ON THE INADEQUACY OF THE SCALING OF LINEAR ELASTICITY FOR 3D–2D ASYMPTOTICS IN A NONLINEAR SETTING

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Abstract. Rescaling of independent and/or dependent variables is the usual first step when performing a 3D–2D asymptotic analysis of elastic equilibrium for an ε–thin three–dimensional domain. The direction transverse to the thickness of the domain is dilated by $\frac{1}{\varepsilon}$ in the linearized setting, as well as in its nonlinear analogue. The dependent variables (i.e., the components of the displacement field) are however left untouched in the nonlinear setting, while the third component is contracted by a factor ε in the linearized setting. We investigate the consequences of adopting the contrary scaling of the dependent variables in both settings and evidence a striking difference at first order in ε: linearized elasticity is only affected through the kinematics of the limit fields on the plate (the resulting 2d–domain), while nonlinear elasticity loses its structure because the resulting plate energy depends on the imposed lateral boundary conditions. Therefore, there is no limit model behavior under such a scaling, at least at first order.

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1. Introduction

In recent years various models of plate and/or thin film behaviors have been derived through the use of Γ–convergence techniques: in a first step, the three–dimensional elastic energy of a transversally thin body is transversally dilated, so as to have unit thickness. In doing so, one is forced to accordingly dilate the transverse derivatives that appear in the energy. The resulting energy, defined on a fixed domain this time, will thus exhibit explicit dependence upon the thickness of the body. The second step is then to take the variational limit of that energy as the thickness tends to 0.

In a linearized setting, the elastic energy is a quadratic function of the linearized strain and if the dilation is to preserve that structure, it must then correspondingly shrink the transverse component of the displacement field, whereas, in a nonlinear setting, the elastic energy is a function of the displacement gradients and the structure is preserved without altering the components of that displacement field. Thus there seems to be a dichotomy in the rescaling process: simple dilation of the transverse direction in the nonlinear setting versus dilation of the transverse direction and corresponding shrinkage of the transverse displacement field in a linearized setting (see [7], [12]).

Of course, the variational limit implicitly imposes a certain order on the applicable loads. Indeed, Γ–convergence is useless if the considered fields do not converge in the space where the Γ–convergence takes place. The convergence of the fields is usually a consequence of a coercivity property of the potential energy (that is of the elastic energy evaluated on the kinematically admissible displacement fields minus the work done by the force–loads). But the force–loads are given in the original configuration, and thus will experience the rescaling. This is why, in a nonlinear setting, transverse force–loads can be of order 1 whereas in the linearized setting, they must be of the order of the thickness (hence vanishingly small as the thickness decreases to 0). This is however no obstacle if considering loads that have zero transverse component.

In this study, we investigate the consequences of adopting the apparently "wrong" rescaling in both linearized and nonlinear settings. The linearized setting (suitably generalized to accommodate non quadratic energy densities) shows its resilience to the scaling: the variational limit in the "wrong" rescaling is the same as that in the usual rescaling, except for the fact that the kinematically admissible fields are different in the limit, which is to be expected since the magnitude of the transverse component of the unscaled displacement field is more rigidly constrained in that case (see Section 4).

The nonlinear setting however reacts strongly to the scaling: the variational limit in the "wrong" rescaling is not clearcut: it is shown in Section 2, with the help of the results derived in Sections 3, to critically depend upon the lateral boundary conditions imposed on the thin body, at least when the limit displacement field is assumed to be 0 in the transverse direction and affine in the in–plane directions on its boundary (the limit model is always two–dimensional in that it only involves the in–plane variables). From a mechanical standpoint, the result can be interpreted as follows: for such boundary conditions on the limit fields, the thin plate wants to deflect, at first order in the thickness, in the transverse direction, which the classical non–linear scaling allows, but not the "wrong" linear scaling which produces no transverse displacement at all. The evidenced dependence of the limit behavior upon the kind of lateral boundary conditions imposed on the thin three–dimensional body demonstrates that, under the "wrong" rescaling, one cannot hope to obtain a reasonable mechanical model of plate and/or thin film behavior at first order in the thickness, because any kind of limit constitutive model should of course be independent of the lateral boundary conditions of the approximating fields. Note that such is indeed the case if the non linear scaling is used.

As a final note, the — possibly useless — question of determining the Γ–limit in the nonlinear setting for the "wrong" rescaling remains open at this time.

2. About two different scalings

The basic tenet of dimensional reduction is a reformulation of the original problem defined on a thin 3d domain $\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon)$, where ω is a bounded domain of $\mathbb{R}^2$, on a fixed domain $\Omega := \Omega(1)$ through a $\frac{1}{\varepsilon}$–dilation of $\Omega(\varepsilon)$ in the transverse direction x_3.

If addressing a problem of nonlinear elasticity on $\Omega(\varepsilon)$ with, for illustration sake, affine boundary data on the lateral boundary $\partial\omega \times (-1, 1)$, i.e.,

$$\begin{cases} u_\gamma(\varepsilon)(x) = \xi_{\gamma\beta} x_\beta, \\ u_3(\varepsilon)(x) = 0, \end{cases}$$

for $x = (x_\alpha, x_3) \in \partial\omega \times (-1, 1)$ (where the index 3 denotes the transverse direction, and Greek indices $\alpha, \beta, \gamma, \lambda$, range between 1 and 2), the equilibrium displacement field $u(\varepsilon)$, a $\mathbb{R}^3$–valued field defined on $\Omega(\varepsilon)$, minimizes the elastic energy

$$\int_{\Omega(\varepsilon)} W(Dw)\, dx \tag{2.1}$$

among all fields $w(x_\alpha, x_3)$ which satisfy the boundary condition. In (2.1), W is a homogeneous (x–independent) elastic energy density.

