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AN ASYMPTOTIC STUDY OF THE DEBONDING OF THIN FILMS

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ABSTRACT. We examine the asymptotic behavior of a bilayer thin film using the notion of $\Gamma-$convergence. We allow for debonding at the interface, but penalize it using an interfacial energy; thus the functional we consider consists of the elastic energy of the two layers and the interfacial energy with penalizes debonding. We show that the asymptotic theory or $\Gamma-$limit depends on the particular form of the interfacial energy, and derive detailed results for both the cohesive and the brittle interface.

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1. INTRODUCTION

Thin films and coatings are being used in an ever increasing variety of applications. These films are often subjected to severe internal stresses due to a host of reasons including differential thermal expansion and epitaxial mismatch. These stresses in turn can lead to failure: the interface between the film and substrate can fail leading to the debonding of the film from the substrate, the film may fracture, the substrate may fracture and finally mismatch dislocations can form in the case of heteroepitaxial films. Clearly such failure is undesirable from the point of view of applications, and thus it has received much attention. A striking feature of thin films is the existence of a critical thickness below which they do not appear to fail, and much theoretical effort has gone into identifying this critical thickness, as well as the factors which enable one to increase it.

The idea of critical thickness is intuitively clear from the point of view of scaling. Consider for example a stretched film bonded to an infinite rigid substrate. The elastic energy of this film scales as its thickness. If the film debonds from the substrate, the elastic energy goes to zero, but it creates a new surface which costs an interfacial energy. Thus debonding relieves elastic energy which decreases with thickness, but adds interfacial energy which is independent of thickness. Consequently, there is a thickness below which the cost of interfacial energy is greater than the gain of elastic energy, thus rendering debonding energetically unfavorable.

A systematic review of the fracture of thin films is provided by Hutchinson and Suo [14]. They find the critical thickness for a variety of configurations and failure modes in terms of some empirical parameters. A typical analysis begins with an ansatz about the (macroscopic) state of stress, assumes the existence of a crack, and finds the conditions under which it is energetically advantageous for the crack to grow. While these results have been found to be very useful in applications, the validity of the ansatz and the assumption of the pre-existing cracks remain to be justified.

Similar issues arise in heteroepitaxial films where the substrate and the film are made of different materials but are atomistically contiguous. Such films are widely used in modern electronics and they fail by the formation of misfit dislocations at the film-substrate interface; but once again there is a critical thickness below which such failure does not occur. This has also been studied quite extensively, see Freund [13] for a recent review. A typical analysis starts with an ansatz about the (macroscopic) state of stress, and studies the growth of a suitable dislocation loop. A slightly different approach was undertaken by Frank and van der Merwe [12] and more recently by Leo and Hu [17]; they too start with an ansatz about the state of stress, assume a particular arrangement of dislocations and compare the energies between the dislocated and undislocated states.

In this paper we examine the asymptotic behavior of a bilayer thin film allowing for the possibility of debonding using the notion of $\Gamma-$convergence. This notion has been used successfully in recent years to rigorously obtain the limit behavior of various thin films (see for example, [3, 5, 6, 11, 16, 18]), including homogeneous, heterogeneous and rough films, but do not allow possible interfacial debonding.

Consider a bilayer film consisting of two regions $\Omega^+_\varepsilon = \omega\times(0,\varepsilon h)$ and $\Omega^-_\varepsilon = \omega\times(-\varepsilon s,0)$ for some given $\omega\subset\mathbb{R}^2$. The total energy of the film under a deformation $y:\Omega^+_\varepsilon\cup\Omega^-_\varepsilon\to\mathbb{R}^3$ is given by

$$\tilde{E}_\varepsilon = \int_{\Omega^+_\varepsilon} W^+(Dy)dz + \int_{\Omega^-_\varepsilon} W^-(Dy)dz + \varepsilon^\alpha \int_\omega \Psi([\![y]\!])dz_1dz_2,$$

