H}^{n-1}(\Gamma_0) < +\infty$. Then it is not always true that Γ_k σ^p-converges to Γ_0 in U. Indeed, using Theorem 4.7 and Remark 2.9 we can prove that in this case Γ_k σ^p-converges to the set Γ characterized by the following properties:

(a) if $v \in SBV^p(U)$ and $S(v) \tilde{\subset} \Gamma_0$, then $S(v) \tilde{\subset} \Gamma$;
(b) there exists a function $u \in SBV^p(U)$ such that $S(u) \cong \Gamma \tilde{\subset} \Gamma_0$.

Therefore Γ_k σ^p-converges to Γ_0 in U if and only if

$$\text{there exists } u \in SBV^p(U) \text{ with } S(u) \cong \Gamma_0 . \tag{4.1}$$

Note that (4.1) is not always true. For instance, if Γ_0 is contained in a smooth manifold M of dimension $n-1$, then (4.1) implies that there exists $u \in W^{1,p}(\Omega \backslash M)$ such that $\Gamma_0 \cong \{x \in M : u^-(x) \neq u^+(x)\}$, where u^- and u^+ are the traces of u on both sides of M. If $p > n$, by the Sobolev embedding theorem u^- and u^+ are continuous on M, hence a set $\Gamma_0 \subset M$ which satisfies (4.1) must be open. Using the notion of capacity associated to $W^{1,p}$ it is possible to prove that also in the case $1 < p \leq n$ there are sets $\Gamma_0 \subset M$ which do not satisfy (4.1).

The following lower semicontinuity theorem is an easy consequence of Theorem 2.8.

Theorem 4.3. *Let κ be as in Theorem 2.8, let Γ_k, Γ, and Γ' be rectifiable subsets of U with $\mathcal{H}^{n-1}(\Gamma') < +\infty$, and let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E) < +\infty$. If Γ_k σ^p-converges to Γ in U, then*

$$\int_{(\Gamma \cup \Gamma') \backslash E} \kappa(x, \nu) \, d\mathcal{H}^{n-1} \leq \liminf_{k \to \infty} \int_{(\Gamma_k \cup \Gamma') \backslash E} \kappa(x, \nu_k) \, d\mathcal{H}^{n-1} ,$$

where ν and ν_k are unit normal vector fields on $\Gamma \cup \Gamma'$ and $\Gamma_k \cup \Gamma'$, respectively.

Proof. By condition (b) there exist a function $u \in SBV^p(U)$ and a sequence u_k converging to u weakly in $SBV^p(U)$ such that $S(u) \cong \Gamma$ and $S(u_k) \tilde{\subset} \Gamma_k$ for every k. The conclusion follows by applying Theorem 2.8 and Remark 2.10. □

Remark 4.4. Assume that Γ_k σ^p-converges to Γ, then

(a) $\mathcal{H}^{n-1}(\Gamma) < +\infty$ by Theorem 4.3;
(b) if further, K is compact and $\Gamma_k \tilde{\subset} K$ for every k, then $\Gamma \tilde{\subset} K$ by (b) of Definition 4.1, together with the fact that $\mathcal{H}^{n-1}(S(u) \setminus K) \leq \liminf_k \mathcal{H}^{n-1}(S(u_k) \setminus K)$.

Lemma 4.5. *Let $\Gamma \subset U$ be a set with $\mathcal{H}^{n-1}(\Gamma) < +\infty$. If there exists a sequence $u^i \in SBV^p(U) \cap L^\infty(U)$ such that $\Gamma \cong \bigcup_i S(u^i)$, then there exists $u \in SBV^p(U) \cap L^\infty(U)$ such that $S(u) \cong \Gamma$. Let Γ_k be a sequence of subsets of U with $\mathcal{H}^{n-1}(\Gamma_k)$ bounded uniformly with respect to k. If, in addition to the previous hypotheses, each function u^i is the weak limit in $SBV^p(U)$ of a sequence u^i_k with $S(u^i_k) \tilde{\subset} \Gamma_k$ for every k, then condition (b) of Definition 4.1 is satisfied.*

Proof. Suppose that there exists a sequence $u^i \in SBV^p(U) \cap L^\infty(U)$ such that $\Gamma \cong \bigcup_i S(u^i)$. It is not restrictive to assume that

$$\|u^i\|_\infty \leq 1 , \qquad \|\nabla u^i\|_p \leq 1 . \tag{4.2}$$

Given a sequence of real numbers $c_i > 0$, with $\sum_i c_i < +\infty$, we define

$$u := \sum_{i=1}^{\infty} c_i u^i \,, \qquad v^\ell := \sum_{i=1}^{\ell} c_i u^i \,.$$

Since $v^\ell \in SBV^p(U) \cap L^\infty(U)$ and $S(v^\ell) \tilde{\subset} \Gamma$, from Theorem 2.2 we obtain that $u \in SBV^p(U) \cap L^\infty(U)$ and v^ℓ converges to u weakly in $SBV^p(U)$. By Remark 2.9 we have $S(u) \tilde{\subset} \Gamma$.

It remains to choose the sequence c_i so that $\Gamma \tilde{\subset} S(u)$. First of all we fix a (Borel) orientation of Γ and for every $v \in SBV^p(U)$ we define the jump of v on Γ as $[v] := v^+ - v^- \in L^1(\Gamma, \mathcal{H}^{n-1})$. We define two sequences c_i and ε_i inductively. We set $c_1 = 1$. Suppose that c_ℓ and $\varepsilon_{\ell-1}$ have already been defined. We choose ε_ℓ such that $0 < \varepsilon_\ell < \varepsilon_{\ell-1}$ (for $\ell = 1$ we require only $0 < \varepsilon_1$) and

$$\mathcal{H}^{n-1}(\{x \in S(v^\ell) : |[v^\ell](x)| < \varepsilon_\ell\}) < 2^{-\ell} \,. \tag{4.3}$$

To choose $c_{\ell+1}$, for every $c > 0$ we consider the set

$$A_c^\ell := \{x \in S(v^\ell) : [v^\ell](x) + c[u^{\ell+1}](x) = 0\} \,,$$

as in the proof of [14, Lemma 3.1]. The family $(A_c^\ell)_{c>0}$ is composed of pairwise disjoint subsets of $S(v^\ell)$. As $\mathcal{H}^{n-1}(S(v^\ell)) < +\infty$, we have that $\mathcal{H}^{n-1}(A_c^\ell) = 0$ except for a countable number of c. We choose $c_{\ell+1}$ such that $0 < c_{\ell+1} < \varepsilon_\ell 2^{-\ell-1}$ and $\mathcal{H}^{n-1}(A^\ell_{c_{\ell+1}}) = 0$. Since $S(v^\ell + c_{\ell+1} u^{\ell+1}) \cup A^\ell_{c_{\ell+1}} \cong S(v^\ell) \cup S(u^{\ell+1})$, we have

$$S(v^{\ell+1}) = S(v^\ell + c_{\ell+1} u^{\ell+1}) \cong S(v^\ell) \cup S(u^{\ell+1}) \,.$$

We deduce by induction that

$$S(v^\ell) \cong \bigcup_{i=1}^{\ell} S(u^i) \tag{4.4}$$

for every ℓ. Let

$$E_\ell := \{x \in S(v^\ell) : |[v^\ell]| < \varepsilon_\ell\} \quad \text{and} \quad F_\ell := \bigcup_{i=\ell}^{\infty} E_i \,.$$

By (4.3) we have $\mathcal{H}^{n-1}(E_\ell) < 2^{-\ell}$, hence $\mathcal{H}^{n-1}(F_\ell) < 2^{1-\ell}$.

Let us prove that

$$S(v^\ell) \tilde{\subset} S(u) \cup E_\ell \,. \tag{4.5}$$

Indeed, if $x \in S(v^\ell) \backslash E_\ell$, then $|[v^\ell](x)| \geq \varepsilon_\ell$. On the other hand, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ we have

$$[u](x) = [v^\ell](x) + \sum_{i=\ell+1}^{\infty} c_i [u^i](x) \,,$$

hence

$$|[u](x)| \geq \varepsilon_\ell - \Big| \sum_{i=\ell+1}^{\infty} c_i [u^i](x) \Big| . \tag{4.6}$$

Since $|[u^i](x)| < 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ we have

$$\Big| \sum_{i=\ell+1}^{\infty} c_i [u^i](x) \Big| \leq \sum_{i=\ell+1}^{\infty} c_i < \varepsilon_\ell \sum_{i=\ell+1}^{\infty} 2^{-i} \leq \varepsilon_\ell \,,$$

and we deduce from (4.6) that $[u](x) \neq 0$, hence $x \in S(u)$ for $\mathcal{H}^{n-1}$-a.e. $x \in S(v^\ell) \backslash E_\ell$. We conclude that (4.5) is satisfied. By (4.4) and (4.5) we obtain that

$$\bigcup_{i=1}^{k} S(u^i) \tilde{\subset} S(u) \cup F_\ell$$

for every $k \geq \ell$. Taking the union with respect to k we get $\Gamma \tilde{\subset} S(u) \cup F_\ell$, which implies $\mathcal{H}^{n-1}(\Gamma \setminus S(u)) \leq 2^{1-\ell}$. By the arbitrariness of ℓ we obtain $\Gamma \tilde{\subset} S(u)$.

Under the additional hypotheses of the second part of the lemma we may assume that

$$\|u_k^i\|_\infty \leq 1\,, \qquad \|\nabla u_k^i\|_p \leq 1\,. \tag{4.7}$$

and we define

$$v_k^\ell := \sum_{i=1}^{\ell} c_i u_k^i\,, \qquad u_k := \sum_{i=1}^{\infty} c_i u_k^i\,.$$

Since $v_k^\ell \in SBV^p(U) \cap L^\infty(U)$ and $S(v_k^\ell) \tilde{\subset} \Gamma_k$, from Theorem 2.2 we obtain that $u_k \in SBV^p(U) \cap L^\infty(U)$ and v_k^ℓ converges to u_k weakly in $SBV^p(U)$ as $\ell \to \infty$ and u_k converges to u weakly in $SBV^p(U)$ as $k \to \infty$. By Remark 2.9 we have $S(u_k) \tilde{\subset} \Gamma_k$ for every k. ☐

Proposition 4.6. *Assume that Γ_k σ^p-converges to Γ in U and that u_k converges to u weakly in $GSBV^p(U;\mathbb{R}^m)$. If $S(u_k) \tilde{\subset} \Gamma_k$ for every k, then $S(u) \tilde{\subset} \Gamma$.*

Proof. Assume that $S(u_k) \tilde{\subset} \Gamma_k$ for every k. Let $\psi_j \in C_c^\infty(\mathbb{R}^m;\mathbb{R}^m)$ with $\psi_j(z) = z$ for $|z| \leq j$. Then $\psi_j(u_k)$ converges to $\psi_j(u)$ weakly in $SBV^p(U;\mathbb{R}^m)$. Since $S(\psi_j(u_k)) \tilde{\subset} S(u_k) \tilde{\subset} \Gamma_k$, if we apply condition (a) of Definition 4.1 to each component we obtain $S(\psi_j(u)) \tilde{\subset} \Gamma$. The conclusion follows from the fact that $S(u) \cong \bigcup_j S(\psi_j(u))$. ☐

4.2. **Compactness properties.** We now prove some compactness properties of the σ^p-convergence.

Theorem 4.7. *Every sequence $\Gamma_k \subset U$, with $\mathcal{H}^{n-1}(\Gamma_k)$ uniformly bounded, has a σ^p-convergent subsequence.*

Proof. Let Γ_k be a sequence of subsets of U with $\mathcal{H}^{n-1}(\Gamma_k) \leq C < +\infty$ for every k. Let w_h be a sequence in $L^\infty(U)$ with the following density property: for every $w \in L^\infty(U)$ there exists a subsequence w_{h_i} which converges to w strongly in $L^p(U)$ and satisfies the inequality $\|w_{h_i}\|_\infty \leq \|w\|_\infty$ for every i. For every positive integers ℓ, h, and k let $u_k^{\ell,h}$ be the solution of the following minimization problem

$$\min\{\|\nabla u\|_p^p + \ell \|u - w_h\|_p^p : u \in SBV^p(U),\ S(u) \tilde{\subset} \Gamma_k\}\,.$$

To prove the existence of a solution it is enough to apply Theorem 2.2 and Remark 2.9 to a minimizing sequence; the uniqueness follows from the strict convexity of the functional. By a truncation argument we obtain $\|u_k^{\ell,h}\|_\infty \leq \|w_h\|_\infty$. By Theorem 2.2 and by a diagonal argument, passing to a subsequence, we may assume that $u_k^{\ell,h}$ converges weakly in $SBV^p(U)$ to a function $u^{\ell,h} \in SBV^p(U)$ as $k \to \infty$. Let

$$\Gamma := \bigcup_{\ell,h=1}^{\infty} S(u^{\ell,h})\,.$$

To prove that $\mathcal{H}^{n-1}(\Gamma) < +\infty$, for every integer $r > 0$ we consider the sequence $v_k^r := (u_k^{\ell,h})_{1 \leq \ell,h \leq r}$ and the function $v^r := (u^{\ell,h})_{1 \leq \ell,h \leq r}$. As v_k^r converges to v^r weakly in $SBV^p(U;\mathbb{M}^{r\times r})$ we have

$$\mathcal{H}^{n-1}\Big(\bigcup_{\ell,h=1}^{r} S(u^{\ell,h})\Big) = \mathcal{H}^{n-1}(S(v^r)) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(S(v_k^r)) =$$

$$= \liminf_{k\to\infty} \mathcal{H}^{n-1}\Big(\bigcup_{\ell,h=1}^{r} S(u_k^{\ell,h})\Big) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(\Gamma_k) \leq C\,.$$

Passing to the limit as $r \to \infty$ we obtain $\mathcal{H}^{n-1}(\Gamma) \leq C$.

We want to prove that Γ_k σ^p-converges to Γ. By construction Γ satisfies all hypotheses of Lemma 4.5, which implies that condition (b) of Definition 4.1 is satisfied.

To prove condition (a) let v_j be a sequence which converges weakly in $SBV^p(U)$ to a function v, and with $S(v_j) \tilde{\subset} \Gamma_{k_j}$ for some sequence $k_j \to \infty$. We have to show that $S(v) \tilde{\subset} \Gamma$. By the density property of w_h there exists a subsequence w_{h_i} which converges to v in $L^p(U)$ and such that $\|w_{h_i}\|_\infty \leq \|v\|_\infty < +\infty$. Let $\ell_i \to \infty$ such that $\ell_i \|w_{h_i} - v\|_p^p \to 0$. From the minimality of $u_{k_j}^{\ell_i,h_i}$ we obtain $\|u_{k_j}^{\ell_i,h_i}\|_\infty \leq \|w_{h_i}\|_\infty \leq \|v\|_\infty$ and

$$\|\nabla u_{k_j}^{\ell_i,h_i}\|_p^p + \ell_i \|u_{k_j}^{\ell_i,h_i} - w_{h_i}\|_p^p \leq \|\nabla v_j\|_p^p + \ell_i \|v_j - w_{h_i}\|_p^p .$$

Passing to the limit as $j \to \infty$ we get $\|u^{\ell_i,h_i}\|_\infty \leq \|v\|_\infty$ and

$$\|\nabla u^{\ell_i,h_i}\|_p^p + \ell_i \|u^{\ell_i,h_i} - w_{h_i}\|_p^p \leq M + \ell_i \|v - w_{h_i}\|_p^p , \tag{4.8}$$

where $M := \sup_j \|\nabla v_j\|_p^p < +\infty$. This inequality implies that $\nabla u^{\ell_i,h_i}$ is bounded in $L^p(U;\mathbb{R}^n)$ uniformly with respect to i and that $u^{\ell_i,h_i} - w_{h_i}$ tends to 0 in $L^p(U)$. Since w_{h_i} converge to v in $L^p(U)$, we conclude that u^{ℓ_i,h_i} converge to v in $L^p(U)$. We now apply Remark 2.9 to the sequence u^{ℓ_i,h_i} and conclude that $S(v) \tilde{\subset} \Gamma$. □

We shall use the following extension of the compactness theorem, that can be proved by adapting the arguments of Helly's theorem.

Theorem 4.8. *Let $t \mapsto \Gamma_k(t)$ be a sequence of increasing set functions defined on an interval $I \subset \mathbb{R}$ with values contained in U, i.e.,*

$$\Gamma_k(s) \tilde{\subset} \Gamma_k(t) \subset U \quad \text{for every } s,\, t \in I \text{ with } s < t .$$

Assume that the measures $\mathcal{H}^{n-1}(\Gamma_k(t))$ are bounded uniformly with respect to k and t. Then there exist a subsequence Γ_{k_j} and an increasing set function $t \mapsto \Gamma(t)$ on I such that

$$\Gamma_{k_j}(t) \quad \sigma^p\text{-converges to } \Gamma(t) \text{ in } U \tag{4.9}$$

for every $t \in I$.

Proof. Let D be a countable dense set in I. Using Theorem 4.7 and a diagonal argument we can extract a subsequence Γ_{k_j} such that $\Gamma_{k_j}(t)$ σ^p-converges to some set $\Gamma(t)$ in U for every $t \in D$. From Remark 4.2 we have $\Gamma(s) \tilde{\subset} \Gamma(t)$ for every $s,\, t \in D$ with $s < t$. By Theorem 4.3 the measures $\mathcal{H}^{n-1}(\Gamma(t))$ are bounded uniformly with respect to $t \in D$. For every $t \in I \backslash D$ we define

$$\Gamma_-(t) := \bigcup_{s<t,\, s \in D} \Gamma(s) , \qquad \Gamma_+(t) := \bigcap_{s>t,\, s\in D} \Gamma(s) .$$

Then $\Gamma_+(s) \tilde{\subset} \Gamma_-(t) \tilde{\subset} \Gamma_+(t)$ for every $s,\ t \in I \setminus D$ with $s < t$, and the measures $\mathcal{H}^{n-1}(\Gamma_+(t))$ are bounded uniformly with respect to $t \in I \backslash D$. This implies that the set $D_1 := \{t \in I \backslash D : \mathcal{H}^{n-1}(\Gamma_+(t) \backslash \Gamma_-(t)) > 0\}$ is at most countable.

For every $t \in I \backslash (D \cup D_1)$ we define $\Gamma(t) := \Gamma_+(t) \cong \Gamma_-(t)$. Given $t \in I \backslash (D \cup D_1)$, let us prove that $\Gamma_{k_j}(t)$ σ^p-converges to $\Gamma(t)$. If u_i converges weakly to u in $SBV^p(U)$ and $S(u_i) \tilde{\subset} \Gamma_{k_{j_i}}(t)$ for some sequence $j_i \to \infty$, then $S(u_i) \tilde{\subset} \Gamma_{k_{j_i}}(s)$ for every $s > t,\ s \in D$; by σ^p-convergence, this implies $S(u) \tilde{\subset} \Gamma(s)$, and taking the intersection for $s > t,\ s \in D$, we obtain $S(u) \tilde{\subset} \Gamma_+(t) \cong \Gamma(t)$, so that condition (a) of Definition 4.1 is satisfied.

To prove condition (b), we observe that for every $s < t,\ s \in D$ there exists a function $u(s) \in SBV^p(U)$ and a sequence $u_k(s)$ converging to $u(s)$ weakly in $SBV^p(U)$ such that $S(u(s)) \cong \Gamma(s)$ and $S(u_k(s)) \tilde{\subset} \Gamma_{k_j}(s) \tilde{\subset} \Gamma_{k_j}(t)$. Condition (b) of Definition 4.1 follows now from Lemma 4.5.