In such a setting the following scaling of the dependent fields is customarily adopted (cf. e.g. [3], [12]):

$$u_\varepsilon(x_\alpha, x_3) := u(\varepsilon)(x_\alpha, \varepsilon x_3), \tag{2.2}$$

and the rescaled problem, formulated on Ω, reads as

$$\Lambda_\varepsilon := \inf\left\{ \int_\Omega W\left(D_\beta v \Big| \frac{1}{\varepsilon} D_3 v\right)\, dx : \quad \begin{cases} v_\gamma = \xi_{\gamma\beta} x_\beta, \\ \\ v_3 = 0 \end{cases} \text{on } \partial\omega \times (-1,1) \right\}, \tag{2.3}$$

where (M_β, M_3) stands for the 3×3 matrix with columns $M_1, M_2, M_3 \in \mathbb{R}^3$. Whenever W is continuous and exhibits p–growth ($1 < p < \infty$), i.e.,

$$\frac{1}{C}|F|^p \le W(F) \le C(1 + |F|^p) \tag{2.4}$$

for some $C > 0$, it is then shown in [12] that, for every sequence $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0$ and for any open subset $A \subset \omega$,

$$E_{\varepsilon_n}(v; A) := \int_{A\times(-1,1)} W\left(D_\beta v \Big| \frac{1}{\varepsilon_n} D_3 v\right)\, dx$$

$\Gamma(L^p)$–converges to

$$E(v; A) := \begin{cases} 2\int_A Q_{2,3}\overline{W}(D_\beta v)\, dx_\alpha, & v \in W^{1,p}(\omega; \mathbb{R}^3), \\ \\ \infty, & \text{otherwise} \end{cases}$$

where, for $\overline{M} \in \mathbb{R}^{3\times 2}$,

$$\overline{W}(\overline{M}) := \inf_{z\in\mathbb{R}^3} W(\overline{M}|z).$$

Here and in the remainder of the paper, for any Borel measurable function $Z : \mathbb{R}^N \mapsto \mathbb{R}^d$, $Q_{N,d}Z$ stands for the *quasiconvexification* of Z, that is, for any $M \in \mathbb{R}^{d\times N}$,

$$Q_{N,d}Z(M) := \inf_\varphi \left\{ \int_{(0,1)^N} Z(M + D\varphi)\, dx : \varphi \in \mathcal{C}_0^\infty\left((0,1)^N; \mathbb{R}^d\right) \right\}. \tag{2.5}$$

For a detailed study of Γ-convergence we refer the reader to [4] and [9].

Remark 2.1. The result of [12] can be easily extended — in a manner similar to that of the second step in the proof of Lemma 3.1 below — to Borel measurable energy functionals.

Remark 2.2. It is actually proved in [12] that, for any $v \in W^{1,p}(\omega; \mathbb{R}^3)$, any sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$, and any open subset $A \subset \omega$, the recovery sequence $\{v_n\}$ with $v_n \underset{n\to\infty}{\longrightarrow} v$ strongly in $L^p(\omega \times (-1,1); \mathbb{R}^3)$ and $E(v; A) = \lim_{n\to\infty} E_{\varepsilon_n}(v_n; A)$ can be chosen such that $v_n - v \in W_0^{1,p}(A \times (-1,1); \mathbb{R}^3)$. As an immediate corollary, any sequence $\{u_n\}$, where u_n is a(n approximate) solution of (2.3) with $\varepsilon = \varepsilon_n$, admits a strong $L^p(\Omega; \mathbb{R}^3)$–converging subsequence that converges to $u \in W^{1,p}(\omega; \mathbb{R}^3)$ where u is a solution for

$$\Lambda := \inf\left\{ \int_\omega Q_{2,3}\overline{W}(D_\beta v)\, dx_\alpha : \quad \begin{cases} v_\gamma = \xi_{\gamma\beta} x_\beta \\ \\ v_3 = 0 \end{cases} \text{on } \partial\omega \right\}. \tag{2.6}$$

Furthermore, $\Lambda = \lim_{n\to\infty} \Lambda_{\varepsilon_n}$.

Remark 2.3. Note that the definition (2.5) of the quasiconvexification of $\overline{W}$ does not depend upon the specific choice of $(0,1)^N$ as base domain (cf. e.g; [2]), and thus that $u_\xi := (\xi_{\alpha\beta}x_\beta, 0)$ is a minimizer for (2.6). In particular,

$$\Lambda = \int_\omega Q_{2,3}\overline{W}(D_\beta u_\xi)\, dx_\alpha = Q_{2,3}\overline{W}\begin{pmatrix} \xi \\ \text{---} \\ 0 \end{pmatrix}\mathcal{L}^2(\omega),$$

where $\begin{pmatrix} \xi \\ \text{---} \\ 0 \end{pmatrix}$ denotes the 3×2 matrix with columns $\begin{pmatrix} \xi_{11} \\ \xi_{21} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \xi_{12} \\ \xi_{22} \\ 0 \end{pmatrix}$.

This paper investigates the consequences of adopting a different scaling of the dependent fields, namely,

$$\begin{cases} (\hat{u}_\varepsilon)_\gamma(x_\alpha, x_3) & := u_\gamma(\varepsilon)(x_1, x_2, \varepsilon x_3), \\ \\ (\hat{u}_\varepsilon)_3(x_\alpha, x_3) & := \varepsilon u_3(\varepsilon)(x_1, x_2, \varepsilon x_3). \end{cases} \tag{2.7}$$

This is the usual scaling adopted in the context of linearized elasticity because it preserves the linearized strain structure of the strain tensor, and it greatly facilitates the ensuing analysis in a linear framework (cf. [7]). Specifically, the matrix with 3×3 entries

$$e_{ij}(u(\varepsilon)) := \frac{1}{2}\left(\frac{\partial u_i(\varepsilon)}{\partial x_j} + \frac{\partial u_j(\varepsilon)}{\partial x_i}\right)$$

in the unscaled configuration $\Omega(\varepsilon)$ becomes

$$e^\varepsilon(\hat{u}_\varepsilon) := \begin{pmatrix} e_{\gamma\beta}(\hat{u}_\varepsilon) & \frac{1}{\varepsilon}e_{\gamma 3}(\hat{u}_\varepsilon) \\ \\ \frac{1}{\varepsilon}e_{3\beta}(\hat{u}_\varepsilon) & \frac{1}{\varepsilon^2}e_{33}(\hat{u}_\varepsilon) \end{pmatrix}, \tag{2.8}$$

with obvious notation.