where ε is a (small) real number, $W^+, W^- : \mathbb{R}^{3\times3}\to\mathbb{R}$ are the elastic energy densities of the two regions, $\varepsilon^\alpha\Psi:\mathbb{R}^3\to\mathbb{R}$ is the interfacial energy which penalizes the jump in the interfacial deformation $[\![y]\!] := y(z_1,z_2,0^+) - y(z_1,z_2,0^-)$ across the interface between Ω^+_ε and Ω^-_ε, and α is a real number. Notice that we have allowed for the interfacial energy to scale with the overall thickness with some exponent α where $\alpha=0$ if the interfacial energy is independent of thickness. We have chosen both layers to be of comparable thickness (i.e., $h/s = O(1)$). Often the film is much smaller than the substrate; we comment on that case in Section 6.

We assume that the equilibrium state of the film is described by the minimizer of this functional over all admissible deformations y. We are interested in the behavior of a very thin film. Therefore, we consider some sequence $\{\varepsilon\}$ of strictly decreasing real numbers with limit 0, and study the asymptotic behavior of the minimizers of $\tilde{E}_\varepsilon$ as $\varepsilon\searrow 0$. In order to do so, it is convenient to make the following change of variables:

$$x_1 = z_1, \quad x_2 = z_2, \quad x_3 = \frac{1}{\varepsilon}z_3,$$
$$u(x) = y(z(x)), \quad E_\varepsilon = \frac{1}{\varepsilon}\tilde{E}_\varepsilon.$$

Thus,

$$E_\varepsilon = \int_{\Omega^+} W^+\Big(D_\alpha u|\frac{1}{\varepsilon}D_3u\Big)dz + \int_{\Omega^-} W^-\Big(D_\alpha u|\frac{1}{\varepsilon}D_3u\Big)dz + \varepsilon^{\alpha-1}\int_\omega \Psi([\![u]\!])dx_1dx_2,$$

where $\Omega^+ := \omega\times(0,h)$, $\Omega^- := \omega\times(-s,0)$, $D_\alpha u$ is the 3×2 matrix of partial derivatives $\partial u_i/\partial x_\alpha$, $i=1,2,3$, $\alpha=1,2$, $(A|a)$ is a 3×3 matrix whose first two columns are those of the 3×2 matrix A, the last column is the $3-$vector a, and, as before, $[\![u]\!] := u(x_1,x_2,0^+) - u(x_1,x_2,0^-)$. In this paper, we study the limit behavior of E_ε as $\varepsilon\to0$ within the framework of $\Gamma-$convergence. See Section 2 for a precise statement of the problem.

We now give a brief and non-technical description of our main results. We begin by considering the case $\alpha<1$, which includes the case $\alpha=0$ when the interfacial energy is independent of thickness. Here, the interfacial energy is very strong and goes to infinity unless the limit deformation is continuous across the interface. Further, under polynomial coercivity conditions on $W^\pm$, we observe that the bulk energy goes to infinity unless the limit transformation satisfies $D_3u=0$. Thus, the limiting functional E (the $\Gamma-$limit of E_ε) is finite only if the limit deformation is continuous across the interface and independent of x_3. We show in Theorem 2.2 that for such transformations the $\Gamma-$limit E of E_ε is 'local' so that we can write

$$E = \int_\omega \tilde{W}(D_\alpha u)dx_\alpha$$

for some function $\tilde{W}$ that we call the effective energy density of the film. The characterization of $\tilde{W}$ depends critically on the interfacial energy.

We consider in detail two types of interfacial energy. The first, studied in Section 3, is the cohesive type which includes the case where the surface energy increases continuously from zero with the jump in the deformation across the interface; for example

$$\Psi(t) = C|t|^\gamma, \quad \gamma > 1-\alpha,\ \alpha < 1.$$

There (see Theorem 3.1)

$$\tilde{W}(\overline{F}) = hQ\overline{W}^+(\overline{F}) + sQ\overline{W}^-(\overline{F})$$

where $\overline{W}^\pm(\overline{F}) := \min_{a\in\mathbb{R}^3} W^\pm(\overline{F}|a)$, and $Q\overline{W}^\pm$ is the quasiconvexification of $\overline{W}^\pm$. We will explain this result momentarily.