Since the set D_1 is at most countable, using Theorem 4.7 and a diagonal argument it is possible to extract a further subsequence, still denoted Γ_{k_j}, such that $\Gamma_{k_j}(t)$ σ^p-converges to some set $\Gamma(t)$ for every $t \in D_1$. This concludes the proof of (4.9) for every $t \in I$. The monotonicity of $t \mapsto \Gamma(t)$ on I follows from Remark 4.2. □

4.3. **Some results in measure theory.** We begin with a lemma concerning perturbations of bounded sequences in L^p spaces.

Lemma 4.9. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $p > 1$, let $m, n \geq 1$, and let $H\colon X \times \mathbb{R}^n \to \mathbb{R}^m$ be a Carathéodory function. Assume that there exist a constant $a \geq 0$ and a nonnegative function $b \in L^{p'}(X)$, with $p' = p/(p-1)$, such that*

$$|H(x,\xi)| \leq a|\xi|^{p-1} + b(x) \tag{4.10}$$

for every $(x,\xi) \in X \times \mathbb{R}^n$. Let Φ_k and Ψ_k be two sequences in $L^p(X;\mathbb{R}^n)$. Assume that Φ_k is bounded in $L^p(X;\mathbb{R}^n)$ and Ψ_k converges to 0 strongly in $L^p(X;\mathbb{R}^n)$. Then

$$\int_X [H(x, \Phi_k(x) + \Psi_k(x)) - H(x, \Phi_k(x))]\, \Phi(x)\, d\mu(x) \to 0 \tag{4.11}$$

for every $\Phi \in L^p(X;\mathbb{R}^m)$.

Proof. There exists a constant $C \geq 0$ such that

$$\|\Phi_k\|_p \leq C \qquad \text{and} \qquad \|\Phi_k + \Psi_k\|_p \leq C \tag{4.12}$$

for every k. Let us fix $\Phi \in L^p(X;\mathbb{R}^m)$. By the absolute continuity of the integral, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$A \in \mathcal{A},\ \ \mu(A) < \delta \qquad \Longrightarrow \qquad \int_A |\Phi(x)|^p d\mu(x) < \varepsilon^p\,. \tag{4.13}$$

Let us fix $M > 0$ such that

$$2\, C^p/M^p < \delta\,. \tag{4.14}$$

For every $x \in X$ and $\eta > 0$ let

$$\omega(x,\eta) := \max\{|H(x,\xi_1) - H(x,\xi_2)| : |\xi_1| \leq M,\ |\xi_2| \leq M,\ |\xi_1 - \xi_2| \leq \eta\}\,. \tag{4.15}$$

Since H is a Carathéodory function and satisfies (4.10), it turns out that ω is a Carathéodory function, $\omega(x,0) = 0$, and

$$0 \leq \omega(x,\eta) \leq 2aM^{p-1} + 2b(x)$$

for every $x \in X$ and every $\eta > 0$. As Ψ_k converges to 0 strongly in $L^p(X;\mathbb{R}^n)$ and $\omega(x,\eta) \to 0$ as $\eta \to 0$, we have

$$\int_X \omega(x, |\Psi_k(x)|)\, |\Phi(x)|\, d\mu(x) \to 0\,. \tag{4.16}$$

Let

$$A_k := \{x \in X : |\Phi_k(x) + \Psi_k(x)| > M\} \cup \{x \in X : |\Phi_k(x)| > M\}\,, \tag{4.17}$$

$$B_k := \{x \in X : |\Phi_k(x) + \Psi_k(x)| \leq M\} \cap \{x \in X : |\Phi_k(x)| \leq M\}\,. \tag{4.18}$$

By (4.10) we have

$$|H(x, \Phi_k(x) + \Psi_k(x)) - H(x, \Phi_k(x))| \leq a|\Phi_k(x) + \Psi_k(x)|^{p-1} + a|\Phi_k(x)|^{p-1} + 2b(x) \tag{4.19}$$

for every $x \in A_k$. By (4.15) and (4.18) we have

$$|H(x, \Phi_k(x) + \Psi_k(x)) - H(x, \Phi_k(x))| \leq \omega(x, |\Psi_k(x)|) \tag{4.20}$$

for every $x \in B_k$. Using the Hölder inequality, from (4.19) and (4.20) we obtain

$$\int_X |H(x, \Phi_k(x) + \Psi_k(x)) - H(x, \Phi_k(x))|\, |\Phi(x)|\, d\mu(x) \leq \tag{4.21}$$

$$\leq K\Big(\int_{A_k} |\Phi(x)|^p d\mu(x)\Big)^{\frac{1}{p}} + \int_{B_k} \omega(x, |\Psi_k(x)|)\, |\Phi(x)|\, d\mu(x)\,,$$

where $K := 2(aC^{p-1} + \|b\|_{p'})$.

By Chebyshev's inequality, (4.12) and (4.17) imply $\mu(A_k) \leq 2\,C^p/M^p$, and thus, by (4.14), $\mu(A_k) < \delta$. Therefore, from (4.13) and (4.21) we deduce that

$$\int_X |H(x,\Phi_k(x)+\Psi_k(x)) - H(x,\Phi_k(x))|\,|\Phi(x)|\,d\mu(x) \leq$$
$$\leq K\varepsilon + \int_X \omega(x,|\Psi_k(x)|)\,|\Phi(x)|\,d\mu(x)\,.$$

Taking (4.16) into account, we obtain (4.11) upon passing to the limit in the previous inequality first as $k\to\infty$ and then as $\varepsilon\to 0$. □

Remark 4.10. Let $(X,\mathcal{A},\mu)$ be a finite nonatomic measure space, let $p>1$, let m, $n\geq 1$, and let $H\colon X\times\mathbb{R}^n \to \mathbb{R}^m$ be a Carathéodory function. Assume that the function $x\mapsto H(x,\Phi(x))\,\Psi(x)$ is μ-integrable for every $\Phi\in L^p(X;\mathbb{R}^n)$ and $\Psi\in L^p(X;\mathbb{R}^m)$. Then, the function $x\mapsto H(x,\Phi(x))$ belongs to $L^{p'}(X;\mathbb{R}^m)$ for every $\Phi\in L^p(X;\mathbb{R}^n)$, and this implies (4.10) by a classical result in the theory of integral operators (see [19, Theorem 2.3 in Chapter I]).

In particular, the conclusion of Lemma 4.9 still holds.

Let U and W be as in Theorem 2.8. Assume in addition that $\xi\mapsto W(x,\xi)$ belongs to $C^1(\mathbb{M}^{m\times n})$ for every $x\in U$. Since $\xi\mapsto W(x,\xi)$ is rank-one convex on $\mathbb{M}^{m\times n}$ for every $x\in U$ (see, e.g., [8]), from (2.5) we can deduce that there exist a constant $a_2>0$ and a nonnegative function $b_2\in L^{p'}(U)$ such that

$$|\partial_\xi W(x,\xi)| \leq a_2|\xi|^{p-1} + b_2(x) \tag{4.22}$$

for every $(x,\xi)\in U\times\mathbb{M}^{m\times n}$.

Let us consider the C^1 functional $\mathcal{W}\colon L^p(U;\mathbb{M}^{m\times n})\to\mathbb{R}$ defined by

$$\mathcal{W}(\Phi) := \int_U W(x,\Phi(x))\,dx$$

whose differential $\partial\mathcal{W}\colon L^p(U;\mathbb{M}^{m\times n})\to L^{p'}(U;\mathbb{M}^{m\times n})$ is given by

$$\langle\partial\mathcal{W}(\Phi),\Psi\rangle = \int_U \partial_\xi W(x,\Phi(x))\Psi(x)\,dx\,,$$

for every Φ, $\Psi\in L^p(U;\mathbb{M}^{m\times n})$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces $L^{p'}(U;\mathbb{M}^{m\times n})$ and $L^p(U;\mathbb{M}^{m\times n})$.

Lemma 4.11. *Assume that u_k converges to u weakly in $GSBV^p(U;\mathbb{R}^m)$ and that $\mathcal{W}(\nabla u_k)$ converges to $\mathcal{W}(\nabla u)$. Then $\partial\mathcal{W}(\nabla u_k)$ converges to $\partial\mathcal{W}(\nabla u)$ weakly in $L^{p'}(U;\mathbb{M}^{m\times n})$.*

Proof. It is enough to prove that

$$\langle\partial\mathcal{W}(\nabla u),\Psi\rangle \leq \liminf_{k\to\infty}\,\langle\partial\mathcal{W}(\nabla u_k),\Psi\rangle \tag{4.23}$$

for every $\Psi\in L^p(U;\mathbb{M}^{m\times n})$. Let η_i be a sequence of positive numbers converging to 0. If we apply the lower semicontinuity theorem for $GSBV$ (Theorem 2.8) to the function $W(x,\xi+\eta_i\Psi(x))$, for every i we obtain

$$\frac{\mathcal{W}(\nabla u+\eta_i\Psi)-\mathcal{W}(\nabla u)}{\eta_i} \leq \liminf_{k\to\infty}\,\frac{\mathcal{W}(\nabla u_k+\eta_i\Psi)-\mathcal{W}(\nabla u_k)}{\eta_i}\,. \tag{4.24}$$

Therefore there exists an increasing sequence of integers k_i such that

$$\frac{\mathcal{W}(\nabla u+\eta_i\Psi)-\mathcal{W}(\nabla u)}{\eta_i} - \frac{1}{i} \leq \frac{\mathcal{W}(\nabla u_k+\eta_i\Psi)-\mathcal{W}(\nabla u_k)}{\eta_i} \tag{4.25}$$

for every $k\geq k_i$. Defining $\varepsilon_k:=\eta_i$ for $k_i\leq k<k_{i+1}$, from (4.25) we obtain

$$\liminf_{k\to\infty}\,\frac{\mathcal{W}(\nabla u+\varepsilon_k\Psi)-\mathcal{W}(\nabla u)}{\varepsilon_k} \leq \liminf_{k\to\infty}\,\frac{\mathcal{W}(\nabla u_k+\varepsilon_k\Psi)-\mathcal{W}(\nabla u_k)}{\varepsilon_k}\,. \tag{4.26}$$

Since $\mathcal{W}$ is of class C^1 on $L^p(U;\mathbb{M}^{m\times n})$, we have

$$\langle \partial\mathcal{W}(\nabla u), \Psi\rangle = \lim_{k\to\infty} \frac{\mathcal{W}(\nabla u + \varepsilon_k \Psi) - \mathcal{W}(\nabla u)}{\varepsilon_k} \tag{4.27}$$

and

$$\frac{\mathcal{W}(\nabla u_k + \varepsilon_k \Psi) - \mathcal{W}(\nabla u_k)}{\varepsilon_k} = \langle \partial\mathcal{W}(\nabla u_k + \tau_k \Psi), \Psi\rangle \tag{4.28}$$

for suitable constants $\tau_k \in [0, \varepsilon_k]$. By (4.22) and by Lemma 4.9 we have

$$\liminf_{k\to\infty} \langle \partial\mathcal{W}(\nabla u_k + \tau_k \Psi), \Psi\rangle = \liminf_{k\to\infty} \langle \partial\mathcal{W}(\nabla u_k), \Psi\rangle\,. \tag{4.29}$$

Inequality (4.23) follows now from (4.26)–(4.29). □

If W is strictly convex, using [24, Theorem 2] one can prove that ∇u_k converges in measure to ∇u on U. By (4.22) this implies that $\partial\mathcal{W}(\nabla u_k)$ converges to $\partial\mathcal{W}(\nabla u)$ weakly in $L^{p'}(U;\mathbb{M}^{m\times n})$. We refer to [6] for similar results with $p = 1$.

4.4. **Approximation by Riemann sums.** We now prove a lemma concerning the approximation of Lebesgue integrals by Riemann sums. The convergence result (4.34) is well-known (see [17]). For the application we have in mind we need the stronger result (4.33), that is related to the Saks-Henstock lemma (see [23] and [18]) used in the theory of Henstock-Kurzweil integral (see, e.g., [21]). We prefer to present an independent proof, based on [12, page 63], which only uses Fubini's theorem.

Lemma 4.12. *Let $[a,b]$ be a closed bounded interval, let X be a Banach space, and let $f\colon [a,b] \to X$ be a Bochner integrable function. Then there exists a sequence of subdivisions $(t_k^i)_{0\le i\le i_k}$ of the interval $[a,b]$, with*

$$a = t_k^0 < t_k^1 < \cdots < t_k^{i_k-1} < t_k^{i_k} = b\,, \tag{4.30}$$

$$\lim_{k\to\infty} \max_{1\le i\le i_k} (t_k^i - t_k^{i-1}) = 0\,, \tag{4.31}$$

such that

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} \|f(t_k^i) - f(t)\|\, dt = 0\,. \tag{4.32}$$

In particular we have

$$\sum_{i=1}^{i_k} \Big\|(t_k^i - t_k^{i-1}) f(t_k^i) - \int_{t_k^{i-1}}^{t_k^i} f(t)\, dt\Big\| \longrightarrow 0\,, \tag{4.33}$$

$$\sum_{i=1}^{i_k} (t_k^i - t_k^{i-1}) f(t_k^i) \longrightarrow \int_a^b f(t)\, dt \qquad \textit{strongly in } X \tag{4.34}$$

as $k\to\infty$.

Proof. We extend f to 0 outside $[a,b]$. Set, for every $m\ge 1$ and $i\in\mathbb{Z}$, $\tau_m^i := i/m$. For every $s\in[0,1]$ we have

$$\sum_{i\in\mathbb{Z}} \int_{s+\tau_m^{i-1}}^{s+\tau_m^i} \|f(s+\tau_m^i) - f(t)\|\, dt =$$

$$= \sum_{i\in\mathbb{Z}} \int_0^{\frac{1}{m}} \|f(s+\tau_m^i) - f(s+\tau_m^i - \tau)\|\, d\tau\,.$$

Note that there are at most $m(b-a+1)+2$ non-zero elements in the above sums, namely those with $i \in I_m := \{i \in \mathbb{Z} : m(a-1) \le i \le mb+1\}$. Integrating with respect to s we obtain

$$\int_0^1 \Big[\sum_{i\in\mathbb{Z}} \int_{s+\tau_m^{i-1}}^{s+\tau_m^i} \|f(s+\tau_m^i) - f(t)\|\, dt\Big]\, ds \le$$

$$\le \sum_{i\in I_m} \int_0^{\frac{1}{m}} \Big[\int_{-\infty}^{+\infty} \|f(s+\tau_m^i) - f(s+\tau_m^i - \tau)\|\, ds\Big]\, d\tau = \tag{4.35}$$

$$= \sum_{i\in I_m} \int_0^{\frac{1}{m}} \Big[\int_{-\infty}^{+\infty} \|f(s) - f(s-\tau)\|\, ds\Big]\, d\tau\,.$$

By continuity of the translations, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{-\infty}^{+\infty} \|f(s) - f(s-\tau)\|\, ds < \varepsilon \tag{4.36}$$

for $0 < \tau < \delta$. Thus, from (4.35) and (4.36) we obtain

$$\lim_{m\to\infty} \int_0^1 \Big[\sum_{i\in\mathbb{Z}} \int_{s+\tau_m^{i-1}}^{s+\tau_m^i} \|f(s+\tau_m^i) - f(t)\|\, dt\Big]\, ds = 0\,.$$

Therefore there exists a sequence $m_k \to \infty$ such that

$$\lim_{k\to\infty} \sum_{i\in\mathbb{Z}} \int_{s+\tau_{m_k}^{i-1}}^{s+\tau_{m_k}^i} \|f(s+\tau_{m_k}^i) - f(t)\|\, dt = 0 \tag{4.37}$$

for a.e. $s \in [0,1]$. Let us fix $s \in [0,1]$ such that (4.37) holds. Let ρ_k be the largest integer i such that $s + \tau_{m_k}^i \le a$, and let σ_k be the smallest integer i such that $s + \tau_{m_k}^i \ge b$, and let $i_k := \sigma_k - \rho_k$. For $i = 1, \dots, i_k - 1$ we define $t_k^i := s + \tau_{m_k}^{\rho_k + i}$ and we set $t_k^0 := a$ and $t_k^{i_k} := b$. Then (4.30) and (4.31) are satisfied. Moreover

$$\sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} \|f(t_k^i) - f(t)\|\, dt =$$

$$= \sum_{i=\rho_k+2}^{\sigma_k-1} \int_{s+\tau_{m_k}^{i-1}}^{s+\tau_{m_k}^i} \|f(s+\tau_{m_k}^i) - f(t)\|\, dt + \tag{4.38}$$

$$+ \int_a^{a_k} \|f(a_k) - f(t)\|\, dt + \int_{b_k}^b \|f(b) - f(t)\|\, dt\,,$$

where $a_k := s + \tau_{m_k}^{\rho_k+1}$ and $b_k := s + \tau_{m_k}^{\sigma_k-1}$. Since all integers between $\rho_k + 2$ and $\sigma_k - 1$ belong to I_{m_k}, the first term in the right hand side of (4.38) tends to 0 by (4.37). The second term is estimated by

$$\int_a^{a_k} \|f(a_k) - f(t)\|\, dt \le \int_{s+\tau_{m_k}^{\rho_k}}^{s+\tau_{m_k}^{\rho_k+1}} \|f(s+\tau_{m_k}^{\rho_k+1}) - f(t)\|\, dt$$

which also tends to 0 by (4.37). The third term tends to 0 by the absolute continuity of the integral, since $b - b_k$ tends to 0 by the choice of σ_k. This concludes the proof of (4.32). ☐

Remark 4.13. If X_j is a sequence of Banach spaces and $f_j \colon [a,b] \to X_j$ is a sequence of Bochner integrable function, then there exists a sequence of subdivisions $(t_k^i)_{0\le i\le i_k}$, independent of j and satisfying (4.30) and (4.31), such that (4.32) is satisfied simultaneously for each function f_j. Indeed, we can consider the Banach space X of all sequences $x := (x_j)$ such that $x_j \in X_j$ for every j and $\sum_j \|x_j\|_{X_j} < +\infty$, endowed with the norm $\|x\|_X := \sum_j \|x_j\|_{X_j}$. To obtain the result it is enough to apply Lemma 4.12 to the function

$g\colon [a,b] \to X$ whose components g_j are given by $g_j(t) := 2^{-j} f_j(t)/\|f_j\|_1$, where $\|f_j\|_1 := \int_a^b \|f_j(t)\|_{X_j}\, dt$.

5. PRELIMINARY RESULTS

We now return to the framework described in Section 3 and adapt to it the tools developed in Section 4. Moreover, we extend the jump transfer results of [14] to the space $GSBV(\Omega;\mathbb{R}^m)$, and use them to prove the stability of minimum energy configurations (see Subsection 3.8).

5.1. **Jump transfer.** To deal with the interaction between cracks and boundary deformations it is convenient to extend all deformations to a bounded open set Ω_0, containing $\overline{\Omega}$ and with Lipschitz boundary. When we speak of σ^p-convergence we always refer to σ^p-convergence in Ω_0.

For every rectifiable set $\Gamma \subset \mathbb{R}^n$ we define

$$\Gamma^N := \Gamma \cup \partial_N \Omega\,. \tag{5.1}$$

From (3.3) we obtain that

$$\mathcal{K}(\Gamma^N) = \mathcal{K}(\Gamma) \tag{5.2}$$

for every rectifiable set $\Gamma \subset \mathbb{R}^n$. Theorem 4.3 implies that

$$\mathcal{K}(\Gamma \cup \Gamma') \le \liminf_{k\to\infty} \mathcal{K}(\Gamma_k \cup \Gamma') \tag{5.3}$$

whenever Γ_k, Γ, and Γ' are rectifiable sets in $\overline{\Omega}$, Γ_k σ^p-converges to Γ, and $\mathcal{H}^{n-1}(\Gamma') < +\infty$.