The problem rescaled as in (2.7) now reads as

(2.9)

$$\Lambda_\varepsilon = \inf\left\{\int_\Omega W\begin{pmatrix} D_\beta v_\gamma & \frac{1}{\varepsilon}D_3 v_\gamma \\ \\ \frac{1}{\varepsilon}D_\beta v_3 & \frac{1}{\varepsilon^2}D_3 v_3 \end{pmatrix} dx : \begin{cases} v_\gamma = \xi_{\gamma\beta}x_\beta \\ \\ v_3 = 0 \end{cases} \text{on } \partial\omega\times(-1,1)\right\}.$$

Remark 2.4. The reader will undoubtedly object that the new rescaling (2.7) and the customary one (2.2) do not correspond to the same class of admissible "loads". Indeed, if for example body loads of the form $\int_{\Omega(\varepsilon)} f\cdot w\, dx$ were added to the elastic energy (2.1), then the scaling in (2.2) would allow for loads of order 1 in ε in all directions, while that in (2.7) would only allow for transverse loads of order ε. Equivalently, it may be said that the Γ–convergence process must be tailored to the size of the loads. Nevertheless, we took care, in our choice of boundary conditions (a form of loading), to impose 0 displacements in the transverse direction, so that both scalings are adequate from the standpoint of the limit process.

Given a specific sequence $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0^+$ and a Borel measurable W with p–growth ($1 < p < \infty$) in the sense of (2.4), we assume in Section 3 that, for any open set $A \subset \omega$, $F_{\{\varepsilon_n\}}(v; A)$, the $\Gamma(L^p)$–liminf of

$$F_{\varepsilon_n}(v; A) := \int_{A\times(-1,1)} W\begin{pmatrix} D_\beta v_\gamma & \frac{1}{\varepsilon_n}D_3 v_\gamma \\ \\ \frac{1}{\varepsilon_n}D_\beta v_3 & \frac{1}{\varepsilon_n^2}D_3 v_3 \end{pmatrix} dx, \tag{2.10}$$

which is, we recall, defined as

(2.11)

$$F_{\{\varepsilon_n\}}(v;A) := \inf_{\{v_n\}} \Big\{ \liminf_{n\to\infty} F_{\varepsilon_n}(v_n;A) : v_n \to v \text{ strongly in } L^p(A\times(-1,1);\mathbb{R}^3) \Big\},$$

is impervious to lateral boundary conditions; specifically we assume that, for any $v \in W^{1,p}(A;\mathbb{R}^2)\times\mathbb{R}$,

$$\begin{aligned} F_{\{\varepsilon_n\}}(v;A) \quad &= \quad \inf_{\{v_n\}} \Big\{ \liminf_{n\to\infty} F_{\varepsilon_n}(v_n;A) :\ v_n \to v \text{ strongly in} \\ &\qquad L^p(A\times(-1,1);\mathbb{R}^3);\ v_n = v \text{ on } \partial A\times(-1,1) \Big\}. \end{aligned} \tag{2.12}$$

Remark 2.5. Note that, in contrast with (2.11), the test sequences in (2.12) are constrained to take the same values as the target v on the lateral boundary of $A\times(-1,1)$.

Also note that the above assumption is met by E_{ε_n} as already pointed out in Remark 2.2.

Then Lemma 3.1 in Section 3 asserts that $F_{\{\varepsilon_n\}}(v;\cdot)$ is a local functional, that is,

$$F_{\{\varepsilon_n\}}(v;A) = \begin{cases} 2\int_A W_{\{\varepsilon_n\}}(D_\beta \hat{v}_\gamma)\, dx_\alpha, & v = (\hat{v}, c) \in W^{1,p}(A;\mathbb{R}^2)\times\mathbb{R}, \\ \infty, & \text{otherwise}, \end{cases} \tag{2.13}$$

for some energy density $W_{\{\varepsilon_n\}} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ and, further, that $W_{\{\varepsilon_n\}}$ is independent of the chosen sequence $\{\varepsilon_n\}$ and given by

$$W_{\{\varepsilon_n\}}(\hat{M}) = Q_{2,2}\hat{W}(\hat{M}),\ \hat{M} \in \mathbb{R}^{2\times 2}, \tag{2.14}$$

with

$$\hat{W}(\hat{M}) := \inf_{y\in\mathbb{R}^2, z\in\mathbb{R}^3} W \begin{pmatrix} \hat{M}_{\gamma\beta} & z_\gamma \\ y_\beta & z_3 \end{pmatrix}. \tag{2.15}$$

Remark 2.6. In view of the coercivity of W (see (2.4)), the presence of the factor $1/\varepsilon_n$ in front of all derivatives of v_3 in the definition (2.10) of F_{ε_n} constrains the target v in a manner such that v_α must be independent of x_3 while v_3 must actually be constant, hence the functional domain specified in (2.13), precisely, $F_{\{\varepsilon_n\}}(v;A) < +\infty$ if and only if $v \in W^{1,p}(A;\mathbb{R}^2)\times\mathbb{R}$.

If (2.12) is assumed to hold true for *any sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \searrow 0^+$, and if $\{\hat{u}_n\}$ is a sequence of approximate minimizers for (2.9) with $\varepsilon_n = 1/n$, so that

$$\Lambda_{\frac{1}{n}} \le F_{\frac{1}{n}}(\hat{u}_n;\omega) \le \Lambda_{\frac{1}{n}} + \frac{1}{n},$$

then, setting

$$\begin{cases} (u_n)_\gamma(x) := (\hat{u}_n)_\gamma(x), \\ \varepsilon(u_n)_3(x) := (\hat{u}_n)_3(x), \end{cases}$$

both sequences $\{\hat{u}_n\}$ and $\{u_n\}$ may be viewed, through "descaling", as minimizing sequences for the original unscaled sequence, or, rather, for

$$\Lambda_{\frac{1}{n}} = \inf \left\{ n \int_{\Omega(1/n)} W(Dv)\, dx : \begin{cases} v_\gamma = \xi_{\gamma\beta} x_\beta \\ v_3 = 0 \end{cases} \text{on } \partial\omega\times(-1/n,1/n) \right\}.$$

Note that the factor n in front of the integral cancels out with the Jacobian of the n–dilation, i.e. $1/n$, during rescaling.