The second, studied in Section 4, is Griffith's brittle type interfacial energy,

$$\Psi(0) = 0 \quad \text{and} \quad \Psi(t) = 1,\ t \neq 0.$$

In Theorem 4.1 we prove that

$$\tilde{W}(\overline{F}) = \hat{W}(\overline{F}) := \inf_{\varphi,\lambda} \frac{1}{s+h}\int_{(-1/2,1/2)^2\times(-s,h)} W(\overline{F} + D_\alpha\phi|\lambda D_3\phi, x_3)dx$$

where

$$W(F,x_3) := \begin{cases} W^+(F) & x_3 > 0, \\ W^-(F) & x_3 < 0, \end{cases}$$

and the infimum is taken over $\lambda > 0$, and $\varphi \in \mathcal{C}^\infty([-1/2,1/2]^2 \times [-s,h])$ with $\varphi = 0$ on $\partial[-1/2,1/2]^2 \times [-s,h]$. We note that we need a irksome technical hypothesis (H8) to prove this result.

We also show that these two types of interfacial energy are extremal. In fact (see Lemma 2.4), for any Ψ satisfying some mild hypothesis,

$$hQ\overline{W}^+(\overline{F}) + sQ\overline{W}^-(\overline{F}) \leq \tilde{W}(\overline{F}) \leq \hat{W}(\overline{F}).$$

To understand these results, recall the case of a homogeneous thin film of thickness 2ε with no fracture [16]. Here, the relevant sequence of functionals, after a change of variables as above, is

$$E_\varepsilon(u) = \int_\Omega W\left(D_\alpha u|\frac{1}{\varepsilon}D_3 u\right)dx,$$

where $\Omega := \omega \times (-1,1)$ and the $\Gamma-$limit of this functional is

$$E(u) = 2\int_\omega Q\overline{W}(D_\alpha u)dx_\alpha.$$

Physically, as the thickness goes to zero, the deformation in the x_3 direction tends to be affine. Subsequently one can minimize this affine deformation out to obtain $\overline{W}$. It turns out that $\bar{W}$ may not be quasiconvex even if W is, due to "wrinkling" of the film. Therefore, quasiconvexification is required.

Let us now return to our bilayer film with a cohesive interface. The effective energy is $hQ\overline{W}^+ + sQ\overline{W}^-$; each part of the film has relaxed independently in spite of the absence of overall debonding. Physically, the cohesive interface allows small debonding with small energy, consequently the deformation in each layer can develop oscillations independently as long as they are small. Thus each part relaxes to the extent allowed by a common overall deformation. This suggests that in this case one should not expect a critical thickness; instead the amount of debonding would become smaller and smaller with thickness.

The situation is quite different in the case of a brittle interface. Here, the effective energy $\hat{W}$ is exactly that obtained in the case of a bilayer film with *no* possibility of debonding [6, 18]. Physically, even a small debond costs large energy. This suggests the existence of a critical thickness below which the minimizer is continuous. Unfortunately our $\Gamma-$convergence results do not prove it, and such a result would require stronger techniques.

We turn to the case $\alpha \geq 1$ in Section 5. The interfacial energy is now weak, and thus the limit energy can be finite even if the limit deformation is not continuous across the interface. However, it is still true that $D_3 u = 0$ for finite limit energy, and thus the meaningful limit deformations are

$$u(x_1, x_2, x_3) = \begin{cases} u^+(x_1, x_2) & x_3 > 0, \\ u^-(x_1, x_2) & x_3 < 0. \end{cases}$$

When $\alpha > 1$, the limit energy is oblivious to the presence of the interfacial energy, while in the critical case $\alpha = 1$ it contains both bulk and interfacial energy terms (see Theorem 5.1).

Finally, Section 6 addresses the case of a thin film on a thick rigid substrate.