From [14, Theorem 2.1] we shall obtain the following result.

Theorem 5.1 (Jump transfer in SBV)**.** *Assume that $\Gamma_k \in \mathcal{R}(\overline{\Omega}_B)$ and that Γ_k^N σ^p-converges to Γ. Then for every function $v \in SBV^p(\Omega_0;\mathbb{R}^m)$ there exists a sequence $v_k \in SBV^p(\Omega_0;\mathbb{R}^m)$ such that*

(a) $v_k = v$ *a.e. in* $\Omega_0 \backslash \Omega_B$,
(b) $v_k \to v$ *strongly in* $L^1(\Omega_0;\mathbb{R}^m)$,
(c) $\nabla v_k \to \nabla v$ *strongly in* $L^p(\Omega_0;\mathbb{M}^{m\times n})$,
(d) $\mathcal{H}^{n-1}((S(v_k)\backslash\Gamma_k^N)\backslash(S(v)\backslash\Gamma^N)) \to 0$.

If, in addition, $v \in L^\infty(\Omega_0;\mathbb{R}^m)$, then we may assume that v_k is bounded in $L^\infty(\Omega_0;\mathbb{R}^m)$.

To prove Theorem 5.1 we need the following lemma.

Lemma 5.2. *Assume that $\Gamma_k \in \mathcal{R}(\overline{\Omega}_B)$ and that Γ_k^N σ^p-converges to Γ. Then there exist a function $w \in SBV^p(\Omega_0)$ and a sequence w_k converging to w weakly in $SBV^p(\Omega_0)$ such that $S(w) \cong \Gamma\backslash\partial_N\Omega$ and $S(w_k) \,\tilde{\subset}\, \Gamma_k$ for every k.*

Proof. Let $\varphi_i \in C^\infty(\mathbb{R}^n)$ with $\varphi_i(x) = 1$ if $\mathrm{dist}(x,\partial_N\Omega) > 1/i$, and $\varphi_i(x) = 0$ in a neighbourhood of $\partial_N\Omega$. From condition (b) in Definition 4.1 there exist a function $v \in SBV^p(\Omega_0)$ and a sequence v_k converging to v weakly in $SBV^p(\Omega_0)$ such that $S(v) \cong \Gamma$ and $S(v_k) \,\tilde{\subset}\, \Gamma_k^N$ for every k. Let $v^i := \varphi_i v$ and $v_k^i := \varphi_i v_k$. Then v_k^i converges to v^i weakly in $SBV^p(\Omega_0)$, $S(v_k^i) \,\tilde{\subset}\, \Gamma_k$, and $\Gamma\backslash\partial_N\Omega \cong \bigcup_i S(v^i)$. As $\mathcal{H}^{n-1}(\Gamma) < +\infty$ by Remark 4.4-(a), the conclusion follows from Lemma 4.5. ☐

Proof of Theorem 5.1. By assumption we have $\Gamma_k^N \,\tilde{\subset}\, \overline{\Omega}_B \cup \partial_N\Omega$ for every k. As $\overline{\Omega}_B \cup \partial_N\Omega$ is closed, we deduce that $\Gamma \,\tilde{\subset}\, \overline{\Omega}_B \cup \partial_N\Omega$. By Lemma 5.2 there exist a function $w \in SBV^p(\Omega_0)$ and a sequence w_k converging to w weakly in $SBV^p(\Omega_0)$ such that $S(w) \cong \Gamma\backslash\partial_N\Omega \,\tilde{\subset}\, \overline{\Omega}_B$ and $S(w_k) \,\tilde{\subset}\, \Gamma_k \,\tilde{\subset}\, \overline{\Omega}_B$ for every k. Let us fix $v = (v^1,\dots,v^m) \in SBV^p(\Omega_0;\mathbb{R}^m)$. If we apply [14, Theorem 2.1] to each component v^i of v, with $\Omega = \Omega_B$, $\Omega' = \Omega_0$, $u_k = w_k$, and

$u = w$, we construct a sequence $v_k = (v_k^1, \dots, v_k^m) \in SBV^p(\Omega_0; \mathbb{R}^m)$ which satisfies (a), (b), (c), and

$$\mathcal{H}^{n-1}((S(v_k^i) \backslash S(w_k)) \backslash (S(v^i) \backslash S(w))) \to 0$$

for $i = 1, \dots, m$. By monotonicity this implies

$$\mathcal{H}^{n-1}((S(v_k^i) \backslash \Gamma_k) \backslash (S(v^i) \backslash (\Gamma \backslash \partial_N \Omega)) \to 0 \,.$$

As $(S(v_k^i) \backslash \Gamma_k^N) \backslash (S(v^i) \backslash \Gamma^N) \,\tilde{\subset}\, (S(v_k^i) \backslash \Gamma_k) \backslash (S(v^i) \backslash (\Gamma \backslash \partial_N \Omega))$, we obtain

$$\mathcal{H}^{n-1}((S(v_k^i) \backslash \Gamma_k^N) \backslash (S(v^i) \backslash \Gamma^N)) \to 0 \,. \tag{5.4}$$

Since $S(v_k) \cong \bigcup_i S(v_k^i)$ and $S(v) \cong \bigcup_i S(v^i)$, we have

$$(S(v_k) \backslash \Gamma_k^N) \backslash (S(v) \backslash \Gamma^N) \,\tilde{\subset}\, \bigcup_{i=1}^m ((S(v_k^i) \backslash \Gamma_k^N) \backslash (S(v^i) \backslash \Gamma^N)) \,,$$

so that (d) follows from (5.4).

If, in addition, $v \in L^\infty(\Omega_0; \mathbb{R}^m)$, we can replace v_k by $\varphi(v_k)$, where $\varphi \in C_0^1(\mathbb{R}^m; \mathbb{R}^m)$ satisfies $\varphi(z) = z$ for $|z| \le \|v\|_\infty$. The new sequence $\varphi(v_k)$ is bounded in $L^\infty(\Omega_0; \mathbb{R}^m)$ and continues to satisfy (a)–(d). □

The following theorem extends the result of Theorem 5.1 to the case of $GSBV$ functions.

Theorem 5.3 (Jump transfer in $GSBV$)**.** *Assume that $\Gamma_k \in \mathcal{R}(\overline{\Omega}_B)$ and that Γ_k^N σ^p-converges to Γ. Then for every function $v \in GSBV_q^p(\Omega_0; \mathbb{R}^m)$ there exists a sequence $v_k \in GSBV_q^p(\Omega_0; \mathbb{R}^m)$ such that*

- (a) *$v_k = v$ a.e. in $\Omega_0 \backslash \Omega_B$,*
- (b) *$v_k \to v$ strongly in $L^q(\Omega_0; \mathbb{R}^m)$,*
- (c) *$\nabla v_k \to \nabla v$ strongly in $L^p(\Omega_0; \mathbb{M}^{m \times n})$,*
- (d) *$\mathcal{H}^{n-1}((S(v_k) \backslash \Gamma_k^N) \backslash (S(v) \backslash \Gamma^N)) \to 0$.*

Proof. Since $\Omega_0 \backslash \overline{\Omega}_B$ has a Lipschitz boundary, by Proposition 2.4 for $\mathcal{H}^{n-1}$-a.e. $x \in \partial \Omega_B$ there exists $\tilde{v}_B(x) \in \mathbb{R}^m$ such that

$$\tilde{v}_B(x) := \operatorname*{ap\,lim}_{y \to x,\ y \notin \overline{\Omega}_B} v(y) \,. \tag{5.5}$$

For every integer $i \ge 1$ let

$$S_i := \{x \in \partial \Omega_B : |\tilde{v}_B(x)| \ge i\} \,. \tag{5.6}$$

Then

$$\mathcal{H}^{n-1}(S_i) \to 0 \,. \tag{5.7}$$

Let $\varphi \in C_c^1(\mathbb{R}^m; \mathbb{R}^m)$ be a function such that $\varphi(z) = z$ for $|z| \le 1$, and let $\varphi_i(z) := i\varphi(z/i)$. Then $\varphi_i \in C_c^1(\mathbb{R}^m; \mathbb{R}^m)$, $\varphi_i(z) = z$ for $|z| \le i$, and $|\nabla \varphi_i| \le C$ for some constant independent of i. Since $v \in GSBV^p(\Omega_0; \mathbb{R}^m)$, the functions $v^i := \varphi_i(v)$ belong to $SBV^p(\Omega_0; \mathbb{R}^m) \cap L^\infty(\Omega_0; \mathbb{R}^m)$.

By Theorem 5.1 for every i there exists a sequence $v_k^i \in SBV^p(\Omega_0; \mathbb{R}^m)$ such that $v_k^i = v^i$ a.e. in $\Omega_0 \backslash \Omega_B$, $v_k^i \to v^i$ strongly in $L^q(\Omega_0; \mathbb{R}^m)$, $\nabla v_k^i \to \nabla v^i$ strongly in $L^p(\Omega_0; \mathbb{M}^{m \times n})$, and $\mathcal{H}^{n-1}((S(v_k^i) \backslash \Gamma_k^N) \backslash (S(v^i) \backslash \Gamma^N)) \to 0$. Therefore there exists an increasing sequence of integers k_i such that

$$\|v_k^i - v^i\|_q < 1/i \tag{5.8}$$

$$\|\nabla v_k^i - \nabla v^i\|_p < 1/i \tag{5.9}$$

$$\mathcal{H}^{n-1}((S(v_k^i) \backslash \Gamma_k^N) \backslash (S(v^i) \backslash \Gamma^N)) < 1/i \tag{5.10}$$

for every $k \ge k_i$.

For $k_i \le k < k_{i+1}$ define $v_k := v_k^i$ a.e. on Ω_B, and $v_k := v$ a.e. on $\Omega \backslash \Omega_B$. Then $v_k \in GSBV_q^p(\Omega_0; \mathbb{R}^m)$ and condition (a) is satisfied.

As $|\varphi_i(z)| \leq C|z|$ for every i and z, the sequence v^i converges to v in $L^q(\Omega_0;\mathbb{R}^m)$. Since $\nabla v^i = \nabla\varphi_i(v)\nabla v$, the sequence ∇v^i converges to ∇v strongly in $L^p(\Omega_0;\mathbb{M}^{m\times n})$. Therefore (b) and (c) follow from (5.8) and (5.9).

For $k_i \leq k < k_{i+1}$ let $x \in \partial\Omega_B \backslash (S(v^i_k) \cup S_i)$ such that (5.5) holds. As $x \in \partial\Omega_B\backslash S_i$, it is easy to deduce from (5.5) that

$$\tilde{v}_B(x) = \operatorname*{ap\,lim}_{y\to x,\ y\notin\overline{\Omega}_B} v^i(y) = \operatorname*{ap\,lim}_{y\to x,\ y\notin\overline{\Omega}_B} v^i_k(y)\,. \tag{5.11}$$

Since $x \notin S(v^i_k)$, the approximate limit of v^i_k at x exists, so that we must have

$$\tilde{v}_B(x) := \operatorname*{ap\,lim}_{y\to x,\ y\in\overline{\Omega}_B} v^i_k(y)\,. \tag{5.12}$$

By the definition of v_k, from (5.5) and (5.12) we deduce that

$$\tilde{v}_B(x) := \operatorname*{ap\,lim}_{y\to x} v_k(y)\,,$$

hence $x \notin S(v_k)$.

This shows that $\partial\Omega_B\backslash(S(v^i_k)\cup S_i) \tilde{\subset} \partial\Omega_B\backslash S(v_k)$, which implies $S(v_k)\cap\partial\Omega_B \tilde{\subset} (S(v^i_k)\cap\partial\Omega_B)\cup S_i$. Since $S(v_k)\cap\Omega_B \cong S(v^i_k)\cap\Omega_B$, we conclude that $S(v_k)\cap\overline{\Omega}_B \tilde{\subset} (S(v^i_k)\cap\overline{\Omega}_B)\cup S_i$. Recalling that $S(v^i) \tilde{\subset} S(v)$, we obtain

$$[(S(v_k)\backslash\Gamma^N_k)\backslash(S(v)\backslash\Gamma^N)]\cap\overline{\Omega}_B \tilde{\subset} [((S(v^i_k)\backslash\Gamma^N_k)\backslash(S(v^i)\backslash\Gamma^N))\cup S_i]\cap\overline{\Omega}_B\,. \tag{5.13}$$

On the other hand, $S(v_k)\backslash\overline{\Omega}_B \tilde{\subset} S(v)\backslash\overline{\Omega}_B$ and $\Gamma^N_k\backslash\overline{\Omega}_B \cong \partial_N\Omega\backslash\overline{\Omega}_B \cong \Gamma^N\backslash\overline{\Omega}_B$ by Remark 4.4-(b), so that

$$[(S(v_k)\backslash\Gamma^N_k)\backslash(S(v)\backslash\Gamma^N)]\backslash\overline{\Omega}_B \cong \emptyset\,. \tag{5.14}$$

From (5.13) and (5.14) we deduce that

$$(S(v_k)\backslash\Gamma^N_k)\backslash(S(v)\backslash\Gamma^N) \tilde{\subset} ((S(v^i_k)\backslash\Gamma^N_k)\backslash(S(v^i)\backslash\Gamma^N))\cup S_i$$

for $k_i \leq k < k_{i+1}$. Therefore (d) follows from (5.7) and (5.10). ☐

Remark 5.4. Condition (d) of Theorem 5.3, together with (3.3) and (3.5), implies that

$$\lim_{k\to\infty} \mathcal{K}((S(v_k)\backslash\Gamma^N_k)\backslash(S(v)\backslash\Gamma^N)) = 0\,,$$

hence

$$\limsup_{k\to\infty} \mathcal{K}(S(v_k)\backslash\Gamma^N_k) \leq \mathcal{K}(S(v)\backslash\Gamma^N)\,.$$

5.2. **Convergence of minima.** We begin by proving the existence Theorems 3.7 and 3.8.

Proof of Theorem 3.7. Let us fix $t \in [0,T]$, let us extend $\psi(t)$ to a function $\psi_0(t) \in W^{1,p}(\Omega_0;\mathbb{R}^m)\cap L^q(\Omega_0;\mathbb{R}^m)$, and let u_k be a minimizing sequence of problem (3.56). We extend also u_k to Ω_0 by setting $u_k := \psi_0(t)$ a.e. on $\Omega_0\backslash\Omega$. The extended functions belong to $GSBV^p_q(\Omega_0;\mathbb{R}^m)$ and satisfy $S(u_k) \tilde{\subset} \Gamma(t)^N$ by Proposition 2.4 and by the definition of $S(u_k)$. Since $\mathcal{E}^{el}(t)(\psi(t)) < +\infty$ by (3.52), the infimum in (3.56) is finite. By (3.51) there exists a constant $C \geq 0$ such that

$$\|\nabla u_k\|^p_{p,\Omega_0} + \|u_k\|^q_{q,\Omega_0} \leq C \tag{5.15}$$

for every k. By Theorem 2.7 there exists a subsequence, still denoted u_k, which converges weakly in $GSBV^p(\Omega_0;\mathbb{R}^m)$ to a function u. Since $u = \psi_0(t)$ a.e. on $\Omega_0\backslash\Omega$ and $S(u) \tilde{\subset} \Gamma(t)^N$ by Remark 2.9, we conclude that $u \in AD(\psi(t),\Gamma(t))$.

By (2.8) in Theorem 2.8 we have

$$\mathcal{W}(\nabla u) \leq \liminf_{k\to\infty} \mathcal{W}(\nabla u_k)\,. \tag{5.16}$$

By Lemma 2.5 the functions u_k and u belong to $W^{1,p}(\Omega_S;\mathbb{R}^m)\cap L^q(\Omega_S;\mathbb{R}^m)$. By (3.54) and (5.15) the sequence u_k is bounded in $W^{1,p}(\Omega_S;\mathbb{R}^m)$, so it converges to u weakly in

$W^{1,p}(\Omega_S;\mathbb{R}^n)$. From the compactness of the trace operator we deduce that u_k converges to u strongly in $L^r(\partial_S\Omega;\mathbb{R}^m)$, and by the continuity of $\mathcal{G}(t)$ we get

$$\mathcal{G}(t)(u) = \lim_{k\to\infty} \mathcal{G}(t)(u_k)\,. \tag{5.17}$$

From (3.15), (5.16), and (5.17) we obtain

$$\mathcal{E}^{el}(t)(u) \le \liminf_{k\to\infty} \mathcal{E}^{el}(t)(u_k)\,.$$

Since $u \in AD(\psi(t),\Gamma(t))$ and u_k is a minimizing sequence, we conclude that u is a minimum point of (3.56). □

Proof of Theorem 3.8. Let us fix $t\in[0,T]$ and $\Gamma_0\in\mathcal{R}(\overline{\Omega}_B)$, and let (u_k,Γ_k) be a minimizing sequence of problem (3.60) with $\Gamma_0 \tilde{\subset} \Gamma_k \tilde{\subset} \overline{\Omega}_B$ for every k. We extend u_k to Ω_0 by setting $u_k := \psi_0(t)$ a.e. on $\Omega_0\backslash\Omega$, where $\psi_0(t)$ is the function introduced at the beginning of the proof of Theorem 3.7. Note that these extensions belong to $GSBV^p_q(\Omega_0;\mathbb{R}^m)$ and satisfy $S(u_k) \tilde{\subset} \Gamma_k^N$ by Proposition 2.4 and by the definition of $S(u_k)$. Since $\mathcal{E}(t)(\psi(t),\Gamma_0) < +\infty$ by (3.52), the infimum in (3.60) is finite. By (3.6) and (3.51) there exists a constant $C\ge 0$ such that

$$\|\nabla u_k\|^p_{p,\Omega_0} + \|u_k\|^q_{q,\Omega_0} + \mathcal{H}^{n-1}(\Gamma_k^N) \le C \tag{5.18}$$

for every k. By Theorem 2.7 there exists a subsequence, still denoted by u_k, which converges weakly in $GSBV^p(\Omega_0;\mathbb{R}^m)$ to a function u which satisfies $u=\psi_0(t)$ a.e. on $\Omega_0\backslash\Omega$.

By Theorem 4.7 there exists a subsequence, still denoted Γ_k, such that Γ_k^N σ^p-converges to a set Γ^*. Since $\Gamma_k^N \tilde{\subset} \overline{\Omega}_B\cup\partial_N\Omega$ and $\partial_N\Omega$ is closed, we deduce, thanks to Remark 4.4-(b), that $\Gamma^* \tilde{\subset} \overline{\Omega}_B\cup\partial_N\Omega$. By Proposition 4.6 we have $S(u)\tilde{\subset}\Gamma^*$. Since $u=\psi_0(t)$ a.e. on $\Omega_0\backslash\Omega$ we deduce that the traces of u and $\psi(t)$ coincide $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega\backslash\Gamma^*$. Let $\Gamma := \Gamma^*\backslash\partial_N\Omega$. Then $\Gamma\in\mathcal{R}(\overline{\Omega}_B)$ and $u\in AD(\psi(t),\Gamma)$, because $\partial_D\Omega\backslash\Gamma = \partial_D\Omega\backslash\Gamma^*$.