Thus, if $u \in W^{1,p}(\omega;\mathbb{R}^3)$ is the strong $L^p(\omega\times(-1,1);\mathbb{R}^3)$–limit of a subsequence of u_{k_n} of u_n, then, according to Remark 2.2,

$$2\int_\omega Q_{2,3}\overline{W}(D_\beta u)\,dx_\alpha = \Lambda, \tag{2.16}$$

where Λ is defined in (2.6). Furthermore, setting $\hat{u} := (u,0)$, the sequence $\{\hat{u}_{k_n}\}$ converges strongly in $L^p(\Omega;\mathbb{R}^3)$ to $\hat{u}$. From the very definition of $\Gamma(L^p)$–liminf, we have

$$F_{\{\frac{1}{k_n}\}}(\hat{u};\omega) \le \liminf_{n\to\infty} F_{\frac{1}{k_n}}(\hat{u}_{k_n};\omega) = \liminf_{n\to\infty}\Lambda_{\frac{1}{k_n}} = \lim_{n\to\infty}\Lambda_{\frac{1}{n}} = \Lambda. \tag{2.17}$$

Now, if $v \in W^{1,p}(A;\mathbb{R}^2)\times\{0\}$, then, according to (2.12) applied to $F_{\{\frac{1}{k_n}\}}(v;\omega)$ there exists a subsequence of $\{k_n\}$ (labelled $\{k_n'\}$) and a corresponding sequence $v_n \in W^{1,p}(\omega\times(-1,1);\mathbb{R}^3)$ such that $v_n \to v$, strongly in $L^p(\omega\times(-1,1);\mathbb{R}^3)$ with

$$\begin{cases} (v_n)_\gamma = \xi_{\gamma\beta}x_\beta \\ & \text{on } \partial\omega\times(-1,1) \\ (v_n)_3 = 0 \end{cases}$$

and

$$F_{\{\frac{1}{k_n}\}}(v;\omega) = \lim_{n\to\infty} F_{\frac{1}{k_n'}}(v_n;\omega).$$

Now,

$$\Lambda_{\frac{1}{k_n'}} \le F_{\frac{1}{k_n'}}(v_n;\omega),$$

so that, in view of (2.17),

$$F_{\{\frac{1}{k_n}\}}(\hat{u};\omega) \le \Lambda = \lim_{n\to\infty}\Lambda_{\frac{1}{k_n}} = \lim_{n\to\infty}\Lambda_{\frac{1}{k_n'}} \le F_{\{\frac{1}{k_n}\}}(v;\omega). \tag{2.18}$$

In particular, taking $v = \hat{u}$, we conclude that

$$F_{\{\frac{1}{k_n}\}}(\hat{u};\omega) = \Lambda. \tag{2.19}$$

Since (2.12) has been assumed to hold true for $\{k_n\}$, (2.18), (2.19), (2.13), (2.14) imply that

$$\Lambda = 2\int_\omega Q_{2,2}\hat{W}(D_\beta\hat{u})\,dx_\alpha \le 2\int_\omega Q_{2,2}\hat{W}(D_\beta v)\,dx_\alpha. \tag{2.20}$$

In other words, $\hat{u}$ is a minimizer for

$$\inf\left\{\int_\omega Q_{2,2}\hat{W}(D_\beta v)\,dx_\alpha : \quad v_\gamma = \xi_{\gamma\beta}x_\beta \text{ on } \partial\omega\right\}.$$

In the spirit of Remark 2.3,

$$\int_\omega Q_{2,2}\hat{W}(D_\beta\hat{u})\,dx_\alpha = Q_{2,2}\hat{W}(\xi)\mathcal{L}^2(\omega), \tag{2.21}$$

where ξ is the 2×2 matrix with entries $\xi_{\alpha\beta}$.

In view of (2.16), (2.20),

$$2\int_\omega Q_{2,3}\overline{W}(D_\beta u)\,dx_\alpha = \Lambda = 2\int_\omega Q_{2,2}\hat{W}(D_\beta\hat{u})\,dx_\alpha,$$

so that, appealing to Remark 2.3 and to (2.21), we finally obtain

$$Q_{2,3}\overline{W}\begin{pmatrix}\xi\\ \text{---}\\ 0\end{pmatrix} = Q_{2,2}\hat{W}(\xi), \quad \xi\in\mathbb{R}^{2\times2}. \tag{2.22}$$

In Section 3 we exhibit a functional with quadratic growth $p = 2$ for which (2.22) *does not hold true*. Consequently, our premise is incorrect and, for at least one sequence $\{\varepsilon_n\}$, $F_{\{\varepsilon_n\}}$, the $\Gamma(L^p)$–liminf of F_{ε_n} defined in (2.10), will fail to be impervious to boundary conditions in the sense of (2.12). We will have thus established the following

Theorem 2.7. *There exists a continuous function* $W : \mathbb{R}^{3\times 3} \mapsto \mathbb{R}$ *satisfying*

$$\frac{1}{C}|F|^p \le W(F) \le C(1+|F|^p), \quad F \in \mathbb{R}^{3\times 3},$$

for some $C > 0$ *such that for all affine functions* $v \in W^{1,p}(\omega;\mathbb{R}^2)\times\mathbb{R}$, *there exists a sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \to 0^+$ *and an open subset* $A \subset \omega$ *for which the* $\Gamma(L^p)$*–liminf* $F_{\{\varepsilon_n\}}(v, A)$ *is* not *equal to*

$$\inf_{\{v_n\}}\Big\{\liminf_{n\to\infty} F_{\varepsilon_n}(v_n; A) : \; v_n \to v \quad \text{strongly in } L^p(A\times(-1,1);\mathbb{R}^3); \\ v_n = v \text{ on } \partial A \times (-1,1)\Big\}.$$

Remark 2.8. The rescaling of linearized elasticity leads to a limit behavior which critically depends upon the kind of boundary conditions that are imposed on the approximating sequences. From a mechanical standpoint, it amounts to a statement of non existence of a limit model, at first order in the thickness, under such a scaling. This is because the limit kinematics that are imposed by the scaling are too stringent: they force the transverse limit displacement to be 0 (or a constant); but the minimum displacement field, on the contrary, wants to experience a transverse deflection, which the scaling (2.2) allows, but not the scaling (2.7).

3. A representation formula for the scaling of linearized elasticity

In this section we consider a fixed sequence $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0$ and an energy density W such that F, the $\Gamma(L^p)$–liminf of F_{ε_n} defined in (2.10), (2.11) is impervious to boundary conditions; specifically, W is assumed to be such that

(H1) (2.4) is satisfied;
(H2) for any affine $v \in W^{1,p}(\omega;\mathbb{R}^3)\times\mathbb{R}$, $F_{\{\varepsilon_n\}}(v;\cdot)$ (defined, for every open subset $A \subset \omega$, by (2.11)) is also given by (2.12).