In the remainder of the paper, $\rightarrow$ will denote strong convergence, while $\rightharpoonup$ will stand for weak, or weak–* convergence; points in ω will be designated by x_α or x_β unless there is some ambiguity.

2. Local character of the limit energy

Consider two continuous non-negative elastic energy densities $W^+, W^- : \mathbb{R}^9 \rightarrow \mathbb{R}$ such that

(H1) $1/C|F|^p - C \leq W^{\pm}(F) \leq C(1 + |F|^p),\ F \in \mathbb{R}^{3\times 3},\ 1 < p < \infty$,

a lower semi–continuous function $\Psi : \mathbb{R}^3 \mapsto \mathbb{R}$ such that

(H2) $\Psi(t) > 0,\ t \neq 0,\ \Psi(0) = 0$,

(H3) Ψ is subadditive: $\Psi(a + b) \leq C(\Psi(a) + \Psi(b)),\ a, b \in \mathbb{R}^3$,

(H4) $t \mapsto \Psi(t)$ increases with $|t|$,

and a real number α with

(H5) $\alpha < 1$.

Let ω be an open, bounded, connected subset of $\mathbb{R}^2$ with Lipschitz boundary. We define, for any open subset A of ω and any $u \in L^p(\Omega; \mathbb{R}^3)$,

$$E_\varepsilon(u; A) := \begin{cases} \int_{A\times(-1,0)} W^-(D_\alpha u | 1/\varepsilon D_3 u)\, dx \quad + \quad \int_{A\times(0,1)} W^+(D_\alpha u|1/\varepsilon D_3 u)\, dx & \\ \qquad + \ \varepsilon^{\alpha-1} \int_A \Psi([\![u]\!]) dx_\alpha, & u \in X(A), \\ \infty, & \text{otherwise}, \end{cases}$$

where

$$X(A) := W^{1,p}(A \times (-1, 0); \mathbb{R}^3) \cap W^{1,p}(A \times (0, 1); \mathbb{R}^3),$$

and consider

$$J_{\{\varepsilon\}}(u; A) := \inf_{\{v_\varepsilon\}} \left\{ \liminf_{\varepsilon} E_\varepsilon(v_\varepsilon; A) : v_\varepsilon \rightarrow u \text{ strongly in } L^p(\Omega; \mathbb{R}^3) \right\}. \tag{2.1}$$

A straightforward rescaling enables us to set $h = s = 1$ with no loss of generality.

Remark 2.1. If $J_{\{\varepsilon\}}(u; \omega) < \infty$, then, in view of the coercivity property (H1), there exists a sequence $\{v_\varepsilon\} \subset X(\omega)$ such that a subsequence — still indexed by ε— satisfies

$$v_\varepsilon \rightharpoonup \begin{cases} u^- \text{ in } W^{1,p}(\omega \times (-1, 0)); \mathbb{R}^3) \\ u^+ \text{ in } W^{1,p}(\omega \times (0, 1)); \mathbb{R}^3) \end{cases} ,u^-, u^+ \in W^{1,p}(\omega; \mathbb{R}^3),$$

so that $D_3 u = 0$ on $\omega \times ((-1, 0) \cup (0, 1))$ where $u := \begin{cases} u^- \text{ on } \omega \times (-1, 0) \\ u^+ \text{ on } \omega \times (0, +1) \end{cases}$. Furthermore, by continuity of the trace and compactness,

$$\begin{cases} v_\varepsilon(x_\alpha, 0^-) \rightarrow u^- \text{ strongly in } L^p(\omega; \mathbb{R}^3), \\ v_\varepsilon(x_\alpha, 0^+) \rightarrow u^+ \text{ strongly in } L^p(\omega; \mathbb{R}^3). \end{cases}$$

Now, the finiteness of $J_{\{\varepsilon\}}(u;\omega)$ also implies, since $\alpha < 1$ (cf. (H5)), that

$$\lim_{\varepsilon} \int_{\omega} \Psi([\![v_\varepsilon]\!]) dx_\alpha = 0,$$

and Fatou's lemma then yields $\int_\omega \Psi(u^+ - u^-) dx_\alpha = 0$; appealing to (H2), we conclude that $u^+ = u^-$, and thus that $u \in W^{1,p}(\omega;\mathbb{R}^3)$. Conversely, if $u \in W^{1,p}(\omega;\mathbb{R}^3)$, take it as test sequence in the definition of $J_{\{\varepsilon\}}(u;\omega)$; (H1) immediately implies that $J_{\{\varepsilon\}}(u;\omega) < \infty$.