Arguing as in the proof of Theorem 3.7 we obtain (5.16) and (5.17). By (5.2) and (5.3) we have also

$$\mathcal{K}(\Gamma\cup\Gamma_0) = \mathcal{K}(\Gamma^*\cup\Gamma_0) \le \liminf_{k\to\infty} \mathcal{K}(\Gamma_k^N\cup\Gamma_0) = \liminf_{k\to\infty}\mathcal{K}(\Gamma_k)\,. \tag{5.19}$$

By (3.15), (5.16), (5.17), and (5.19) we have

$$\mathcal{E}(t)(u,\Gamma\cup\Gamma_0) \le \liminf_{k\to\infty}\mathcal{E}(t)(u_k,\Gamma_k)\,.$$

Since $\Gamma\cup\Gamma_0\in\mathcal{R}(\overline{\Omega}_B)$ and $u\in AD(\psi(t),\Gamma\cup\Gamma_0)$, our assumption on (u_k,Γ_k) implies that $(u,\Gamma\cup\Gamma_0)$ is a minimum point of (3.60). □

We now prove the stability of minimizers with respect to the σ^p-convergence.

Theorem 5.5. *Let $t_k\in[0,T]$ and $\Gamma_k\in\mathcal{R}(\overline{\Omega}_B)$. Assume that $t_k\to t_\infty$ and Γ_k^N σ^p-converges to Γ^*_∞, and define $\Gamma_\infty := \Gamma^*_\infty\backslash\partial_N\Omega$. For every k let $u_k\in AD(\psi(t_k),\Gamma_k)$ be a function such that*

$$\mathcal{E}(t_k)(u_k,\Gamma_k) \le \mathcal{E}(t_k)(v,\Gamma) \tag{5.20}$$

for every $\Gamma\in\mathcal{R}(\overline{\Omega}_B)$, with $\Gamma_k\tilde{\subset}\Gamma$, and every $v\in AD(\psi(t_k),\Gamma)$. Then $\Gamma_\infty\in\mathcal{R}(\overline{\Omega}_B)$ and there exist a subsequence of u_k, not relabelled, and a function $u_\infty\in AD(\psi(t_\infty),\Gamma_\infty)$ such that

$$u_k \rightharpoonup u_\infty \ \textit{weakly in}\ GSBV^p(\Omega;\mathbb{R}^m)\,, \tag{5.21}$$

$$u_k \rightharpoonup u_\infty \ \textit{weakly in}\ L^q(\Omega;\mathbb{R}^m)\,, \tag{5.22}$$

$$u_k \to u_\infty \ \textit{strongly in}\ L^{\dot q}(\Omega;\mathbb{R}^m)\,, \tag{5.23}$$

$$u_k \to u_\infty \ \textit{strongly in}\ L^r(\partial_S\Omega;\mathbb{R}^m)\,, \tag{5.24}$$

$$\nabla u_k \rightharpoonup \nabla u_\infty \ \textit{weakly in}\ L^p(\Omega;\mathbb{M}^{m\times n})\,. \tag{5.25}$$

Moreover

$$\mathcal{E}(t_\infty)(u_\infty, \Gamma_\infty) \le \mathcal{E}(t_\infty)(v, \Gamma) \tag{5.26}$$

for every $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, *with* $\Gamma_\infty \,\tilde{\subset}\, \Gamma$, *and every* $v \in AD(\psi(t_\infty), \Gamma)$. *Finally*

$$\mathcal{W}(\nabla u_k) \to \mathcal{W}(\nabla u_\infty)\,, \tag{5.27}$$

$$\mathcal{F}(t_k)(u_k) \to \mathcal{F}(t_\infty)(u_\infty)\,, \tag{5.28}$$

$$\mathcal{G}(t_k)(u_k) \to \mathcal{G}(t_\infty)(u_\infty)\,. \tag{5.29}$$

Proof. Taking $\Gamma := \Gamma_k$ and $v = \psi(t_k)$ in (5.20) we obtain

$$\mathcal{E}(t_k)(u_k, \Gamma_k) \le \mathcal{E}(t_k)(\psi(t_k), \Gamma_k)\,.$$

By (3.6) and (3.52) we have

$$\mathcal{E}(t_k)(\psi(t_k), \Gamma_k) \le \alpha_1^{\mathcal{E}}(\|\nabla\psi(t_k)\|_p^p + \|\psi(t_k)\|_q^q + \|\psi(t_k)\|_{r,\partial_S\Omega}^r) + \kappa_2\mathcal{H}^{n-1}(\Gamma_k) + \beta_1^{\mathcal{E}}\,,$$

so that $\mathcal{E}(t_k)(u_k, \Gamma_k)$ is bounded uniformly with respect to k (recall that $\mathcal{H}^{n-1}(\Gamma_k^N)$ is bounded by the definition of the σ^p-convergence). By (3.6) and (3.51) there exists a constant $C \ge 0$ such that

$$\|\nabla u_k\|_p^p + \|u_k\|_q^q + \mathcal{H}^{n-1}(\Gamma_k) \le C \tag{5.30}$$

for every k.

Using an extension operator it is possible to construct an absolutely continuous function $t \mapsto \psi_0(t)$ from $[0,T]$ into $W^{1,p}(\Omega_0;\mathbb{R}^m) \cap L^q(\Omega_0;\mathbb{R}^m)$ such that $\psi_0(t) = \psi(t)$ a.e. in Ω for every t. Then we extend u_k to Ω_0 by setting $u_k := \psi_0(t_k)$ a.e. on $\Omega_0\setminus\Omega$. The extended functions belong to $GSBV_q^p(\Omega_0;\mathbb{R}^m)$ and satisfy $S(u_k) \,\tilde{\subset}\, \Gamma_k^N$ by Proposition 2.4 and by the definition of $S(u_k)$. By (5.30) and by Theorem 2.7 there exists a subsequence, still denoted by u_k, which converges to a function u_∞ weakly in $GSBV^p(\Omega_0;\mathbb{R}^m)$ and weakly in $L^q(\Omega_0;\mathbb{R}^m)$. Since $\psi_0(t_k)$ converges to $\psi_0(t_\infty)$ in $W^{1,p}(\Omega_0;\mathbb{R}^m)$ we conclude that $u_\infty = \psi_0(t_\infty)$ a.e. on $\Omega_0\setminus\Omega$. By Proposition 4.6 we have $S(u_\infty) \,\tilde{\subset}\, \Gamma_\infty^*$, hence the traces of u_∞ and $\psi(t_\infty)$ agree $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega\setminus\Gamma_\infty^*$.

Since $\Gamma_k^N \,\tilde{\subset}\, \overline{\Omega}_B \cup \partial_N\Omega$ and $\partial_N\Omega$ is closed, we deduce that $\Gamma_\infty^* \,\tilde{\subset}\, \overline{\Omega}_B \cup \partial_N\Omega$ and by Remark 4.4-(a) we have $\mathcal{H}^{n-1}(\Gamma_\infty^*) < +\infty$, therefore $\Gamma_\infty \in \mathcal{R}(\overline{\Omega}_B)$ and $u_\infty \in AD(\psi(t_\infty), \Gamma_\infty)$ because $\partial_D\Omega\setminus\Gamma_\infty = \partial_D\Omega\setminus\Gamma_\infty^*$.

Properties (5.21) and (5.22) have already been proved, and (5.25) is a consequence of (5.21). Since $\dot{q} < q$, property (5.23) follows from the fact that u_k converges to u_∞ pointwise a.e. on Ω, and that u_k is bounded in $L^q(\Omega;\mathbb{R}^m)$ by (5.22).

By Lemma 2.5 the functions u_k and u_∞ belong to $W^{1,p}(\Omega_S;\mathbb{R}^m) \cap L^q(\Omega_S;\mathbb{R}^m)$. By (3.54) and (5.30) the sequence u_k is bounded in $W^{1,p}(\Omega_S;\mathbb{R}^m)$, so it converges to u_∞ weakly in $W^{1,p}(\Omega_S;\mathbb{R}^m)$. Property (5.24) follows from the fact that the trace operator is compact from $W^{1,p}(\Omega_S;\mathbb{R}^m)$ into $L^r(\partial_S\Omega;\mathbb{R}^m)$.

To prove (5.26), we fix $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$ with $\Gamma_\infty \,\tilde{\subset}\, \Gamma$. Given $v \in AD(\psi(t_\infty), \Gamma)$, we extend v by setting $v := \psi_0(t_\infty)$ a.e. on $\Omega_0\setminus\Omega$, and define $w := v - \psi_0(t_\infty)$ a.e. on Ω_0.

Since $(\Gamma_\infty^*)^N = \Gamma_\infty^N$, by the Jump Transfer Theorem 5.3 there exists a sequence $w_k \in GSBV_q^p(\Omega_0;\mathbb{R}^m)$ such that $w_k = w$ a.e. in $\Omega_0\setminus\Omega_B$, $w_k \to w$ strongly in $L^q(\Omega_0;\mathbb{R}^m)$, $\nabla w_k \to \nabla w$ strongly in $L^p(\Omega_0;\mathbb{M}^{m\times n})$, $\mathcal{H}^{n-1}((S(w_k)\setminus\Gamma_k^N)\setminus(S(w)\setminus\Gamma_\infty^N)) \to 0$, and $S(w_k) \,\tilde{\subset}\, \overline{\Omega}_B \cup \partial_N\Omega$.

Let $v_k := w_k + \psi_0(t_k)$. Then $v_k \in GSBV^p(\Omega_0;\mathbb{R}^m)$, $v_k = \psi_0(t_k)$ a.e. in $\Omega_0\setminus\Omega$, $v_k \to v$ strongly in $L^q(\Omega_0;\mathbb{R}^m)$, $\nabla v_k \to \nabla v$ strongly in $L^p(\Omega_0;\mathbb{M}^{m\times n})$, $\mathcal{H}^{n-1}((S(v_k)\setminus\Gamma_k^N)\setminus(S(v)\setminus\Gamma_\infty^N)) \to 0$, and $S(v_k) \,\tilde{\subset}\, \overline{\Omega}_B \cup \partial_N\Omega$. By Lemma 2.5 the functions v_k and v belong to $W^{1,p}(\Omega_S;\mathbb{R}^m)$ and by (3.54) v_k converges to v strongly in $W^{1,p}(\Omega_S;\mathbb{R}^m)$. We conclude by (3.31) that v_k converges to v strongly in $L^r(\partial_S\Omega;\mathbb{R}^m)$.

Let $\Gamma_k' := \Gamma_k \cup (S(v_k)\setminus\partial_N\Omega)$. As $S(v_k)\setminus\partial_N\Omega \,\tilde{\subset}\, \overline{\Omega}_B\setminus\partial_N\Omega$, we have $\Gamma_k' \in \mathcal{R}(\overline{\Omega}_B)$. Since $S(v_k)\cap\Omega \,\tilde{\subset}\, \Gamma_k'$, $S(v_k)\cap\partial_D\Omega \,\tilde{\subset}\, \Gamma_k'$, and $v_k = \psi_0(t_k)$ a.e. on $\Omega_0\setminus\Omega$, by Proposition 2.4 and by the definition of $S(v_k)$ we have $v_k \in AD(\psi(t_k), \Gamma'_k)$. By the minimality condition (5.20) we have

$$\begin{aligned} &\mathcal{W}(\nabla u_k) + \mathcal{K}(\Gamma_k) - \mathcal{F}(t_k)(u_k) - \mathcal{G}(t_k)(u_k) \leq \\ &\leq \mathcal{W}(\nabla v_k) + \mathcal{K}(\Gamma'_k) - \mathcal{F}(t_k)(v_k) - \mathcal{G}(t_k)(v_k)\,, \end{aligned}$$

which implies

$$\begin{aligned} &\mathcal{W}(\nabla u_k) - \mathcal{F}(t_k)(u_k) - \mathcal{G}(t_k)(u_k) \leq \\ &\leq \mathcal{W}(\nabla v_k) + \mathcal{K}(S(v_k) \setminus \Gamma^N_k) - \mathcal{F}(t_k)(v_k) - \mathcal{G}(t_k)(v_k)\,. \end{aligned} \tag{5.31}$$

By (2.8) in Theorem 2.8 we have

$$\mathcal{W}(\nabla u_\infty) \leq \liminf_{k\to\infty} \mathcal{W}(\nabla u_k)\,. \tag{5.32}$$

Since ∇v_k converges to ∇v strongly in $L^p(\Omega_0; \mathbb{M}^{m\times n})$, by the continuity of $\mathcal{W}$ we have

$$\mathcal{W}(\nabla v) = \lim_{k\to\infty} \mathcal{W}(\nabla v_k)\,. \tag{5.33}$$

By (3.16) and (3.20) we have

$$|\mathcal{F}(t_k)(u_k) - \mathcal{F}(t_\infty)(u_k)| \leq \int_{t_\infty}^{t_k} |\dot{\mathcal{F}}(s)(u_k)|\, ds \leq \int_{t_\infty}^{t_k} (\alpha_3^{\mathcal{F}}(s)\|u_k\|_{\dot q}^{\dot q} + \beta_3^{\mathcal{F}}(s))\, ds\,, \tag{5.34}$$

$$|\mathcal{F}(t_k)(v_k) - \mathcal{F}(t_\infty)(v_k)| \leq \int_{t_\infty}^{t_k} |\dot{\mathcal{F}}(s)(v_k)|\, ds \leq \int_{t_\infty}^{t_k} (\alpha_3^{\mathcal{F}}(s)\|v_k\|_{\dot q}^{\dot q} + \beta_3^{\mathcal{F}}(s))\, ds\,. \tag{5.35}$$

As u_k converges to u_∞ strongly in $L^{\dot q}(\Omega; \mathbb{R}^m)$ by (5.23), using (3.15) and (5.34) we obtain

$$\mathcal{F}(t_\infty)(u_\infty) \geq \limsup_{k\to\infty} \mathcal{F}(t_\infty)(u_k) = \limsup_{k\to\infty} \mathcal{F}(t_k)(u_k)\,. \tag{5.36}$$

As $\mathcal{F}(t_\infty)$ is continuous in $L^q(\Omega; \mathbb{R}^m)$ and v_k converges to v_∞ strongly in $L^q(\Omega; \mathbb{R}^m)$, using (5.35) we obtain

$$\mathcal{F}(t_\infty)(v) = \lim_{k\to\infty} \mathcal{F}(t_\infty)(v_k) = \lim_{k\to\infty} \mathcal{F}(t_k)(v_k)\,. \tag{5.37}$$

In the same way we prove that

$$\mathcal{G}(t_\infty)(u_\infty) \geq \limsup_{k\to\infty} \mathcal{G}(t_\infty)(u_k) = \limsup_{k\to\infty} \mathcal{G}(t_k)(u_k)\,, \tag{5.38}$$

$$\mathcal{G}(t_\infty)(v) = \lim_{k\to\infty} \mathcal{G}(t_\infty)(v_k) = \lim_{k\to\infty} \mathcal{G}(t_k)(v_k)\,. \tag{5.39}$$

From (5.32), (5.36), and (5.38) we obtain

$$\begin{aligned} &\mathcal{W}(\nabla u_\infty) - \mathcal{F}(t_\infty)(u_\infty) - \mathcal{G}(t_\infty)(u_\infty) \leq \\ &\leq \liminf_{k\to\infty} \big[\mathcal{W}(\nabla u_k) - \mathcal{F}(t_k)(u_k) - \mathcal{G}(t_k)(u_k)\big]\,. \end{aligned} \tag{5.40}$$

From Remark 5.4, and from (5.33), (5.37), and (5.39) we obtain

$$\begin{aligned} &\limsup_{k\to\infty} \big[\mathcal{W}(\nabla v_k) + \mathcal{K}(S(v_k) \setminus \Gamma^N_k) - \mathcal{F}(t_k)(v_k) - \mathcal{G}(t_k)(v_k)\big] \leq \\ &\leq \mathcal{W}(\nabla v) + \mathcal{K}(S(v) \setminus \Gamma^N_\infty) - \mathcal{F}(t_\infty)(v) - \mathcal{G}(t_\infty)(v)\,. \end{aligned} \tag{5.41}$$

By (5.31), (5.40), and (5.41) we have

$$\begin{aligned} &\mathcal{W}(\nabla u_\infty) - \mathcal{F}(t_\infty)(u_\infty) - \mathcal{G}(t_\infty)(u_\infty) \leq \\ &\leq \mathcal{W}(\nabla v) + \mathcal{K}(S(v) \setminus \Gamma^N_\infty) - \mathcal{F}(t_\infty)(v) - \mathcal{G}(t_\infty)(v)\,. \end{aligned} \tag{5.42}$$

As $S(v) \tilde{\subset} \Gamma^N$, we have $\mathcal{K}(S(v) \setminus \Gamma^N_\infty) \leq \mathcal{K}(\Gamma^N \setminus \Gamma^N_\infty) = \mathcal{K}(\Gamma \setminus \Gamma_\infty)$, so that inequality (5.42) implies (5.26).

Taking $v := u_\infty$, from (5.31) and (5.41) we obtain

$$\begin{aligned} &\limsup_{k\to\infty} \big[\mathcal{W}(\nabla u_k) - \mathcal{F}(t_k)(u_k) - \mathcal{G}(t_k)(u_k)\big] \leq \\ &\leq \mathcal{W}(\nabla u_\infty) - \mathcal{F}(t_\infty)(u_\infty) - \mathcal{G}(t_\infty)(u_\infty)\,. \end{aligned}$$

This inequality, together with (5.32), (5.36), and (5.38), gives (5.27), (5.28), and (5.29). □

5.3. **Convergence of Riemann sums.** To prove Theorem 3.13 we need to approximate the time integrals of $\dot{\mathcal{F}}(t)(u)$ and $\dot{\mathcal{G}}(t)(u)$ by Riemann sums, uniformly when u varies in a compact set. As usual, if f is a measurable function defined a.e. on $[0,T]$, the same symbol f denotes its extension by 0 to $[0,T]$. We begin by the following remark.

Remark 5.6. Let $\mathcal{V}$ be a countable dense subset of $W^{1,p}(\Omega;\mathbb{R}^m)\cap L^{\dot q}(\Omega;\mathbb{R}^m)$. By Remark 4.13 for every $t\in(0,T]$ there exists a sequence of subdivisions $(s_k^i)_{0\le i\le i_k}$, with

$$0=s_k^0<s_k^1<\cdots<s_k^{i_k-1}<s_k^{i_k}=t\,, \tag{5.43}$$

$$\lim_{k\to\infty}\max_{1\le i\le i_k}(s_k^i-s_k^{i-1})=0\,, \tag{5.44}$$

such that

$$\lim_{k\to\infty}\sum_{i=1}^{i_k}\Big|(s_k^i-s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u)-\int_{s_k^{i-1}}^{s_k^i}\dot{\mathcal{F}}(s)(u)\,ds\Big|=0\,, \tag{5.45}$$

$$\lim_{k\to\infty}\sum_{i=1}^{i_k}\Big|(s_k^i-s_k^{i-1})\dot{\mathcal{G}}(s_k^i)(u)-\int_{s_k^{i-1}}^{s_k^i}\dot{\mathcal{G}}(s)(u)\,ds\Big|=0 \tag{5.46}$$

for every $u\in\mathcal{V}$.