We then prove the following

Lemma 3.1. *Under hypotheses (H1), (H2), for any* $v = (\hat{v}, c) \in W^{1,p}(\omega;\mathbb{R}^2)\times\mathbb{R}$ *and for every open set* $A \subset \omega$, *the* $\Gamma(L^p)$*–liminf* $F_{\{\varepsilon_n\}}(v; A)$ *is given by*

$$F_{\{\varepsilon_n\}}(v; A) = 2\int_A Q_{2,2}\hat{W}(D_\beta \hat{v})\, dx_\alpha$$

with $\hat{W}$ *defined by (2.15) and its quasiconvexification* $Q_{2,2}\hat{W}$ *by (2.5).* □

Proof. Step 1. Assume first that W is continuous. Fix $v \in W^{1,p}(\omega;\mathbb{R}^2)\times\mathbb{R}$.

It is immediate from the sequentially $W^{1,p}$–weak lower semi–continuous character of the quasiconvexification of $\hat{W}$ (cf. e.g. [1]) and the coercivity of W (see (H1)) that $F_{\{\varepsilon_n\}}(v; A) \ge 2\int_A Q_{3,3}\hat{W}(D_\beta v)\, dx_\alpha$. But a straightforward application of Fubini's theorem in (2.5) would show that $Q_{3,3}\hat{W}(\hat{M}) \ge Q_{2,2}\hat{W}(\hat{M})$, $\hat{M} \in \mathbb{R}^{2\times 2}$. Consequently,

$$F_{\{\varepsilon_n\}}(v; A) \ge 2\int_A Q_{2,2}\hat{W}(D_\beta v)\, dx_\alpha.$$

We must prove the opposite inequality. For notational convenience, we identify, from now onward in the proof, a target v with its first two components since the third is a constant.

It suffices to prove that

$$F_{\{\varepsilon_n\}}(v;A) \le 2\int_A \hat{W}(D_\beta v)\,dx_\alpha, \tag{3.1}$$

because, since $F_{\{\varepsilon_n\}}(\cdot;A)$ is sequentially $W^{1,p}$–weak lower semi–continuous, we then obtain

$$F_{\{\varepsilon_n\}}(v;A) \le 2\int_A Q_{2,2}\hat{W}(D_\beta v)\,dx_\alpha.$$

In order to prove (3.1) we first consider the case where $v := \hat{M}x$. For any n, there exists y_n, z_n, such that

$$\hat{W}(\hat{M}) + 1/n \ge W\begin{pmatrix} \hat{M}_{\beta\gamma} & z_{n\gamma} \\ y_{n\beta} & z_{n3} \end{pmatrix}.$$

Note that, by virtue of the coercivity of W (see (H1)), $\{y_n\}, \{z_n\}$ are bounded sequences. At the expense of the possible extraction of a subsequence still indexed by n, we are thus at liberty to further assume that

$$y_n \to y, \quad z_n \to z.$$

Set

$$u_{m,n}(x_\alpha, x_3) := \begin{pmatrix} v_\gamma(x_\alpha) \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon_m z_{n\gamma} x_3 \\ \varepsilon_m y_{n,\lambda} x_\lambda + \varepsilon_m^2 z_{n3} x_3 \end{pmatrix}.$$

Then, through diagonalization, there exists a sequence

$$u_n := u_{m(n),n}$$

such that

$$\lim_{n\to\infty} u_n = \begin{pmatrix} v_\gamma(x_\alpha) \\ 0 \end{pmatrix} \quad \text{strongly in } L^p(A\times(-1,1);\mathbb{R}^3),$$

and, since $\{\varepsilon_{m(n)}\}$ is a subsequence of $\{\varepsilon_n\}$,

$$\begin{aligned} &F_{\{\varepsilon_n\}}(v;A) \\ &\le \liminf_{n\to\infty} \int_{A\times(-1,1)} W\begin{pmatrix} D_\beta(u_n)_\gamma & \frac{1}{\varepsilon_m(n)} D_3(u_n)_\gamma \\ \frac{1}{\varepsilon_m(n)} D_\beta(u_n)_3 & \frac{1}{\varepsilon_m(n)^2} D_3(u_n)_3 \end{pmatrix} dx \\ &= \liminf_{n\to\infty} 2\mathcal{L}^2(A) W\begin{pmatrix} \hat{M}_{\beta\gamma} & z_{n\gamma} \\ y_{n\beta} & z_{n3} \end{pmatrix} \le 2\mathcal{L}^2(A)\hat{W}(\hat{M}). \end{aligned}$$

Thus, we have established that, whenever v is affine, i.e., $v = \hat{M}x$, then

$$F_{\{\varepsilon_n\}}(v;A) \le 2\mathcal{L}^2(A)\hat{W}(\hat{M}). \tag{3.2}$$

In particular, if $Q'(a,r)$ is a given open square in ω, we conclude, in view of (3.2) and (H2) to the existence of a subsequence $\{\varepsilon'_n\}$ of $\{\varepsilon_n\}$ and of a sequence $v_n \to v$ strongly in $L^p(A\times(-1,1);\mathbb{R}^3)$ such that $v_n = v$ on $\partial A \times (-1,1)$, with

$$\lim_{n\to\infty} F_{\varepsilon'_n}(v_n; Q'(a,r)) \le 2\mathcal{L}^2(Q'(a,r))\hat{W}(\hat{M}). \tag{3.3}$$

Then, upon rescaling and up to a translation, the same conclusion holds true for *the same sequence* $\{\varepsilon'_n\}$ *on any square* $Q'(a,r) \subset \omega$.

If we now consider an open set $A \subset \omega$, an integer m and appeal to Vitali's Covering theorem, we may cover A with squares like $Q'(a,r)$, up to a set of measure at most $1/m$, so that

$$\begin{cases} A_m := \cup_{i=1}^{N(m)} \overline{Q'(a_i^m, r_i^m)} \subset\subset A, \\ \overline{Q'(a_i^m, r_i^m)} \cap \overline{Q'(a_j^m, r_j^m)} = \emptyset, \ i \neq j, \\ \mathcal{L}^2(A \setminus A_m) \leq \frac{1}{m}. \end{cases}$$

In view of (3.3), there exist $N(m)$ sequences $\{v_{n,m}^i\}$ with

$$\begin{cases} v_{n,m}^i \underset{n\to\infty}{\longrightarrow} v \text{ strongly in } L^p(Q'(a_i^m, r_i^m) \times (-1,1); \mathbb{R}^3), \\ v_{n,m}^i = v \text{ on } \partial Q'(a_i^m, r_i^m) \times (-1,1), \\ \lim_{n\to\infty} F_{\varepsilon'_n}(v_{n,m}^i; Q'(a_i^m, r_i^m)) \leq 2\mathcal{L}^2(Q'(a_i^m, r_i^m))\hat{W}(\hat{M}). \end{cases}$$

Set

$$v_{n,m} := \sum_{i=1}^{N(m)} v_{n,m}^i 1_{Q'(a_i^m, r_i^m)} + v 1_{A\setminus A_m}, \tag{3.4}$$

where, for any set A, 1_A denotes the characteristic function of that set. Then, in view of (H1), (3.3), as $n \to \infty$,

$$\begin{aligned} &F_{\{\varepsilon_n\}}(v; A) \leq \liminf_{m\to\infty} \liminf_{n\to\infty} F_{\varepsilon'_n}(v_{n,m}; A) \\ &\leq \liminf_{m\to\infty} \left\{ 2 \sum_{i=1}^{N(m)} \lim_{n\to\infty} F_{\varepsilon'_n}(v_{n,m}^i; Q'(a_i^m, r_i^m)) + \mathcal{C}\mathcal{L}^2(A\setminus A_m) \right\} \\ &\leq \lim_{m\to\infty} \left\{ 2 \sum_{i=1}^{N(k(m))} \lim_{n\to\infty} F_{\varepsilon'_n}(v_{n,k(m)}^i; Q'(a_i^{k(m)}, r_i^{k(m)})) + \mathcal{C}\mathcal{L}^2(A\setminus A_{k(m)}) \right\} \\ &\leq \lim_{m\to\infty} \lim_{n\to\infty} \left\{ 2 \sum_{i=1}^{N(k(m))} F_{\varepsilon'_n}(v_{n,k(m)}^i; Q'(a_i^{k(m)}, r_i^{k(m)})) + \mathcal{C}\mathcal{L}^2(A\setminus A_{k(m)}) \right\} \\ &\leq 2\mathcal{L}^2(A)\hat{W}(\hat{M}), \end{aligned}$$

where, in the third and fourth inequalities, the liminf has been replaced by a limit upon extracting an appropriate subsequence of the m–indexed sequence

$$\left\{ \sum_{i=1}^{N(m)} \lim_{n\to\infty} F_{\varepsilon'_n}(v_{n,m}^i; Q'(a_i^m, r_i^m)) \right\}.$$

In view of (3.4), (3.5), with

$$\lambda := \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{N(m)} F_{\varepsilon'_n}(v_{n,m}^i; Q'(a_i^m, r_i^m)),$$

by means of a diagonalization argument, there exists an increasing subsequence $\{m(n)\}$ with $m(n) \underset{n\to\infty}{\longrightarrow} \infty$ such that, upon setting $v_n := v_{n,k(m(n))}$, we have

$v_n = v$ on $\partial A \times (-1,1)$, and

$$\begin{cases} \lim_{n\to\infty} ||v_n - v||_{L^p(A\times(-1,1);\mathbb{R}^3)} = 0, \\ \lambda = \lim_{n\to\infty} 2 \sum_{i=1}^{N(k(m(n)))} F_{\varepsilon'_n}(v^i_{n,k(m(n))}; Q'(a_i^{k(m(n))}, r_i^{k(m(n))})) + \mathcal{C}\mathcal{L}^2(A \setminus A_{k(m(n))}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \leq 2\mathcal{L}^2(A)\hat{W}(\hat{M}). \end{cases}$$

We conclude that

$$\begin{aligned} &F_{\{\varepsilon_n\}}(v; A) \leq \lim_{n\to\infty} F_{\varepsilon'_n}(v_n; A) \\ &\leq \lim_{n\to\infty} \left\{ 2 \sum_{i=1}^{N(k(m(n)))} F_{\varepsilon'_n}(v^i_{n,k(m(n))}; Q'(a_i^{k(m(n))}, r_i^{k(m(n))})) + \mathcal{C}\mathcal{L}^2(A \setminus A_{k(m(n))}) \right\} \\ &\leq \mathcal{L}^2(A)\hat{W}(\hat{M}). \end{aligned}$$

We have thus shown that if A is any open subset of ω there exists a sequence $\{v_n\} \subset W^{1,p}(A \times (-1,1); \mathbb{R}^3)$ such that $v_n \to v$ strongly in $L^p(A \times (-1,1); \mathbb{R}^3)$ and $v_n = v$ on $\partial A \times (-1,1)$, and also such that

$$F_{\{\varepsilon_n\}}(v; A) \leq \lim_{n\to\infty} F_{\varepsilon'_n}(v_n; A) \leq 2\mathcal{L}^2(A)\hat{W}(\hat{M}).$$

Let us emphasize the essential fact that the sequence $\{\varepsilon'_n\}$ is *the same* whatever A is. Since the relevant sequences match the target on the lateral boundary $\partial T \times (-1,1)$, the same result clearly holds true if v is *continuous and piecewise affine on a triangulation of the plane*: there exists a sequence $v_n \to v$ strongly in $L^p(A \times (-1,1); \mathbb{R}^3)$, $v_n = v$ on $\partial A \times (-1,1)$, such that

$$F_{\{\varepsilon_n\}}(v; A) \leq \lim_{n\to\infty} F_{\varepsilon'_n}(v_n; A) \leq \int_{A\times(-1,1)} \hat{W}(D_\beta v)\, dx_\alpha.$$

It now remains to consider the case where v is an arbitrary element of $W^{1,p}(\omega; \mathbb{R}^2)$. In such a case, let $\{v_k\}$ be a sequence of piecewise affine and continuous functions such that

$$v_k \underset{k\to\infty}{\longrightarrow} v \text{ strongly in } W^{1,p}(A \times (-1,1); \mathbb{R}^3).$$

For each k, there exists a sequence $v_{n,k} \underset{n\to\infty}{\longrightarrow} v_k$ strongly in $L^p(A \times (-1,1); \mathbb{R}^3)$ such that

$$F_{\{\varepsilon_n\}}(v_k; A) \leq \lim_{n\to\infty} F_{\varepsilon'_n}(v_{n,k}; A) \leq \int_{A\times(-1,1)} \hat{W}(D_\beta v_k)\, dx_\alpha.$$

By virtue of the weak lower semi–continuous character of $F_{\{\varepsilon_n\}}(\cdot; A)$,

$$F_{\{\varepsilon_n\}}(v; A) \leq \liminf_{k\to\infty} F_{\varepsilon_n}(v_k; A) \leq \liminf_{k\to\infty} \int_{A\times(-1,1)} \hat{W}(D_\beta v_k)\, dx_\alpha.$$

Since W is continuous so is $\hat{W}$, thus, by virtue of (H1), Lebesgue's Dominated Convergence Theorem implies that

$$F_{\{\varepsilon_n\}}(v; A) \leq 2 \int_A \hat{W}(D_\beta v)\, dx_\alpha,$$

which establishes (3.1).