Lemma 5.7. *Let $t\in(0,T]$, let $\mathcal{V}$ be a countable dense subset of $W^{1,p}(\Omega;\mathbb{R}^m)\cap L^{\dot q}(\Omega;\mathbb{R}^m)$, and let $(s_k^i)_{0\le i\le i_k}$ be a sequence of subdivisions satisfying (5.43)–(5.46) for every $u\in\mathcal{V}$. Assume that*

$$\lim_{k\to\infty}\sum_{i=1}^{i_k}\Big|(s_k^i-s_k^{i-1})\varphi(s_k^i)-\int_{s_k^{i-1}}^{s_k^i}\varphi(s)\,ds\Big|=0 \tag{5.47}$$

whenever φ is any one of the four functions $\alpha_4^{\mathcal{F}}$, $\beta_4^{\mathcal{F}}$, $\alpha_4^{\mathcal{G}}$, and $\beta_4^{\mathcal{G}}$ which appear in (3.21) and (3.37). Then

$$\lim_{k\to\infty}\sum_{i=1}^{i_k}\sup_{u\in\mathcal{H}}\Big|(s_k^i-s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u)-\int_{s_k^{i-1}}^{s_k^i}\dot{\mathcal{F}}(s)(u)\,ds\Big|=0 \tag{5.48}$$

for every compact subset $\mathcal{H}$ of $L^{\dot q}(\Omega;\mathbb{R}^m)$, and

$$\lim_{k\to\infty}\sum_{i=1}^{i_k}\sup_{u\in\mathcal{H}}\Big|(s_k^i-s_k^{i-1})\dot{\mathcal{G}}(s_k^i)(u)-\int_{s_k^{i-1}}^{s_k^i}\dot{\mathcal{G}}(s)(u)\,ds\Big|=0 \tag{5.49}$$

for every compact subset $\mathcal{H}$ of $L^r(\partial_S\Omega;\mathbb{R}^m)$.

Proof. Let us fix a compact subset $\mathcal{H}$ of $L^{\dot q}(\Omega;\mathbb{R}^m)$ and let

$$M:=\max\{\|u\|_{\dot q}:u\in\mathcal{H}\}+1\,.$$

For every $\varepsilon\in(0,1)$ there exists a finite number $u_1,\ldots,u_h\in\mathcal{V}$, with $\|u_j\|_{\dot q}<M$, such that for every $u\in\mathcal{H}$ there exists j with $\|u-u_j\|_{\dot q}<\varepsilon$. Therefore for every $u\in\mathcal{H}$ and every $s\in[0,t]$ there exists $v_j(s)\in L^{\dot q}(\Omega;\mathbb{R}^m)$, with $\|v_j(s)\|_{\dot q}<M$, such that

$$|\dot{\mathcal{F}}(s)(u)-\dot{\mathcal{F}}(s)(u_j)|\le|\langle\partial\dot{\mathcal{F}}(s)(v_j(s)),u_j-u\rangle|\,.$$

By (3.21) this implies that

$$|\dot{\mathcal{F}}(s)(u)-\dot{\mathcal{F}}(s)(u_j)|\le\varepsilon\,(\alpha_4^{\mathcal{F}}(s)M^{\dot q-1}+\beta_4^{\mathcal{F}}(s)) \tag{5.50}$$

for every $s \in [0,t]$. Consequently

$$\begin{aligned}
&\Big|(s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u)\, ds\Big| \le \\
&\le \Big|(s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u_j) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u_j)\, ds\Big| + \\
&\quad + \varepsilon\,(s_k^i - s_k^{i-1})\,(\alpha_4^{\mathcal{F}}(s_k^i) M^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s_k^i)) + \\
&\quad + \varepsilon \int_{s_k^{i-1}}^{s_k^i} (\alpha_4^{\mathcal{F}}(s) M^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s))\, ds\,.
\end{aligned}$$

This yields

$$\begin{aligned}
&\sup_{u\in\mathcal{H}} \Big|(s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u)\, ds\Big| \le \\
&\le \sum_{j=1}^{h} \Big|(s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u_j) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u_j)\, ds\Big| + \\
&\quad + \varepsilon\,(s_k^i - s_k^{i-1})\,(\alpha_4^{\mathcal{F}}(s_k^i) M^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s_k^i)) + \\
&\quad + \varepsilon \int_{s_k^{i-1}}^{s_k^i} (\alpha_4^{\mathcal{F}}(s) M^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s))\, ds\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{i=1}^{i_k} \sup_{u\in\mathcal{H}} \Big|(s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u)\, ds\Big| \le \\
&\le \sum_{j=1}^{h} \sum_{i=1}^{i_k} \Big|(s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u_j) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u_j)\, ds\Big| + \\
&\quad + \varepsilon \sum_{i=1}^{i_k} (s_k^i - s_k^{i-1})\,(\alpha_4^{\mathcal{F}}(s_k^i) M^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s_k^i)) + \\
&\quad + \varepsilon \int_0^t (\alpha_4^{\mathcal{F}}(s) M^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s))\, ds\,.
\end{aligned}$$

By (5.45) and (5.47) we have

$$\begin{aligned}
&\limsup_{k\to\infty} \sum_{i=1}^{i_k} \sup_{u\in\mathcal{H}} \Big|(s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u)\, ds\Big| \le \\
&\le 2\varepsilon \int_0^t (\alpha_4^{\mathcal{F}}(s) M^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s))\, ds\,,
\end{aligned}$$

and taking the limit as $\varepsilon \to 0$ we obtain (5.48).

The proof of (5.49) is similar. □

6. THE DISCRETE-TIME PROBLEMS

Theorem 3.13 will be proved by time discretization. In this section we study the discrete-time problems and prove a fundamental energy estimate, which will be crucial in the proof of the nondissipativity condition for the solution of the continuous-time quasistatic evolution (condition (c) in Subsection 3.9).

Let us fix a sequence of subdivisions $(t_k^i)_{0\le i\le k}$ of the interval $[0,T]$, with

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T\,, \tag{6.1}$$

$$\lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0\,. \tag{6.2}$$

For $i = 1,\dots,k$ we set

$$\mathcal{F}_k^i := \mathcal{F}(t_k^i)\,, \qquad \mathcal{G}_k^i := \mathcal{G}(t_k^i)\,, \qquad \psi_k^i := \psi(t_k^i)\,, \qquad \mathcal{E}_k^i := \mathcal{E}(t_k^i)\,. \tag{6.3}$$

Let (u_0, Γ_0) be an initial configuration that satisfies the minimality property (3.65). For every k we define u_k^i and Γ_k^i by induction. We set $(u_k^0, \Gamma_k^0) := (u_0, \Gamma_0)$, and for $i = 1,\dots,k$ we define (u_k^i, Γ_k^i) as a solution of the problem

$$\min \{\mathcal{E}_k^i(u, \Gamma) : \Gamma \in \mathcal{R}(\overline{\Omega}_B),\ \Gamma_k^{i-1} \mathrel{\tilde{\subset}} \Gamma,\ u \in AD(\psi_k^i, \Gamma)\}\,. \tag{6.4}$$

The existence of a solution to this problem is proved in Theorem 3.8. Note that

$$\mathcal{E}_k^i(u_k^i, \Gamma_k^i) \le \mathcal{E}_k^i(u, \Gamma) \tag{6.5}$$

for every $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, with $\Gamma_k^i \mathrel{\tilde{\subset}} \Gamma$ and every $u \in AD(\psi_k^i, \Gamma)$.

Since $\psi_k^i \in AD(\psi_k^i, \Gamma_k^{i-1})$, by (6.4) we have $\mathcal{E}^{el}(t_k^i)(u_k^i) + \mathcal{K}(\Gamma_k^i) \le \mathcal{E}^{el}(t_k^i)(\psi_k^i) + \mathcal{K}(\Gamma_k^{i-1})$, which gives $\mathcal{E}^{el}(t_k^i)(u_k^i) \le \mathcal{E}^{el}(t_k^i)(\psi_k^i)$. By (3.52) this implies

$$\mathcal{E}^{el}(t_k^i)(u_k^i) \le \alpha_1^{\mathcal{E}}(\|\nabla\psi_k^i\|_p^p + \|\psi_k^i\|_q^q + \|\psi_k^i\|_{r,\partial_S\Omega}^r) + \beta_1^{\mathcal{E}}\,,$$

so that by (3.51) there exists a constant $C > 0$ such that

$$\|\nabla u_k^i\|_p \le C\,, \quad \|u_k^i\|_{\dot{q}} \le C\,, \quad \|u_k^i\|_q \le C \tag{6.6}$$

for every k and i. By Lemma 2.5 the functions u_k^i belong to $W^{1,p}(\Omega_S;\mathbb{R}^m) \cap L^q(\Omega_S;\mathbb{R}^m)$. Therefore (3.55) implies that, if we change the constant, we may assume also that

$$\|u_k^i\|_{r,\partial_S\Omega} \le C \tag{6.7}$$

for every k and i.

For every $t \in [0,T]$ we define

$$\begin{gathered} \tau_k(t) := t_k^i\,, \qquad u_k(t) := u_k^i\,, \qquad \Gamma_k(t) := \Gamma_k^i\,, \\ \mathcal{F}_k(t) := \mathcal{F}_k^i := \mathcal{F}(t_k^i)\,, \quad \mathcal{G}_k(t) := \mathcal{G}_k^i := \mathcal{G}(t_k^i)\,, \quad \mathcal{E}_k(t) := \mathcal{E}_k^i := \mathcal{E}(t_k^i)\,, \end{gathered} \tag{6.8}$$

where i is the largest integer such that $t_k^i \le t$. Note that $u_k(t) = u_k(\tau_k(t))$ and $\Gamma_k(t) = \Gamma_k(\tau_k(t))$. By (6.6) and (6.7) we have

$$\|\nabla u_k(t)\|_p \le C\,, \quad \|u_k(t)\|_{\dot{q}} \le C\,, \quad \|u_k(t)\|_q \le C\,, \quad \|u_k(t)\|_{r,\partial_S\Omega} \le C\,, \tag{6.9}$$

for every k and every $t \in [0,T]$.

We introduce now a sequence of functions which play an important role in our estimates. For a.e. $t \in [0,T]$ we set

$$\begin{aligned} \theta_k(t) :=\ & \langle \partial\mathcal{W}(\nabla u_k(t)), \nabla\dot\psi(t)\rangle - \langle \partial\mathcal{F}_k(t)(u_k(t)), \dot\psi(t)\rangle - \\ & - \dot{\mathcal{F}}(t)(u_k(t)) - \langle \partial\mathcal{G}_k(t)(u_k(t)), \dot\psi(t)\rangle - \dot{\mathcal{G}}(t)(u_k(t))\,. \end{aligned} \tag{6.10}$$

The main result of this section is the energy estimate given by the following lemma.

Lemma 6.1. *There exists a sequence $R_k \to 0$ such that*

$$\mathcal{E}_k(t)(u_k(t), \Gamma_k(t)) \le \mathcal{E}(0)(u_0, \Gamma_0) + \int_0^{\tau_k(t)} \theta_k(s)\,ds + R_k \tag{6.11}$$

for every k and every $t \in [0,T]$.

Proof. We have to prove that there exists a sequence $R_k \to 0$ such that

$$\mathcal{E}_k^i(u_k^i, \Gamma_k^i) \le \mathcal{E}(0)(u_0, \Gamma_0) + \int_0^{t_k^i} \theta_k(s)\, ds + R_k \tag{6.12}$$

for every k and for every $i = 1, \dots, k$.

Let us fix j and k with $1 \le j \le k$. Since $u_k^{j-1} + \psi_k^j - \psi_k^{j-1} \in AD(\psi_k^j, \Gamma_k^{j-1})$, by the minimality condition (6.4) we have

$$\mathcal{E}_k^j(u_k^j, \Gamma_k^j) \le \mathcal{E}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1}, \Gamma_k^{j-1}) \tag{6.13}$$

We now estimate $\mathcal{E}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1}, \Gamma_k^{j-1})$ in terms of $\mathcal{E}_k^{j-1}(u_k^{j-1}, \Gamma_k^{j-1})$.

Let us consider first $\mathcal{W}(u_k^{j-1} + \psi_k^j - \psi_k^{j-1})$. There exists a constant $\rho_k^j \in [0,1]$ such that

$$\begin{aligned} &\mathcal{W}(\nabla u_k^{j-1} + \nabla\psi_k^j - \nabla\psi_k^{j-1}) - \mathcal{W}(\nabla u_k^{j-1}) = \\ &= \langle \partial\mathcal{W}(\nabla u_k^{j-1} + \rho_k^j(\nabla\psi_k^j - \nabla\psi_k^{j-1})), \nabla\psi_k^j - \nabla\psi_k^{j-1}\rangle\,, \end{aligned} \tag{6.14}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^{p'}(\Omega; \mathbb{M}^{m\times n})$ and $L^p(\Omega; \mathbb{M}^{m\times n})$. Let us consider the piecewise constant function $\Psi_k \in L^\infty([0,T]; L^p(\Omega; \mathbb{M}^{m\times n}))$ defined by

$$\Psi_k(s) := \rho_k^j(\nabla\psi_k^j - \nabla\psi_k^{j-1}) = \rho_k^j \int_{t_k^{j-1}}^{t_k^j} \nabla\dot\psi(\tau)\, d\tau \qquad \text{for } t_k^{j-1} \le s < t_k^j\,. \tag{6.15}$$

Since $s \mapsto \nabla\dot\psi(s)$ belongs to $L^1([0,T]; L^p(\Omega; \mathbb{M}^{m\times n}))$, by the absolute continuity of the integral we have that

$$\|\Psi_k(s)\|_p \to 0 \text{ uniformly with respect to } s \in [0,T]\,. \tag{6.16}$$

Therefore by (6.9) we may assume that $\|\nabla u_k(s) + \Psi_k(s)\|_p \le C + 1$ for every $s \in [0,T]$. From (6.14) and (6.15) we obtain

$$\begin{aligned} &\mathcal{W}(\nabla u_k^{j-1} + \nabla\psi_k^j - \nabla\psi_k^{j-1}) - \mathcal{W}(\nabla u_k^{j-1}) = \\ &= \int_{t_k^{j-1}}^{t_k^j} \langle \partial\mathcal{W}(\nabla u_k(s) + \Psi_k(s)), \nabla\dot\psi(s)\rangle\, ds\,. \end{aligned} \tag{6.17}$$

Let us consider now $\mathcal{F}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1})$. There exists a constant $\sigma_k^j \in [0,1]$ such that

$$\mathcal{F}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1}) - \mathcal{F}_k^j(u_k^{j-1}) = \langle \partial\mathcal{F}_k^j(u_k^{j-1} + \sigma_k^j(\psi_k^j - \psi_k^{j-1})), \psi_k^j - \psi_k^{j-1}\rangle\,, \tag{6.18}$$

where $\langle\cdot,\cdot\rangle$ denotes now the duality pairing between $L^{q'}(\Omega; \mathbb{R}^m)$ and $L^q(\Omega; \mathbb{R}^m)$. Let us consider the piecewise constant function $\varphi_k \in L^\infty([0,T]; L^q(\Omega; \mathbb{R}^m))$ defined by

$$\varphi_k(s) := \sigma_k^j(\psi_k^j - \psi_k^{j-1}) = \sigma_k^j \int_{t_k^{j-1}}^{t_k^j} \dot\psi(\tau)\, d\tau \qquad \text{for } t_k^{j-1} \le s < t_k^j\,. \tag{6.19}$$

Since $s \mapsto \dot\psi(s)$ belongs to $L^1([0,T]; L^q(\Omega; \mathbb{R}^m))$, by the absolute continuity of the integral we have that

$$\|\varphi_k(s)\|_q \to 0 \text{ uniformly with respect to } s \in [0,T]\,. \tag{6.20}$$

Therefore by (6.9) we may assume that $\|u_k(s) + \varphi_k(s)\|_q \le C + 1$ for every $s \in [0,T]$.

From (6.18) and (6.19) we obtain

$$\begin{aligned} &\mathcal{F}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1}) - \mathcal{F}_k^j(u_k^{j-1}) = \\ &= \int_{t_k^{j-1}}^{t_k^j} \langle \partial\mathcal{F}_k^j(u_k(s) + \varphi_k(s)), \dot\psi(s)\rangle\, ds\,. \end{aligned} \tag{6.21}$$

By (3.17) for every $s \in [t_k^{j-1}, t_k^j)$ we have

$$\langle \partial \mathcal{F}_k^j(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle - \langle \partial \mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle = \\ = \int_s^{t_k^j} \langle \partial \dot{\mathcal{F}}(\sigma)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle \, d\sigma \,,$$

$$\langle \partial \mathcal{F}(s)(u_k(s)), \dot\psi(s)\rangle - \langle \partial \mathcal{F}_k^{j-1}(u_k(s)), \dot\psi(s)\rangle = \int_{t_k^{j-1}}^{s} \langle \partial \dot{\mathcal{F}}(\sigma)(u_k(s)), \dot\psi(s)\rangle \, d\sigma \,,$$

By (3.21) this implies that for every $s \in [t_k^{j-1}, t_k^j)$

$$|\langle \partial \mathcal{F}_k^j(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle - \langle \partial \mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle| \le \gamma_k^{\mathcal{F}} \|\dot\psi(s)\|_{\dot q} \,, \tag{6.22}$$

$$|\langle \partial \mathcal{F}(s)(u_k(s)), \dot\psi(s)\rangle - \langle \partial \mathcal{F}_k(s)(u_k(s)), \dot\psi(s)\rangle| \le \gamma_k^{\mathcal{F}} \|\dot\psi(s)\|_{\dot q} \,, \tag{6.23}$$

where

$$\gamma_k^{\mathcal{F}} := \max_{1\le i\le k} \int_{t_k^{i-1}}^{t_k^i} (\alpha_4^{\mathcal{F}}(s)(C+1)^{\dot q-1} + \beta_4^{\mathcal{F}}(s))\, ds \,.$$

By the absolute continuity of the integral we have

$$\gamma_k^{\mathcal{F}} \to 0 \,, \tag{6.24}$$

and by (6.21) and (6.22) we have

$$\mathcal{F}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1}) - \mathcal{F}_k^j(u_k^{j-1}) \ge \\ \ge \int_{t_k^{j-1}}^{t_k^j} \langle \partial \mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle \, ds - \gamma_k^{\mathcal{F}} \int_{t_k^{j-1}}^{t_k^j} \|\dot\psi(s)\|_{\dot q} \, ds \,. \tag{6.25}$$

Since by (3.16)

$$\mathcal{F}_k^j(u_k^{j-1}) - \mathcal{F}_k^{j-1}(u_k^{j-1}) = \int_{t_k^{j-1}}^{t_k^j} \dot{\mathcal{F}}(s)(u_k^{j-1})\, ds = \int_{t_k^{j-1}}^{t_k^j} \dot{\mathcal{F}}(s)(u_k(s))\, ds \,, \tag{6.26}$$

from (6.25) and (6.26) we obtain

$$\mathcal{F}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1}) - \mathcal{F}_k^{j-1}(u_k^{j-1}) \ge \\ \ge \int_{t_k^{j-1}}^{t_k^j} \langle \partial \mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle \, ds + \\ + \int_{t_k^{j-1}}^{t_k^j} \dot{\mathcal{F}}(s)(u_k(s))\, ds - \gamma_k^{\mathcal{F}} \int_{t_k^{j-1}}^{t_k^j} \|\dot\psi(s)\|_{\dot q} \, ds \,. \tag{6.27}$$