Step 2. Finally, we assume that W is Borel measurable and satisfies (H1) and (H2). It is a known fact — although hard to find explicitly stated in the literature; it can be found in a piecemeal manner in [4] and in [5]: see Propositions 6.7, 9.2

and theorem 12.5 in [4]; see also Theorems 2.1, 2.3 in [8] and Remark 1.6(iii) in [5]— that for all open subset $A \subset \Omega$ and for all $v \in W^{1,p}(A \times (-1,1); \mathbb{R}^2) \times \mathbb{R}$

$$\int_{A\times(-1,1)} Q_{3,3}W(Dv)\,dx$$
$$= \inf\left\{ \liminf_{n\to\infty} \int_{A\times(-1,1)} W(Dv_n)\,dx : v_n \rightharpoonup v \text{ in } W^{1,p}(A\times(-1,1);\mathbb{R}^3) \right\}.$$

Hence

$$\begin{aligned} F_{\{\varepsilon_n\}}(v;A) &= \Gamma(L^p)\text{–}\liminf F_{\varepsilon_n}(v;A) \\ &= \Gamma(L^p)\text{–}\liminf Q_{3,3}F_{\varepsilon_n}(v;A), \end{aligned}$$

where

$$Q_{3,3}F_{\varepsilon_n}(v;A) := \int_{A\times(-1,1)} Q_{3,3}W \begin{pmatrix} D_\beta v_\gamma & \frac{1}{\varepsilon_n} D_3 v_\gamma \\ \frac{1}{\varepsilon_n} D_\beta v_3 & \frac{1}{\varepsilon_n^2} D_3 v_3 \end{pmatrix} dx.$$

Now $Q_{3,3}W$ is a continuous function which still satisfies the growth hypothesis (H1), therefore Step 1 of this proof applies and we have

$$F_{\{\varepsilon_n\}}(v;A) = \int_A Q_{2,2}[\widehat{Q_{3,3}W}](D_\beta v)\,dx_\alpha.$$

It remains to show that

$$Q_{2,2}[\widehat{Q_{3,3}W}](\hat{M}) = Q_{2,2}\hat{W}(\hat{M})$$

for all $\hat{M} \in \mathbb{R}^{2\times 2}$. Fix $\hat{M} \in \mathbb{R}^{2\times 2}$. It is clear that

$$Q_{2,2}[\widehat{Q_{3,3}W}](\hat{M}) \le Q_{2,2}\hat{W}(\hat{M}).$$

Conversely, given $\varepsilon > 0$ find $y \in \mathbb{R}^2, z \in \mathbb{R}^3$, such that

$$\widehat{Q_{3,3}W}(\hat{M}) + \varepsilon \ge Q_{3,3}W \begin{pmatrix} \hat{M}_{\gamma\beta} & z_\gamma \\ y_\beta & z_3 \end{pmatrix},$$

and choose $\varphi \in C_0^\infty((0,1)^3;\mathbb{R}^3)$ such that

$$Q_{3,3}W \begin{pmatrix} \hat{M}_{\gamma\beta} & z_\gamma \\ y_\beta & z_3 \end{pmatrix} + \varepsilon \ge \int_{(0,1)^3} W\left(\begin{pmatrix} \hat{M}_{\gamma\beta} & z_\gamma \\ y_\beta & z_3 \end{pmatrix} + D\varphi \right) dx.$$

We conclude that

$$\begin{aligned} \widehat{Q_{3,3}W}(\hat{M}) + 2\varepsilon \quad &\ge \int_{(0,1)^3} W\left(\begin{pmatrix} \hat{M}_{\gamma\beta} & z_\gamma \\ y_\beta & z_3 \end{pmatrix} + D\varphi \right) dx \\ &\ge \int_0^1 \Big[\int_{(0,1)^2} \hat{W}(\hat{M} + D_\beta\varphi_\gamma)\,dx_\alpha \Big] dx_3 \\ &\ge Q_{2,2}\hat{W}(\hat{M}). \end{aligned}$$

Letting $\varepsilon \to 0^+$ we get

$$\widehat{Q_{3,3}W}(\hat{M}) \ge Q_{2,2}\hat{W}(\hat{M}),$$

and thus

$$Q_{2,2}[\widehat{Q_{3,3}W}](\hat{M}) \ge Q_{2,2}\hat{W}(\hat{M}).$$

□

Proof of Theorem 2.7. We first remark that, since

$$\hat{W}(\xi) = \inf_{z\in\mathbb{R}^2} \overline{W} \begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix}, \quad \xi \in \mathbb{R}^{2\times 2},$$

then

$$Q_{2,2}\hat{W}(\xi) = Q_{2,2}\left[\inf_{z\in\mathbb{R}^2} \overline{W}\begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix}\right] \le \inf_{z\in\mathbb{R}^2} Q_{2,3}\overline{W}\begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix},$$

thus, in view of (2.22), which holds true if (H1) (that is (2.4)) and (H2) (that is (2.12)) hold true,

$$Q_{2,3}\overline{W}\begin{pmatrix} \xi \\ \text{—} \\ 0 \end{pmatrix} = \min_{z\in\mathbb{R}^2} Q_{2,3}\overline{W}\begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix}. \tag{3.5}$$

Now, take

$$W\begin{pmatrix} \xi & y \\ z & t \end{pmatrix} := Z\begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix} + |y|^p + t^p, \quad \xi \in \mathbb{R}^{2\times 2}, y \in \mathbb{R}^{2\times 1}, z \in \mathbb{R}^{1\times 2}, t \in \mathbb{R},$$

with Z convex and satisfying, for some $C > 0$,

$$\frac{1}{C}(|\xi|^p + |z|^p) \le Z\begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix} \le C(1 + |\xi|^p + |z|^p),$$

so that

$$\overline{W} = Z,\ Q_{2,3}Z = Z = Q_{2,3}\overline{W},$$

and (3.5) becomes

$$Z\begin{pmatrix} \xi \\ \text{—} \\ 0 \end{pmatrix} = \min_{z\in\mathbb{R}^2} Z\begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix}. \tag{3.6}$$

But (3.6) is violated for example by

$$Z\begin{pmatrix} \xi \\ \text{—} \\ z \end{pmatrix} = |\xi|^p + |z - z_0|^p + K, z_0 \ne 0, K \text{ large enough.}$$

□

Remark 3.2. A proof based on a blow–up argument in the spirit of [11] would show that, if (H2) is replaced by

(H3) for any $v \in W^{1,p}(\omega;\mathbb{R}^3)$, $F_{\{\varepsilon_n\}}(v;\cdot)$ extends to a nonnegative Radon measure on ω,

then the conclusion of Lemma 3.1 also holds true. In other words, if (H2) holds true, then $F_{\{\varepsilon_n\}}(v;\cdot)$ is local and it is given, for any $A \subset \omega$, by $F_{\{\varepsilon_n\}}(v;A) = 2\int_A Q_{2,2}\hat{W}(D_\beta v)\, dx_\alpha$. But even if it is merely supposed that $F_{\{\varepsilon_n\}}(v;\cdot)$ is local, then it must be given by that expression.

It would thus be tempting to revisit the proof of Theorem 2.7 with the hypothesis that, for every sequence $\{\varepsilon_n\}$, (H3) holds in lieu of (H2) (or (2.12)). Unfortunately, we are unable at this point to state a similar theorem, that is to assert the existence of a sequence $\{\varepsilon_n\}$ for which $F_{\{\varepsilon_n\}}(v;\cdot)$ is not a local functional.

4. The case of symmetrized gradients.

The pathology described in Section 2 and summarized in Theorem 2.7 disappears in the symmetric setting, that is that where the energy is *a priori* a function of the symmetrized gradients. In the latter setting, the equilibrium displacement field $u(\varepsilon)$, a $\mathbb{R}^3$–valued field defined on $\Omega(\varepsilon)$, then minimizes the elastic energy

$$\int_{\Omega(\varepsilon)} W(e(w))\, dx, \tag{4.1}$$

where the strain tensor $e(w)$ is $e(w) := 1/2(\nabla w + \nabla w^T)$. In such a case, the scaling of linearized elasticity preserves the strain structure as already noted in (2.8). Considering an elastic energy $W : \mathbb{R}^{3\times 3}_s \mapsto \mathbb{R}$ with p–growth (on the space $\mathbb{R}^{3\times 3}_s$ of 3×3–symmetric matrices) in the sense of (2.4), it is then straightforward, in the spirit of [12], to prove that the $\Gamma(L^p)$–limit of

$$G_\varepsilon(v; A) := \int_{A\times(-1,1)} W(e^\varepsilon(\hat{u}_\varepsilon))\, dx$$

is given by

$$G(v; A) := \begin{cases} 2\int_A Q_{2,2}\hat{W}_s(e_{\gamma\beta}(v))\, dx_\alpha, & v \in KL_p(A), \\ \\ \infty, & \text{otherwise}, \end{cases} \tag{4.2}$$

where

$$\hat{W}_s(\hat{M}) := \inf_{y\in\mathbb{R}^2, z\in\mathbb{R}} W\begin{pmatrix} \hat{M}_{\gamma\beta} & y_\gamma \\ y_\beta & z \end{pmatrix}, \quad \hat{M} \in \mathbb{R}^{2\times 2}_s.$$

In (4.2), the space $KL_p(A)$ (of Kirchhoff–Love type displacements) is defined, for any open set $A \subset \omega$, as

$$KL_p(A) := \{w \in W^{1,p}(A\times(-1,1); \mathbb{R}^3) : e_{\alpha 3}(w) = e_{33}(w) = 0\}.$$

Remark 4.1. The set $KL_p(A)$ is equivalently characterized as

$$KL_p(A) := \Big\{w \in W^{1,p}(A\times(-1,1); \mathbb{R}^3) : w_\gamma = \zeta_\gamma(x_\beta) - x_3\frac{\partial\psi}{\partial x_\gamma}(x_\alpha), \\ w_3 = \psi(x_\alpha), \zeta \in W^{1,p}(A; \mathbb{R}^2),\ \psi \in W^{2,p}(A)\Big\}.$$

If, however, we adopt for the rescaling of (4.1) the nonlinear scaling (2.2), then, in contrast to what happens in the case of an energy of the type (2.1), the rescaled functional

$$H_\varepsilon(v; A) := \int_{A\times(-1,1)} W\begin{pmatrix} e_{\gamma\beta}(v) & \frac{1}{2}(\frac{1}{\varepsilon}\frac{\partial v_\gamma}{\partial x_3} + \frac{\partial v_3}{\partial x_\gamma}) \\ \\ \frac{1}{2}(\frac{1}{\varepsilon}\frac{\partial v_\beta}{\partial x_3} + \frac{\partial v_3}{\partial x_\beta}) & \frac{1}{\varepsilon}\frac{\partial v_3}{\partial x_3} \end{pmatrix} dx$$

is easily seen to $\Gamma(L^p)$–converge to

$$H(v; A) := \begin{cases} 2\int_A Q_{2,2}\hat{W}_s(e_{\gamma\beta}(v))\, dx_\alpha, & v \in W^{1,p}(A; \mathbb{R}^3), \\ \\ \infty, & \text{otherwise}. \end{cases}$$

Note that the target field is not of the Kirchhoff–Love type; this is because the bounds on the approximating sequence (converging strongly to v in $L^p(A\times(-1,1); \mathbb{R}^3)$) in the definition of $H(v; A)$ only imply that v is independent of x_3 and not that $e_{\alpha 3}(v) = e_{33}(v) = 0$.

Remark 4.2. In the symmetric case, Γ–convergence *does not* give rise to the pathology evidenced in the gradient (nonsymmetric) case. Only the limit kinematics (Kirchhoff–Love versus x_3–independent fields) permit to distinguish between the various rescalings which always lead to an expressly computable $\Gamma(L^p)$–limit.

Remark 4.3. In the spirit of Remark 3.2, it is fortunate that the computation of the $\Gamma(L^p)$–limit of $H_\varepsilon(v; A)$ in the case of symmetrized gradient can be performed globally using arguments similar to those in [12], that is, without first proving the local character of the limit — an expression similar to (2.13) — because we do not know how to prove the latter directly.

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