Finally, let us consider $\mathcal{G}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1})$. The same arguments used for $\mathcal{F}_k^j$ show that there exist a sequence $\gamma_k^{\mathcal{G}}$ of real numbers, with

$$\gamma_k^{\mathcal{G}} \to 0 \,, \tag{6.28}$$

and a sequence $\zeta_k \in L^\infty([0,T]; L^r(\partial_S\Omega; \mathbb{R}^m))$, with

$$\|\zeta_k(s)\|_{r,\partial_S\Omega} \to 0 \ \text{ uniformly with respect to } \ s \in [0,T] \,, \tag{6.29}$$

such that

$$|\langle \partial \mathcal{G}(s)(u_k(s)), \dot\psi(s)\rangle - \langle \partial \mathcal{G}_k(s)(u_k(s)), \dot\psi(s)\rangle| \le \gamma_k^{\mathcal{G}} \|\dot\psi(s)\|_{r,\partial_S\Omega} \tag{6.30}$$

and

$$\mathcal{G}_k^j(u_k^{j-1}+\psi_k^j-\psi_k^{j-1})-\mathcal{G}_k^{j-1}(u_k^{j-1})\geq$$
$$\geq\int_{t_k^{j-1}}^{t_k^j}\langle\partial\mathcal{G}(s)(u_k(s)+\zeta_k(s)),\dot\psi(s)\rangle\,ds+ \tag{6.31}$$
$$+\int_{t_k^{j-1}}^{t_k^j}\dot{\mathcal{G}}(s)(u_k(s))\,ds-\gamma_k^{\mathcal{G}}\int_{t_k^{j-1}}^{t_k^j}\|\dot\psi(s)\|_{r,\partial_S\Omega}\,ds\,.$$

By (6.13), (6.17), (6.27), and (6.31) we have

$$\mathcal{E}_k^j(u_k^j,\Gamma_k^j)-\mathcal{E}_k^{j-1}(u_k^{j-1},\Gamma_k^{j-1})\leq$$
$$\leq\int_{t_k^{j-1}}^{t_k^j}\langle\partial\mathcal{W}(\nabla u_k(s)+\Psi_k(s)),\nabla\dot\psi(s)\rangle\,ds-$$
$$-\int_{t_k^{j-1}}^{t_k^j}\langle\partial\mathcal{F}(s)(u_k(s)+\varphi_k(s)),\dot\psi(s)\rangle\,ds-\int_{t_k^{j-1}}^{t_k^j}\dot{\mathcal{F}}(s)(u_k(s))\,ds- \tag{6.32}$$
$$-\int_{t_k^{j-1}}^{t_k^j}\langle\partial\mathcal{G}(s)(u_k(s)+\zeta_k(s)),\dot\psi(s)\rangle\,ds-\int_{t_k^{j-1}}^{t_k^j}\dot{\mathcal{G}}(s)(u_k(s))\,ds+$$
$$+\gamma_k^{\mathcal{F}}\int_{t_k^{j-1}}^{t_k^j}\|\dot\psi(s)\|_{\dot q}\,ds+\gamma_k^{\mathcal{G}}\int_{t_k^{j-1}}^{t_k^j}\|\dot\psi(s)\|_{r,\partial_S\Omega}\,ds\,.$$

Let us fix now i with $1\leq i\leq k$. By summing for $j=1,\dots,i$ we obtain

$$\mathcal{E}_k^i(u_k^i,\Gamma_k^i)-\mathcal{E}(0)(u_0,\Gamma_0)\leq$$
$$\leq\int_0^{t_k^i}\langle\partial\mathcal{W}(\nabla u_k(s)+\Psi_k(s)),\nabla\dot\psi(s)\rangle\,ds-$$
$$-\int_0^{t_k^i}\langle\partial\mathcal{F}(s)(u_k(s)+\varphi_k(s)),\dot\psi(s)\rangle\,ds-\int_0^{t_k^i}\dot{\mathcal{F}}(s)(u_k(s))\,ds- \tag{6.33}$$
$$-\int_0^{t_k^i}\langle\partial\mathcal{G}(s)(u_k(s)+\zeta_k(s)),\dot\psi(s)\rangle\,ds-\int_0^{t_k^i}\dot{\mathcal{G}}(s)(u_k(s))\,ds+$$
$$+\gamma_k^{\mathcal{F}}\int_0^{t_k^i}\|\dot\psi(s)\|_{\dot q}\,ds+\gamma_k^{\mathcal{G}}\int_0^{t_k^i}\|\dot\psi(s)\|_{r,\partial_S\Omega}\,ds\,.$$

Using Lemma 4.9, from (3.8), (3.10), (6.9), and (6.16) for a.e. $s\in[0,T]$ we obtain

$$\langle\partial\mathcal{W}(\nabla u_k(s)+\Psi_k(s)),\nabla\dot\psi(s)\rangle-\langle\partial\mathcal{W}(\nabla u_k(s)),\nabla\dot\psi(s)\rangle\to 0\,,$$

and by (3.12) and (6.9) we have

$$|\langle\partial\mathcal{W}(\nabla u_k(s)+\Psi_k(s)),\nabla\dot\psi(s)\rangle-\langle\partial\mathcal{W}(\nabla u_k(s)),\nabla\dot\psi(s)\rangle|\leq$$
$$\leq 2(\alpha_2^{\mathcal{W}}(C+1)^{p-1}+\beta_2^{\mathcal{W}})\,\|\nabla\dot\psi(s)\|_p\,.$$

As $s\mapsto\nabla\dot\psi(s)$ belongs to $L^1([0,T];L^p(\Omega;\mathbb{M}^{m\times n}))$, we deduce that

$$\int_0^T|\langle\partial\mathcal{W}(\nabla u_k(s)+\Psi_k(s)),\nabla\dot\psi(s)\rangle-\langle\partial\mathcal{W}(\nabla u_k(s)),\nabla\dot\psi(s)\rangle|\,ds\to 0\,. \tag{6.34}$$

By (3.14) and Remark 4.10 we can apply Lemma 4.9. Therefore from (6.9) and (6.20) for a.e. $s \in [0,T]$ we obtain

$$\langle \partial\mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{F}(s)(u_k(s)), \dot\psi(s)\rangle \to 0\,,$$

and by (3.19) and (6.9) we have

$$|\langle \partial\mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{F}(s)(u_k(s)), \dot\psi(s)\rangle| \leq \\ \leq 2(\alpha_2^{\mathcal{F}}(C+1)^{q-1}+\beta_2^{\mathcal{F}})\,\|\dot\psi(s)\|_q\,.$$

As $s\mapsto\dot\psi(s)$ belongs to $L^1([0,T];L^q(\Omega;\mathbb{R}^m))$, we deduce that

$$\int_0^T |\langle \partial\mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{F}(s)(u_k(s)), \dot\psi(s)\rangle|\,ds \to 0\,. \tag{6.35}$$

Finally, by (3.30) and Remark 4.10 we can apply Lemma 4.9 again. Therefore from (6.9) and (6.29) for a.e. $s \in [0,T]$ we obtain

$$\langle \partial\mathcal{G}(s)(u_k(s)+\zeta_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{G}(s)(u_k(s)), \dot\psi(s)\rangle \to 0\,,$$

and by (3.35) and (6.9) we have

$$|\langle \partial\mathcal{G}(s)(u_k(s)+\zeta_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{G}(s)(u_k(s)), \dot\psi(s)\rangle| \leq \\ \leq 2(\alpha_2^{\mathcal{G}}(C+1)^{r-1}+\beta_2^{\mathcal{G}})\,\|\dot\psi(s)\|_{r,\partial_S\Omega}\,.$$

As $s\mapsto\dot\psi(s)$ belongs to $L^1([0,T];L^r(\partial_S\Omega;\mathbb{R}^m))$, we deduce that

$$\int_0^T |\langle \partial\mathcal{G}(s)(u_k(s)+\zeta_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{G}(s)(u_k(s)), \dot\psi(s)\rangle|\,ds \to 0\,. \tag{6.36}$$

Let

$$\begin{aligned} R_k := &\int_0^T |\langle \partial\mathcal{W}(\nabla u_k(s)+\Psi_k(s)), \nabla\dot\psi(s)\rangle - \langle \partial\mathcal{W}(\nabla u_k(s)), \nabla\dot\psi(s)\rangle|\,ds + \\ &+\int_0^T |\langle \partial\mathcal{F}(s)(u_k(s)+\varphi_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{F}(s)(u_k(s)), \dot\psi(s)\rangle|\,ds + \\ &+\int_0^T |\langle \partial\mathcal{F}(s)(u_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{F}_k(s)(u_k(s)), \dot\psi(s)\rangle|\,ds + \\ &+\int_0^T |\langle \partial\mathcal{G}(s)(u_k(s)+\zeta_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{G}(s)(u_k(s)), \dot\psi(s)\rangle|\,ds + \\ &+\int_0^T |\langle \partial\mathcal{G}(s)(u_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{G}_k(s)(u_k(s)), \dot\psi(s)\rangle|\,ds + \\ &+\gamma_k^{\mathcal{F}}\int_0^T \|\dot\psi(s)\|_{\dot q}\,ds + \gamma_k^{\mathcal{G}}\int_0^T \|\dot\psi(s)\|_{r,\partial_S\Omega}\,ds\,. \end{aligned} \tag{6.37}$$

Then $R_k \to 0$ by (6.23), (6.24), (6.28), (6.30), (6.34), (6.35), and (6.36). Inequality (6.12) follows from (6.33) and (6.37). ☐

Remark 6.2. By (3.12), (3.19), (3.20), (3.35), (3.36), and (6.9), the integral term in (6.12) is bounded uniformly with respect to k and i, therefore the same property holds for $\mathcal{E}_k^i(u_k^i,\Gamma_k^i)$. By (3.6), (3.46), and (3.51), this implies that there exists a constant $M>0$ such that

$$\mathcal{H}^{n-1}(\Gamma_k(t)) \leq M \tag{6.38}$$

for every $t\in[0,T]$ and every k, where $\Gamma_k(t)$ is defined in (6.8).

Remark 6.3. We notice that the integral which appears in (6.12) can be written as a sum which involves $\mathcal{F}(t)$, $\mathcal{G}(t)$, $\psi(t)$ only at the discrete times $t = t_k^j$. Indeed we have

$$\int_0^{t_k^i} \theta_k(s)\, ds = \sum_{j=1}^{i} \langle \partial\mathcal{W}(\nabla u_k^{j-1}), \nabla\psi_k^j - \nabla\psi_k^{j-1}\rangle -$$
$$- \sum_{j=1}^{i} \langle \partial\mathcal{F}_k^{j-1}(u_k^{j-1}), \psi_k^j - \psi_k^{j-1}\rangle - \sum_{j=1}^{i} [\mathcal{F}_k^j(u_k^{j-1}) - \mathcal{F}_k^{j-1}(u_k^{j-1})] -$$
$$- \sum_{j=1}^{i} \langle \partial\mathcal{G}_k^{j-1}(u_k^{j-1}), \psi_k^j - \psi_k^{j-1}\rangle - \sum_{j=1}^{i} [\mathcal{G}_k^j(u_k^{j-1}) - \mathcal{G}_k^{j-1}(u_k^{j-1})]$$

for every k and for every $i = 1, \dots, k$.

7. Proof of the main result

We are now in a position to prove the main result of the paper.

Proof of Theorem 3.13. Let us fix a sequence of subdivisions $(t_k^i)_{0\le i\le k}$ of the interval $[0,T]$ satisfying (6.1) and (6.2), and let (u_0, Γ_0) be an initial configuration satisfying the minimality property (3.65). For every k let (u_k^i, Γ_k^i), $i = 1, \dots, k$ be defined inductively as solutions of the discrete problems (6.4), with $(u_k^0, \Gamma_k^0) := (u_0, \Gamma_0)$, and let $\tau_k(t)$, $u_k(t)$, $\Gamma_k(t)$, $\mathcal{F}_k(t)$, $\mathcal{G}_k(t)$, $\mathcal{E}_k(t)$ be defined by (6.8). By (6.38) and Theorem 4.8 there exist a subsequence, still denoted Γ_k, and an increasing set function $t \mapsto \Gamma^*(t)$, such that

$$\Gamma_k^N(t) \ \ \sigma^p\text{-converges to } \Gamma^*(t) \tag{7.1}$$

for every $t \in [0,T]$, where $\Gamma_k^N(t) := \Gamma_k(t) \cup \partial_N\Omega$, according to (5.1). Since $\Gamma_k^N(t) \mathrel{\tilde\subset} \overline{\Omega}_B \cup \partial_N\Omega$ and $\partial_N\Omega$ is closed, we deduce, thanks to Remark 4.4-(b), that $\Gamma^*(t) \mathrel{\tilde\subset} \overline{\Omega}_B \cup \partial_N\Omega$ for every $t \in [0,T]$. Let $\Gamma\colon [0,T] \to \mathcal{R}(\overline{\Omega}_B)$ be the increasing set function defined by

$$\Gamma(t) := \Gamma^*(t) \backslash \partial_N\Omega \tag{7.2}$$

By (5.2) and (5.3) for every $t \in [0,T]$ we have

$$\mathcal{K}(\Gamma(t)) = \mathcal{K}(\Gamma^*(t)) \le \liminf_{k\to\infty} \mathcal{K}(\Gamma_k^N(t)) = \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t))\,. \tag{7.3}$$

For a.e. $t \in [0,T]$ we set

$$\theta_\infty(t) := \limsup_{k\to\infty}\, \theta_k(t)\,, \tag{7.4}$$

where θ_k is defined by (6.10). By (3.12), (3.19), (3.20), (3.35), (3.36), and (6.9) we have

$$\begin{aligned} |\theta_k(t)| \le (\alpha_2^{\mathcal{W}} C^{p-1} + \beta_2^{\mathcal{W}})\|\nabla\dot\psi(t)\|_p + (\alpha_2^{\mathcal{F}} C^{q-1} + \beta_2^{\mathcal{F}})\, \|\dot\psi(t)\|_q + \\ + \alpha_3^{\mathcal{F}}(t) C^{\dot q} + \beta_3^{\mathcal{F}}(t) + (\alpha_2^{\mathcal{G}} C^{r-1} + \beta_2^{\mathcal{G}})\, \|\dot\psi(t)\|_{r,\partial_S\Omega} + \alpha_3^{\mathcal{G}}(t) C^r + \beta_3^{\mathcal{G}}(t)\,. \end{aligned} \tag{7.5}$$

As the right hand side of this inequality belongs to $L^1([0,T])$, the function θ_∞ belongs to $L^1([0,T])$ and, by the Fatou lemma,

$$\limsup_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\, ds \le \int_0^t \theta_\infty(s)\, ds\,. \tag{7.6}$$

For every $t \in [0,T]$ can extract a subsequence θ_{k_j} of θ_k, depending on t, such that

$$\theta_\infty(t) := \lim_{j\to\infty}\, \theta_{k_j}(t)\,. \tag{7.7}$$

By (6.5) and (7.1) we can apply Theorem 5.5 to $\tau_{k_j}(t)$, $\Gamma_{k_j}(t)$, and $u_{k_j}(t)$. Therefore there exist a further subsequence, still denoted u_{k_j}, and a function $u(t)$ in $AD(\psi(t), \Gamma(t))$

such that

$$u_{k_j}(t) \rightharpoonup u(t) \text{ weakly in } GSBV^p(\Omega;\mathbb{R}^m)\,, \tag{7.8}$$

$$u_{k_j}(t) \rightharpoonup u(t) \text{ weakly in } L^q(\Omega;\mathbb{R}^m)\,, \tag{7.9}$$

$$u_{k_j}(t) \to u(t) \text{ strongly in } L^{\dot q}(\Omega;\mathbb{R}^m)\,, \tag{7.10}$$

$$u_{k_j}(t) \to u(t) \text{ strongly in } L^r(\partial_S\Omega;\mathbb{R}^m)\,, \tag{7.11}$$

$$\nabla u_{k_j}(t) \rightharpoonup \nabla u(t) \text{ weakly in } L^p(\Omega;\mathbb{M}^{m\times n})\,. \tag{7.12}$$

Moreover

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(v,\Gamma) \tag{7.13}$$

for every $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, with $\Gamma(t) \,\tilde{\subset}\, \Gamma$, and every $v \in AD(\psi(t),\Gamma)$. This proves that $t \mapsto (u(t),\Gamma(t))$ satisfies properties (a) and (b) of the definition of quasistatic evolution (Subsection 3.9).

By (7.9)–(7.12) there exists a constant $C > 0$ such that

$$\|\nabla u(t)\|_p \le C\,, \quad \|u(t)\|_{\dot q} \le C\,, \quad \|u(t)\|_q \le C\,, \quad \|u(t)\|_{r,\partial_S\Omega} \le C \tag{7.14}$$

for every $t \in [0,T]$.

Finally Theorem 5.5 implies that

$$\mathcal{W}(\nabla u_{k_j}(t)) \to \mathcal{W}(\nabla u(t))\,, \tag{7.15}$$

$$\mathcal{F}_{k_j}(t)(u_{k_j}(t)) \to \mathcal{F}(t)(u(t))\,, \tag{7.16}$$

$$\mathcal{G}_{k_j}(t)(u_{k_j}(t)) \to \mathcal{G}(t)(u(t))\,. \tag{7.17}$$

Using Lemma 4.11, from (7.8) and (7.15), we obtain

$$\langle \partial\mathcal{W}(\nabla u_{k_j}(t)), \nabla\dot\psi(t)\rangle \to \langle \partial\mathcal{W}(\nabla u(t)), \nabla\dot\psi(t)\rangle\,. \tag{7.18}$$

By (7.10) the sequence $\partial_z F(t,x,u_{k_j}(t))$ converges to $\partial_z F(t,x,u(t))$ in measure on Ω. By (3.19) and (7.9) it is bounded in $L^{q'}(\Omega;\mathbb{R}^m)$. Therefore $\partial_z F(t,x,u_{k_j}(t))$ converges to $\partial_z F(t,x,u(t))$ weakly in $L^{q'}(\Omega;\mathbb{R}^m)$, and consequently by (3.14)

$$\langle \partial\mathcal{F}(t)(u_{k_j}(t)), \dot\psi(t)\rangle \to \langle \partial\mathcal{F}(t)(u(t)), \dot\psi(t)\rangle\,, \tag{7.19}$$

so that by (6.23) and (6.24) we obtain

$$\langle \partial\mathcal{F}_{k_j}(t)(u_{k_j}(t)), \dot\psi(t)\rangle \to \langle \partial\mathcal{F}(t)(u(t)), \dot\psi(t)\rangle\,. \tag{7.20}$$

Observing that $\dot{\mathcal{F}}(t)$ is continuous on $L^{\dot q}(\Omega;\mathbb{R}^m)$, while $\partial\mathcal{G}(t)$ and $\dot{\mathcal{G}}(t)$ are continuous on $L^r(\partial_S\Omega;\mathbb{R}^m)$, by (7.10) and (7.11) we have

$$\dot{\mathcal{F}}(t)(u_{k_j}(t)) \to \dot{\mathcal{F}}(t)(u(t))\,, \tag{7.21}$$

$$\langle \partial\mathcal{G}(t)(u_{k_j}(t)), \dot\psi(t)\rangle \to \langle \partial\mathcal{G}(t)(u(t)), \dot\psi(t)\rangle\,, \tag{7.22}$$

$$\dot{\mathcal{G}}(t)(u_{k_j}(t)) \to \dot{\mathcal{G}}(t)(u(t))\,, \tag{7.23}$$

so that by (6.28) and (6.30) we obtain

$$\langle \partial\mathcal{G}_{k_j}(t)(u_{k_j}(t)), \dot\psi(t)\rangle \to \langle \partial\mathcal{G}(t)(u(t)), \dot\psi(t)\rangle\,. \tag{7.24}$$

For a.e. $t \in [0,T]$ we set

$$\begin{aligned} \theta(t) :=& \langle \partial\mathcal{W}(\nabla u(t)), \nabla\dot\psi(t)\rangle - \langle \partial\mathcal{F}(t)(u(t)), \dot\psi(t)\rangle - \\ &- \dot{\mathcal{F}}(t)(u(t)) - \langle \partial\mathcal{G}(t)(u(t)), \dot\psi(t)\rangle - \dot{\mathcal{G}}(t)(u(t)) = \\ &= \langle g(t), \dot\psi(t)\rangle - \dot{\mathcal{F}}(t)(u(t)) - \dot{\mathcal{G}}(t)(u(t))\,, \end{aligned} \tag{7.25}$$

where $g(t)$ is defined by (3.58). From (6.10), (7.7), (7.18), (7.20), (7.21), (7.23), and (7.24) we obtain

$$\theta_\infty(t) = \theta(t) \tag{7.26}$$

for a.e. $t \in [0,T]$.

By (7.3) and (7.15)–(7.17) we have

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \liminf_{j\to\infty} \mathcal{E}_{k_j}(t)(u_{k_j}(t),\Gamma_{k_j}(t)) \le \limsup_{k\to\infty} \mathcal{E}_k(t)(u_k(t),\Gamma_k(t))\,. \tag{7.27}$$

From (6.11), (7.6), and (7.26) we obtain

$$\limsup_{k\to\infty} \mathcal{E}_k(t)(u_k(t),\Gamma_k(t)) \le \mathcal{E}(0)(u_0,\Gamma_0) + \int_0^t \theta(s)\,ds\,. \tag{7.28}$$

By (7.27) and (7.28) we have

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(0)(u_0,\Gamma_0) + \int_0^t \theta(s)\,ds\,. \tag{7.29}$$

To conclude the proof of the theorem it is enough to show that

$$\mathcal{E}(t)(u(t),\Gamma(t)) \ge \mathcal{E}(0)(u_0,\Gamma_0) + \int_0^t \theta(s)\,ds\,. \tag{7.30}$$

for every $t \in [0,T]$. Indeed (7.29) and (7.30) imply that $t \mapsto E(t) := \mathcal{E}(t)(u(t),\Gamma(t))$ is absolutely continuous in $[0,T]$ and that $\dot E(t) = \theta(t)$ for a.e. $t \in [0,T]$. By (7.25) this yields condition (c) of the definition of quasistatic evolution (Subsection 3.9). ☐

In order to prove (7.30) we need the following lemma.

Lemma 7.1. *Assume that $t \mapsto (u(t),\Gamma(t))$ satisfies conditions (a) and (b) in the definition of quasistatic evolution (Subsection 3.9). Let $t \in [0,T]$ and let s^i_k be a subdivision of $[0,t]$ satisfying (5.43) and (5.44). Then there exists a sequence $S_k(t) \to 0$ such that*

$$\begin{aligned}\mathcal{E}(t)(u(t),\Gamma(t)) \ge{}& \mathcal{E}(0)(u_0,\Gamma_0) + \sum_{i=1}^{i_k}\int_{s^{i-1}_k}^{s^i_k} \langle \partial\mathcal{W}(\nabla u(s^i_k)), \nabla\dot\psi(s)\rangle\,ds - \\ &- \sum_{i=1}^{i_k}\int_{s^{i-1}_k}^{s^i_k} \langle \partial\mathcal{F}(s^i_k)(u(s^i_k)), \dot\psi(s)\rangle\,ds - \sum_{i=1}^{i_k}\int_{s^{i-1}_k}^{s^i_k} \dot{\mathcal{F}}(s)(u(s^i_k))\,ds - \\ &- \sum_{i=1}^{i_k}\int_{s^{i-1}_k}^{s^i_k} \langle \partial\mathcal{G}(s^i_k)(u(s^i_k)), \dot\psi(s)\rangle\,ds - \sum_{i=1}^{i_k}\int_{s^{i-1}_k}^{s^i_k} \dot{\mathcal{G}}(s)(u(s^i_k))\,ds - S_k(t)\,.\end{aligned} \tag{7.31}$$

Proof. The minimality property in condition (a) gives $\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(\psi(t),\Gamma(t))$, hence $\mathcal{E}^{el}(t)(u(t)) \le \mathcal{E}^{el}(t)(\psi(t))$. By (3.51) and (3.52), this implies that there exists a constant $C>0$ such that

$$\|\nabla u(t)\|_p \le C\,, \quad \|u(t)\|_{\dot q} \le C\,, \quad \|u(t)\|_q \le C \tag{7.32}$$

for every $t \in [0,T]$. By Lemma 2.5 the functions $u(t)$ belong to $W^{1,p}(\Omega_S;\mathbb{R}^m) \cap L^q(\Omega_S;\mathbb{R}^m)$. Therefore (3.55) implies that, if we change the constant, we may assume also that

$$\|u(t)\|_{r,\partial_S\Omega} \le C \tag{7.33}$$

for every $t \in [0,T]$.

For every k let $\sigma_k\colon (0,t] \to (0,t]$ be the piecewise constant function defined by

$$\sigma_k(s) = s^i_k \qquad \text{for } s^{i-1}_k < s \le s^i_k\,,$$

and let $v_k\colon (0,t] \to GSBV^p_q(\Omega;\mathbb{R}^m)$ be the piecewise constant function defined by

$$v_k(s) := u(s^i_k) = u(\sigma_k(s)) \qquad \text{for } s^{i-1}_k < s \le s^i_k\,. \tag{7.34}$$

For every $i = 1,\dots,i_k$ we have $u(s^i_k) - \psi(s^i_k) + \psi(s^{i-1}_k) \in AD(\psi(s^{i-1}_k),\Gamma(s^i_k))$ and $\Gamma(s^{i-1}_k) \mathrel{\tilde\subset} \Gamma(s^i_k)$. By the minimality property in condition (a) we have

$$\mathcal{E}(s^{i-1}_k)(u(s^{i-1}_k),\Gamma(s^{i-1}_k)) \le \mathcal{E}(s^{i-1}_k)(u(s^i_k) - \psi(s^i_k) + \psi(s^{i-1}_k),\Gamma(s^i_k))\,. \tag{7.35}$$

Arguing as in the proof of (6.33) we obtain that there exist two sequences $\delta_k^{\mathcal{F}}$ and $\delta_k^{\mathcal{G}}$ of real numbers, with

$$\delta_k^{\mathcal{F}} \to 0\,, \qquad \delta_k^{\mathcal{G}} \to 0\,, \tag{7.36}$$

and three sequences $X_k \in L^\infty([0,T]; L^p(\Omega;\mathbb{M}^{m\times n}))$, $\chi_k \in L^\infty([0,T]; L^q(\Omega;\mathbb{R}^m))$, and $\eta_k \in L^\infty([0,T]; L^r(\partial_S\Omega;\mathbb{R}^m))$, with

$$\|X_k(s)\|_p + \|\chi_k(s)\|_q + \|\eta_k(s)\|_{r,\partial_S\Omega} \to 0 \text{ uniformly with respect to } s \in [0,T]\,, \tag{7.37}$$

such that

$$\begin{gathered}
\mathcal{E}(t)(u(t),\Gamma(t)) - \mathcal{E}(0)(u_0,\Gamma_0) \geq \\
\geq \int_0^t \langle \partial\mathcal{W}(\nabla v_k(s) + X_k(s)), \nabla\dot\psi(s)\rangle\, ds - \\
- \int_0^t \langle \partial\mathcal{F}(\sigma_k(s))(v_k(s)+\chi_k(s)), \dot\psi(s)\rangle\, ds - \int_0^t \dot{\mathcal{F}}(s)(v_k(s))\, ds - \\
- \int_0^t \langle \partial\mathcal{G}(\sigma_k(s))(v_k(s)+\eta_k(s)), \dot\psi(s)\rangle\, ds - \int_0^t \dot{\mathcal{G}}(s)(v_k(s))\, ds - \\
-\delta_k^{\mathcal{F}} \int_0^t \|\dot\psi(s)\|_{\dot q}\, ds - \delta_k^{\mathcal{G}} \int_0^t \|\dot\psi(s)\|_{r,\partial_S\Omega}\, ds\,.
\end{gathered} \tag{7.38}$$

Let

$$\begin{gathered}
S_k(t) := \int_0^T |\langle \partial\mathcal{W}(\nabla v_k(s) + X_k(s)), \nabla\dot\psi(s)\rangle - \langle \partial\mathcal{W}(\nabla v_k(s)), \nabla\dot\psi(s)\rangle|\, ds + \\
+ \int_0^T |\langle \partial\mathcal{F}(\sigma_k(s))(v_k(s)+\chi_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{F}(\sigma_k(s))(v_k(s)), \dot\psi(s)\rangle|\, ds + \\
+ \int_0^T |\langle \partial\mathcal{G}(\sigma_k(s))(v_k(s)+\eta_k(s)), \dot\psi(s)\rangle - \langle \partial\mathcal{G}(\sigma_k(s))(v_k(s)), \dot\psi(s)\rangle|\, ds + \\
+ \delta_k^{\mathcal{F}} \int_0^T \|\dot\psi(s)\|_{\dot q}\, ds + \delta_k^{\mathcal{G}} \int_0^T \|\dot\psi(s)\|_{r,\partial_S\Omega}\, ds\,.
\end{gathered} \tag{7.39}$$

Using Lemma 4.9 as in the last part of the proof of Lemma 6.1, from (7.36) and (7.37) we obtain that $S_k(t) \to 0$. Inequality (7.31) follows from (7.38) and (7.39). □

Proof of Theorem 3.13 continued. Let us fix $t \in (0,T]$. By Lemmas 4.12 and 5.7 and Remarks 4.13 and 5.6 there exists a subdivision s_k^i of $[0,t]$, satisfying (5.43), (5.44), (5.48), and (5.49), such that

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| (s_k^i - s_k^{i-1})\theta(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \theta(s)\, ds \Big| = 0\,, \tag{7.40}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big\| (s_k^i - s_k^{i-1})\nabla\dot\psi(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \nabla\dot\psi(s)\, ds \Big\|_p = 0\,, \tag{7.41}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big\| (s_k^i - s_k^{i-1})\dot\psi(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \dot\psi(s)\, ds \Big\|_q = 0\,, \tag{7.42}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big\| (s_k^i - s_k^{i-1})\dot\psi(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \dot\psi(s)\, ds \Big\|_{r,\partial_S\Omega} = 0\,, \tag{7.43}$$

Let

$$\mathcal{H} := \{u \in GSBV_q^p(\Omega;\mathbb{R}^m) : S(u) \,\tilde\subset\, \overline\Omega_B,\ \mathcal{H}^{n-1}(S(u)) \leq M,\ \|\nabla u\|_p \leq C,\ \|u\|_q \leq C\}\,,$$

where M and C are the constants which appear in (6.38) and (7.32). Arguing as in Theorem 5.5 it is easy to prove that $\mathcal{H}$ is compact in $L^{\dot q}(\Omega;\mathbb{R}^m)$ and in $L^r(\partial_S\Omega;\mathbb{R}^m)$.

Let

$$\begin{aligned}\omega_k(t) := \sum_{i=1}^{i_k} \Big\| \partial\mathcal{W}(\nabla u(s_k^i))\Big\|_{p'} \Big\| (s_k^i - s_k^{i-1})\nabla\dot\psi(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \nabla\dot\psi(s)\,ds\Big\|_p + \\ + \sum_{i=1}^{i_k} \Big\| \partial\mathcal{F}(s_k^i)(u(s_k^i))\Big\|_{q'} \Big\| (s_k^i - s_k^{i-1})\dot\psi(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \dot\psi(s)\,ds\Big\|_q + \\ + \sum_{i=1}^{i_k} \sup_{u\in\mathcal{H}} \Big| (s_k^i - s_k^{i-1})\dot{\mathcal{F}}(s_k^i)(u) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{F}}(s)(u)\,ds\Big| + \\ + \sum_{i=1}^{i_k} \Big\| \partial\mathcal{G}(s_k^i)(u(s_k^i))\Big\|_{r',\partial_S\Omega} \Big\| (s_k^i - s_k^{i-1})\dot\psi(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \dot\psi(s)\,ds\Big\|_{r,\partial_S\Omega} + \\ + \sum_{i=1}^{i_k} \sup_{u\in\mathcal{H}} \Big| (s_k^i - s_k^{i-1})\dot{\mathcal{G}}(s_k^i)(u) - \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{G}}(s)(u)\,ds\Big| + S_k(t)\,,\end{aligned} \tag{7.44}$$

where $S_k(t)$ is given by Lemma 7.1. By (3.12), (3.19), (3.35), (7.32), and (7.33) the terms $\|\partial\mathcal{W}(\nabla u(s_k^i))\|_{p'}$, $\|\partial\mathcal{F}(s_k^i)(u(s_k^i))\|_{q'}$, and $\|\partial\mathcal{G}(s_k^i)(u(s_k^i))\|_{r',\partial_S\Omega}$ are bounded uniformly with respect to k and i. Therefore, using (5.48), (5.49), (7.41)–(7.43), and Lemma 7.1 we obtain

$$\lim_{k\to\infty} \omega_k(t) = 0\,. \tag{7.45}$$

By (6.38) and (7.32) we have $u(s_k^i)\in\mathcal{H}$ for every k and i. Taking (7.25) and (7.44) into account, the estimate from below in Lemma 7.1 gives

$$\mathcal{E}(t)(u(t),\Gamma(t)) \ge \mathcal{E}(0)(u_0,\Gamma_0) + \sum_{i=1}^{i_k}(s_k^i - s_k^{i-1})\theta(s_k^i) - \omega_k(t)\,. \tag{7.46}$$

Passing to the limit as $k\to\infty$, by (7.40) we obtain (7.30), which, together with (7.25) and (7.29), yields condition (c) in the definition of quasistatic evolution. ☐

8. CONVERGENCE OF THE DISCRETE-TIME PROBLEMS

In this section we show that for every $t\in[0,T]$ the elastic energies and the crack energies of the solutions to the discrete-time problems converge to the corresponding energies for the continuous-time problem. Note that this result can be proved even if the minimum energy deformations corresponding to a crack $\Gamma(t)$ are not unique, but that it only holds for the discretizations that produce a given crack $\Gamma(t)$.

Let t_k^i be as in Section 6, let (u_0,Γ_0) be an initial configuration that satisfies the minimality property (3.65), and let (u_k^i,Γ_k^i) be the solutions of the minimum problems (6.4), with $(u_k^0,\Gamma_k^0) := (u_0,\Gamma_0)$. Let τ_k, u_k, Γ_k, $\mathcal{F}_k$, $\mathcal{G}_k$, and $\mathcal{E}_k$ be the piecewise constant functions introduced in (6.8), and let $\mathcal{E}_k^{el}\colon[0,T]\to\mathbb{R}$ be the piecewise constant function defined by

$$\mathcal{E}_k^{el}(t) := \mathcal{E}^{el}(t_k^i) = \mathcal{E}^{el}(\tau_k(t))\,, \tag{8.1}$$

where i is the largest integer such that $t_k^i\le t$.

Theorem 8.1. *Let $t\mapsto(v(t),\Gamma(t))$ be a quasistatic evolution, let θ_k be defined by (6.10), and let*

$$\begin{aligned}\theta(t) := \langle\partial\mathcal{W}(\nabla v(t)),\nabla\dot\psi(t)\rangle - \langle\partial\mathcal{F}(t)(v(t)),\dot\psi(t)\rangle - \\ - \dot{\mathcal{F}}(t)(v(t)) - \langle\partial\mathcal{G}(t)(v(t)),\dot\psi(t)\rangle - \dot{\mathcal{G}}(t)(v(t))\,.\end{aligned} \tag{8.2}$$

Assume that $\Gamma_k(t)$ and $\Gamma(t)$ satisfy (7.1) and (7.2) for every $t \in [0,T]$. Then

$$\mathcal{E}^{el}(t)(v(t)) = \lim_{k\to\infty} \mathcal{E}^{el}_k(t)(u_k(t)), \tag{8.3}$$

$$\mathcal{K}(\Gamma(t)) = \lim_{k\to\infty} \mathcal{K}(\Gamma_k(t)) \tag{8.4}$$

for every $t \in [0,T]$. Moreover

$$\theta_k \to \theta \quad in\ L^1([0,T]), \tag{8.5}$$

so that there exists a subsequence of θ_k which converges to θ a.e. in $[0,T]$.

Proof. For a.e. $t \in [0,T]$ let $\theta_\infty(t)$ be defined by (7.4). In the proof of Theorem 3.13 for every $t \in [0,T]$ we constructed a function $u(t) \in AD(\psi(t),\Gamma(t))$ such that $t \mapsto (u(t),\Gamma(t))$ is a quasistatic evolution and

$$\mathcal{E}(t)(u(t),\Gamma(t)) = \mathcal{E}(0)(u_0,\Gamma_0) + \int_0^t \theta_\infty(s)\,ds \tag{8.6}$$

for every $t \in [0,T]$ (see (7.26), (7.29), and (7.30)). Since $(u(t),\Gamma(t))$ and $(v(t),\Gamma(t))$ satisfy the minimality condition (a) in the definition of quasistatic evolution (see Subsection 3.9), we have

$$\mathcal{E}(t)(u(t),\Gamma(t)) = \mathcal{E}(t)(v(t),\Gamma(t)) \tag{8.7}$$

for every $t \in [0,T]$. By condition (c) for $t \mapsto (v(t),\Gamma(t))$ we have

$$\mathcal{E}(t)(v(t),\Gamma(t)) = \mathcal{E}(0)(u_0,\Gamma_0) + \int_0^t \theta(s)\,ds \tag{8.8}$$

From (8.6)–(8.8) we deduce that

$$\theta(t) = \theta_\infty(t) \tag{8.9}$$

for a.e. $t \in [0,T]$.

By Lemma 6.1 for every $t \in [0,T]$ we have

$$\liminf_{k\to\infty} \mathcal{E}_k(t)(u_k(t),\Gamma_k(t)) \le \mathcal{E}(0)(u_0,\Gamma_0) + \liminf_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,ds, \tag{8.10}$$

$$\limsup_{k\to\infty} \mathcal{E}_k(t)(u_k(t),\Gamma_k(t)) \le \mathcal{E}(0)(u_0,\Gamma_0) + \limsup_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,ds. \tag{8.11}$$

Let us fix $t \in [0,T]$ and let $u_{k_j}(t)$ be a subsequence of $u_k(t)$ such that

$$\lim_{j\to\infty} \mathcal{E}^{el}_{k_j}(t)(u_{k_j}(t)) = \liminf_{k\to\infty} \mathcal{E}^{el}_k(t)(u_k(t)). \tag{8.12}$$

Since $\Gamma^N_{k_j}(t)$ σ^p-converges to $\Gamma^*(t)$, using (6.5) we can apply Theorem 5.5 to $\tau_{k_j}(t)$, $\Gamma_{k_j}(t)$, and $u_{k_j}(t)$. Therefore there exist a further subsequence, still denoted $u_{k_j}(t)$, and a function $u^*(t) \in AD(\psi(t),\Gamma(t))$ such that

$$\mathcal{W}(\nabla u_{k_j}(t)) \to \mathcal{W}(\nabla u^*(t)), \tag{8.13}$$

$$\mathcal{F}_{k_j}(t)(u_{k_j}(t)) \to \mathcal{F}(t)(u^*(t)), \tag{8.14}$$

$$\mathcal{G}_{k_j}(t)(u_{k_j}(t)) \to \mathcal{G}(t)(u^*(t)). \tag{8.15}$$

Moreover

$$\mathcal{E}(t)(u^*(t),\Gamma(t)) \le \mathcal{E}(t)(v,\Gamma)$$

for every $\Gamma \in \mathcal{R}(\overline{\Omega}_B)$, with $\Gamma(t) \mathrel{\tilde{\subset}} \Gamma$, and for every $v \in AD(\psi(t),\Gamma)$. Since $(v(t),\Gamma(t))$ satisfies the same minimality property by condition (a) in Subsection 3.9, we have

$$\mathcal{E}^{el}(t)(v(t)) = \mathcal{E}^{el}(t)(u^*(t)). \tag{8.16}$$

From (8.13)–(8.15) we obtain

$$\mathcal{E}^{el}(t)(u^*(t)) = \lim_{j\to\infty} \mathcal{E}^{el}_{k_j}(t)(u_{k_j}(t)),$$

which, together with (8.12) and (8.16) gives

$$\mathcal{E}^{el}(t)(v(t)) = \liminf_{k\to\infty} \mathcal{E}^{el}_k(t)(u_k(t)) \,. \tag{8.17}$$

By (5.2) and (5.3) we have

$$\mathcal{K}(\Gamma(t)) = \mathcal{K}(\Gamma^*(t)) \le \liminf_{k\to\infty} \mathcal{K}(\Gamma^N_k(t)) = \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t)) \,, \tag{8.18}$$

so that (8.17) and (8.18) yield

$$\mathcal{E}(t)(v(t),\Gamma(t)) \le \liminf_{k\to\infty} \mathcal{E}_k(t)(u_k(t),\Gamma_k(t)) \,. \tag{8.19}$$

From (7.6), (8.6), (8.7), (8.9), (8.10), (8.11), and (8.19) we obtain

$$\mathcal{E}(t)(v(t),\Gamma(t)) = \lim_{k\to\infty} \mathcal{E}_k(t)(u_k(t),\Gamma_k(t)) \tag{8.20}$$

$$\int_0^t \theta(s)\,ds = \lim_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,ds \tag{8.21}$$

for every $t\in[0,T]$. Equalities (8.3) and (8.4) follow easily from (8.17), (8.18), and (8.20).

By (8.21) we have

$$\int_0^T \theta(t)\,dt = \lim_{k\to\infty} \int_0^T \theta_k(t)\,dt \,. \tag{8.22}$$

By (7.4) and (8.9) $\theta_k\vee\theta$ converges to θ pointwise on $[0,T]$, so that by (7.5)

$$\theta_k\vee\theta \to \theta \quad \text{in } L^1([0,T]) \,. \tag{8.23}$$

Since $\theta_k+\theta = (\theta_k\vee\theta)+(\theta_k\wedge\theta)$, from (8.22) and (8.23) we obtain

$$\int_0^T \theta(t)\,dt = \lim_{k\to\infty} \int_0^T (\theta_k\wedge\theta)(t)\,dt \,.$$

As $\theta_k\wedge\theta\le\theta$, this implies that $\theta_k\wedge\theta$ converges to θ in $L^1([0,T])$, which, together with (8.23), gives (8.5). □

9. APPENDIX

In Remarks 3.3 and 3.5 we introduced elementary conditions on F and G that ensure that $\mathcal{F}$, $\dot{\mathcal{F}}$, $\mathcal{G}$, and $\dot{\mathcal{G}}$ satisfy all properties required in Subsections 3.4 and 3.5. It is easy to see that (3.23) and (3.39) are much stronger than what we need for this purpose. Indeed, it is enough to assume that the partial derivatives $\partial_z F$, $\partial_t F$, $\partial_z\partial_t F$, $\partial_z G$, $\partial_t G$, and $\partial_z\partial_t G$ exist, are measurable with respect to x, and continuous with respect to (t,z). In the rest of this section we show that the same results can be obtained under a less regular dependence on time, in the spirit of the hypotheses considered for the boundary deformations in Subsection 3.6.

9.1. **Weaker hypotheses on the body forces.** We are interested in particular in the case of *dead loads*, in which the density of the body force $f\colon[0,T]\times\Omega\to\mathbb{R}^m$ per unit volume in the reference configuration does not depend on the deformation. In this case the simplest choice for the potential is $F(t,x,z) := f(t,x)z$. But this linear dependence on z can not be accepted, because it violates the first inequality in (3.18), where $\alpha_0^{\mathcal{F}}>0$. We may consider a slight variant, namely

$$F(t,x,z) := f(t,x)z + F_0(x,z) \,, \tag{9.1}$$

where $F_0\colon\Omega\times\mathbb{R}^m\to\mathbb{R}$ is a Carathéodory function. We assume that for every $x\in\Omega$ the function $z\mapsto F_0(x,z)$ belongs to $C^1(\mathbb{R}^m)$ and that for every $(x,z)\in\Omega\times\mathbb{R}^m$

$$a_0|z|^q - b_0(x) \le -F_0(x,z) \le a_1|z|^q + b_1(x) \,, \tag{9.2}$$

$$|\partial_z F_0(x,z)| \le a_2|z|^{q-1} + b_2(x) \,, \tag{9.3}$$

where $q > 1$, a_0, a_1, a_2 are positive constants, $b_0, b_1 \in L^1(\Omega)$, $b_2 \in L^{q'}(\Omega)$, and $q' := q/(q-1)$.

When the body is subject to a deformation u, the body force acting at time t has a density per unit volume in the reference configuration given by $f(t,x) + \partial_z F_0(x, u(x))$. The term $f(t,x)$ is a time dependent dead load (that we may think of as determined by an experimental device), while $\partial_z F_0(x, u(x))$ can be interpreted as a background time independent body force. As a consequence of our hypotheses on F_0, this force will prevent broken parts of the body from finding infinity as only equilibrium configuration.

Let $\dot q$ be a constant in $(1,q)$ and let $\dot q' = \dot q/(\dot q - 1)$. If $t \mapsto f(t,\cdot)$ is absolutely continuous from $[0,T]$ into $L^{\dot q'}(\Omega;\mathbb{R}^m)$ and $t \mapsto \dot f(t,\cdot)$ is its time derivative, which belongs to $L^1([0,T]; L^{\dot q'}(\Omega;\mathbb{R}^m))$, we can consider the functional $\mathcal{F}(t)\colon L^q(\Omega;\mathbb{R}^m) \to \mathbb{R}$ defined for every $t \in [0,T]$ by (3.13), with F given by (9.1), and the functional $\dot{\mathcal{F}}(t)\colon L^{\dot q}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ defined for a.e. $t \in [0,T]$ by

$$\dot{\mathcal{F}}(t)(u) := \int_\Omega \dot f(t,x)\, u(x)\, dx\,.$$

We can easily check that in this case $\mathcal{F}$ and $\dot{\mathcal{F}}$ satisfy all properties required in Subsection 3.4.

Note that these hypotheses do not guarantee the existence of the partial derivative $\partial_t F(t,x,z)$ for a.e. $t \in [0,T]$ and for every $(x,z) \in \Omega\times\mathbb{R}^m$, so that Remark 3.3 can not be applied.

We now present a more general set of hypotheses, which includes this case as well as those considered in Remark 3.3. More precisely, we assume that $F\colon [0,T]\times\Omega\times\mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions

$$\text{for every } z \in \mathbb{R}^m \text{ the function } (t,x) \mapsto F(t,x,z) \text{ is } \mathcal{L}^1\times\mathcal{L}^n\text{-measurable on } [0,T]\times\Omega, \tag{9.4}$$

$$\text{for every } (t,x) \in [0,T]\times\Omega \text{ the function } z \mapsto F(t,x,z) \text{ belongs to } C^1(\mathbb{R}^m). \tag{9.5}$$

Moreover, we assume that that there exist four constants $q > 1$, $a_0^F > 0$, $a_1^F > 0$, $a_2^F > 0$ and three nonnegative functions b_0^F, $b_1^F \in C^0([0,T]; L^1(\Omega))$, $b_2^F \in C^0([0,T]; L^{q'}(\Omega))$, such that

$$a_0^F |z|^q - b_0^F(t,x) \le -F(t,x,z) \le a_1^F |z|^q + b_1^F(t,x)\,, \tag{9.6}$$

$$|\partial_z F(t,x,z)| \le a_2^F |z|^{q-1} + b_2^F(t,x)\,, \tag{9.7}$$

for every $(t,x,z) \in [0,T]\times\Omega\times\mathbb{R}^m$.

To deal with the dependence of F on t, we assume that there exists a function $\dot F\colon [0,T]\times\Omega\times\mathbb{R}^m \to \mathbb{R}$ such that for every $t \in [0,T]$ and for every $z \in \mathbb{R}^m$

$$F(t,x,z) = F(0,x,z) + \int_0^t \dot F(s,x,z)\, ds \quad \text{for a.e. } x \in \Omega, \tag{9.8}$$

$$\partial_z F(t,x,z) = \partial_z F(0,x,z) + \int_0^t \partial_z \dot F(s,x,z)\, ds \quad \text{for a.e. } x \in \Omega. \tag{9.9}$$

The integrals in (9.8) and (9.9) are well defined since we assume also that

$$\text{for every } z \in \mathbb{R}^m \text{ the function } (t,x) \mapsto \dot F(t,x,z) \text{ is } \mathcal{L}^1\times\mathcal{L}^n\text{-measurable on } [0,T]\times\Omega, \tag{9.10}$$

$$\text{for every } (t,x) \in [0,T]\times\Omega \text{ the function } z \mapsto \dot F(t,x,z) \text{ belongs to } C^1(\mathbb{R}^m), \tag{9.11}$$

and that there exist a constant $\dot q \in [1,q)$ and four nonnegative functions $a_3^F, a_4^F \in L^1([0,T])$, $b_3^F \in L^1([0,T]; L^1(\Omega))$, and $b_4^F \in L^1([0,T]; L^{\dot q'}(\Omega))$ such that

$$|\dot F(t,x,z)| \le a_3^F(t) |z|^{\dot q} + b_3^F(t,x)\,, \tag{9.12}$$

$$|\partial_z \dot F(t,x,z)| \le a_4^F(t) |z|^{\dot q - 1} + b_4^F(t,x) \tag{9.13}$$

for every $(t,x,z) \in [0,T]\times\Omega\times\mathbb{R}^m$.

The functionals $\mathcal{F}(t)\colon L^q(\Omega;\mathbb{R}^m)\to\mathbb{R}$ and $\dot{\mathcal{F}}(t)\colon L^{\dot q}(\Omega;\mathbb{R}^m)\to\mathbb{R}$ are defined by (3.13) and by

$$\dot{\mathcal{F}}(t)(u) := \int_\Omega \dot F(t,x,u(x))\,dx\,. \tag{9.14}$$

Using (9.4)–(9.7) it is easy to see that $\mathcal{F}(t)$ is of class C^1 on $L^q(\Omega;\mathbb{R}^m)$ and that (3.14) and (3.15) hold. By (9.10)–(9.14) for a.e. $t\in[0,T]$ the functional $\dot{\mathcal{F}}(t)$ is of class C^1 on $L^{\dot q}(\Omega;\mathbb{R}^m)$ and

$$\langle\partial\dot{\mathcal{F}}(t)(u),v\rangle = \int_\Omega \partial_z\dot F(t,x,u(x))\,v(x)\,dx \tag{9.15}$$

for every u, $v\in L^{\dot q}(\Omega;\mathbb{R}^m)$, so that the functions $t\mapsto\dot{\mathcal{F}}(t)(u)$ and $t\mapsto\langle\partial\dot{\mathcal{F}}(t)(u),v\rangle$ are measurable on $[0,T]$ for every u, $v\in L^q(\Omega;\mathbb{R}^m)$. From (9.8) and (9.9) we obtain (3.16) and (3.17) for every pair of simple functions u and v from Ω into $\mathbb{R}^m$. An easy approximation argument shows that (3.16) and (3.17) hold for every u, $v\in L^q(\Omega;\mathbb{R}^m)$. Inequalities (3.18)–(3.21) follow immediately from (3.13), (3.14), (9.6), (9.7), and (9.12)–(9.15).

Remark 9.1. Let us check that, under the hypotheses on F_0 considered above, if the function $t\mapsto f(t,\cdot)$ is absolutely continuous from $[0,T]$ into $L^{\dot q'}(\Omega;\mathbb{R}^m)$ and $1<\dot q<1$, then the function F defined by (9.1) satisfies (9.4)–(9.13) with $\dot F(t,x,z):=\dot f(t,x)z$. Properties (9.5) and (9.11) are trivial; (9.4) and (9.8)–(9.10) follow from well-known properties of absolutely continuous functions with values in reflexive Banach spaces (see, e.g., [5, Appendix]). By the Cauchy inequality we have

$$-F(t,x,z)\ge\frac{a_0}{q'}|z|^q-\frac{1}{q'}a_0^{\frac{1}{1-q}}|f(t,x)|^{q'}-b_0(x)\,,$$

so that the first inequality in (9.6) is satisfied with $a_0^F:=a_0/q'$ and $b_0^F(t,x):=b_0(x)+(a_0^{1/(1-q)}/q')|f(t,x)|^{q'}$. The second inequality in (9.6) can be obtained in a similar way, while (9.3) yields (9.7) with $a_2^F:=a_2$ and $b_2^F(t,x):=b_2(x)+|f(t,x)|$.

To prove (9.12) we observe that by the Cauchy inequality we have

$$|\dot F(t,x,z)|=|\dot f(t,x)z|\le\frac{1}{\dot q'}\frac{|\dot f(t,x)|^{\dot q'}}{\|\dot f(t,\cdot)\|_{\dot q'}^{\dot q'-1}}+\frac{1}{\dot q}\|\dot f(t,\cdot)\|_{\dot q'}\,|z|^{\dot q}\,,$$

so that inequality (9.12) is satisfied with $b_3^F(t,x):=(1/\dot q')|\dot f(t,x)|^{\dot q'}\|\dot f(t,\cdot)\|_{\dot q'}^{1-\dot q'}$ and $a_3^F(t)$ $:=(1/\dot q)\|\dot f(t,\cdot)\|_{\dot q'}$. Finally, (9.13) holds with $a_4^F(t,x):=0$ and $b_4^F(t,x):=|\dot f(t,x)|$.

9.2. **Weaker hypotheses on the surface forces.** In the case of a time dependent dead load, the density $g\colon[0,T]\times\partial_S\Omega\to\mathbb{R}^m$ of the applied surface force per unit area in the reference configuration is independent of the deformation u. Then, the simplest choice for the potential is $G(t,x,z):=g(t,x)z$. Let r and r' be as in Subsection 3.5. If $t\mapsto g(t,\cdot)$ is absolutely continuous from $[0,T]$ into $L^{r'}(\partial_S\Omega;\mathbb{R}^m)$, and $t\mapsto\dot g(t,\cdot)$ is its time derivative, which belongs to $L^1([0,T];L^{r'}(\partial_S\Omega;\mathbb{R}^m))$, we can consider the functional $\mathcal{G}(t)\colon L^r(\partial_S\Omega;\mathbb{R}^m)\to\mathbb{R}$ defined for every $t\in[0,T]$ by

$$\mathcal{G}(t)(u):=\int_{\partial_S\Omega}g(t,x)\,u(x)\,d\mathcal{H}^{n-1}(x)\,,$$

and the functional $\dot{\mathcal{G}}(t)\colon L^r(\Omega;\mathbb{R}^m)\to\mathbb{R}$ defined for a.e. $t\in[0,T]$ by

$$\dot{\mathcal{G}}(t)(u):=\int_{\partial_S\Omega}\dot g(t,x)\,u(x)\,d\mathcal{H}^{n-1}(x)\,.$$

We can easily check that in this case $\mathcal{G}$ and $\dot{\mathcal{G}}$ satisfy all properties required in Subsection 3.5.

Note that these hypotheses do not guarantee the existence of the partial derivative $\partial_t G(t,x,z)$ for a.e. $t\in[0,T]$ and for every $(x,z)\in\partial_S\Omega\times\mathbb{R}^m$, so that Remark 3.5 can not be applied.

We present now a more general set of hypotheses which includes this case as well as those considered in Remark 3.5. More precisely, we assume that $G\colon [0,T]\times\partial_S\Omega\times\mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions:

$$\text{for every } z \in \mathbb{R}^m \text{ the function } (t,x) \mapsto G(t,x,z) \text{ is } \mathcal{L}^1\times\mathcal{H}^{n-1}\text{-measurable,} \tag{9.16}$$

$$\text{for every } (t,x) \in [0,T]\times\partial_S\Omega \text{ the function } z \mapsto G(t,x,z) \text{ belongs to } C^1(\mathbb{R}^m). \tag{9.17}$$

Moreover, we assume that there exist two constants $a_1^G \geq 0$, $a_2^G \geq 0$ and four non-negative functions $a_0^G \in L^\infty([0,T];L^{r'}(\partial_S\Omega))$, b_0^G, $b_1^G \in C^0([0,T];L^1(\partial_S\Omega))$, and $b_2^G \in C^0([0,T];L^{r'}(\partial_S\Omega))$ such that

$$-a_0^G(t,x)|z| - b_0^G(t,x) \leq -G(t,x,z) \leq a_1^G|z|^r + b_1^G(t,x)\,, \tag{9.18}$$

$$|\partial_z G(t,x,z)| \leq a_2^G|z|^{r-1} + b_2^G(t,x) \tag{9.19}$$

for every $(t,x,z) \in [0,T]\times\partial_S\Omega\times\mathbb{R}^m$.

To deal with the dependence of G on t, we assume that there exists a function $\dot G\colon [0,T]\times\partial_S\Omega\times\mathbb{R}^m \to \mathbb{R}$ such that for every $t \in [0,T]$ and for every $z \in \mathbb{R}^m$

$$G(t,x,z) = G(0,x,z) + \int_0^t \dot G(s,x,z)\,ds \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial_S\Omega, \tag{9.20}$$

$$\partial_z G(t,x,z) = \partial_z G(0,x,z) + \int_0^t \partial_z \dot G(s,x,z)\,ds \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial_S\Omega. \tag{9.21}$$

The integrals in (9.20) and (9.21) are well defined since we assume also that

$$\text{for every } z \in \mathbb{R}^m \text{ the function } (t,x) \mapsto \dot G(t,x,z) \text{ is } \mathcal{L}^1\times\mathcal{H}^{n-1}\text{-measurable,} \tag{9.22}$$

$$\text{for every } (t,x) \in [0,T]\times\partial_S\Omega \text{ the function } z \mapsto \dot G(t,x,z) \text{ belongs to } C^1(\mathbb{R}^m), \tag{9.23}$$

and that there exist four nonnegative functions $a_3^G, a_4^G \in L^1([0,T])$, $b_3^G \in L^1([0,T];L^1(\partial_S\Omega))$, and $b_4^G \in L^1([0,T],L^{r'}(\Omega))$ such that

$$|\dot G(t,x,z)| \leq a_3^G(t)|z|^r + b_3^G(t,x)\,, \tag{9.24}$$

$$|\partial_z \dot G(t,x,z)| \leq a_4^G(t)|z|^{r-1} + b_4^G(t,x) \tag{9.25}$$

for every $(t,x,z) \in [0,T]\times\partial_S\Omega\times\mathbb{R}^m$.

The functionals $\mathcal{G}(t)$ and $\dot{\mathcal{G}}(t)\colon L^r(\partial_S\Omega;\mathbb{R}^m) \to \mathbb{R}$ are now defined by (3.29) and by

$$\dot{\mathcal{G}}(t)(u) := \int_{\partial_S\Omega} \dot G(t,x,u(x))\,d\mathcal{H}^{n-1}(x)\,.$$

Arguing as in Subsection 9.1 it is easy to prove that $\mathcal{G}$ and $\dot{\mathcal{G}}$ satisfy all properties required in Subsection 3.5.

Remark 9.2. As in Remark 9.1 we can prove that, if the function $t \mapsto g(t,\cdot)$ is absolutely continuous from $[0,T]$ into $L^{r'}(\partial_S\Omega;\mathbb{R}^m)$, then the function $G(t,x,z) := g(t,x)z$ satisfies (9.16)–(9.25) with $\dot G(t,x,z) := \dot g(t,x)z$.

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