In conclusion, provided that (H5) holds true, $J_{\{\varepsilon\}}(u;\omega) < \infty$ if and only if $u \in W^{1,p}(\omega;\mathbb{R}^3)$. We will restric our analysis to such us in Sections 3 and 4, and will refrain from explicitly imposing (H5) unless necessary. In other words we will in those sections only discuss the $\Gamma(L^p)$–liminf of $J_{\{\varepsilon\}}(\cdot;A)$ for target functions $u \in W^{1,p}(\omega;\mathbb{R}^3)$. As mentioned before, this is the only non trivial case if $\alpha < 1$. Section 5 will partially fill the gap when $\alpha \geq 1$.

In the spirit of [6], Section 2, we introduce a countable collection $\mathcal{C}$ of open subsets of ω such that, for any $\delta > 0$ and any open subset A of ω, there exists a finite union C_A of disjoint elements of $\mathcal{C}$ with $\overline{C}_A \subset A$ and $\mathcal{L}^2(A) \leq \mathcal{L}^2(C_A) + \delta$. If $\mathcal{R}$ denotes the countable collection of all finite unions of elements of $\mathcal{C}$, then, with the help of a general argument of Γ–convergence (cf. [4]), we conclude that there exists a subsequence $\{\varepsilon^{\mathcal{R}}\} \subset \{\varepsilon\}$ such that $J_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;C)$ is actually the $\Gamma(L^p)$–limit of $E_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;C)$, *for any* $C \in \mathcal{R}$.

We are now in a position to prove that the limit functional is local.

Theorem 2.2. *Assume that (H1), (H3), (H4) hold true. For any open subset $A \subset \omega$, $J_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;A)$ is the $\Gamma(L^p)$–limit of $E_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;A)$, and there exists a Carathéodory function $W_{\{\varepsilon^{\mathcal{R}}\}} : \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \mapsto \mathbb{R}$ such that, for any $u \in W^{1,p}(\omega;\mathbb{R}^3)$,*

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = \int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha; D_\alpha u) dx_\alpha.$$

Proof. The proof is very close to that of a related result in [6] (see Theorem 2.5 there). We therefore merely outline it and emphasize the differences. In a first step a slicing method à la De Giorgi is used to prove that attainment sequences for a given $u \in W^{1,p}(\omega;\mathbb{R}^3)$, $C \in \mathcal{R}$, can be modified so as to coincide with u in a neighborhood of the boundary ∂C. A second step proves the claim of $\Gamma(L^p)$–convergence. A third one establishes that $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ extends to a finite non–negative Radon measure. A short fourth step establishes the integral representation through a direct application of Theorem 4.3.2 in [7].

Step 1.

Lemma 2.3. *Given $u \in W^{1,p}(\omega;\mathbb{R}^3)$, $C \in \mathcal{R}$, there exists a sequence $\{w_{\varepsilon^{\mathcal{R}}}\}$ such that $w_{\varepsilon^{\mathcal{R}}} \to u$ strongly in $L^p(\Omega;\mathbb{R}^3)$,*

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;C) = \lim_{\varepsilon^{\mathcal{R}}} E_{\varepsilon^{\mathcal{R}}}(w_{\varepsilon^{\mathcal{R}}};C),$$

and

$$w_{\varepsilon^{\mathcal{R}}} = u \quad \text{on } C \setminus K$$

for some compact subset $K \subset C$.

Proof. The proof relies on De Giorgi's slicing argument and is identical to that of Lemma 2.6 in [6], up to the addition of the term involving the surface energy, i.e.,

$$(\varepsilon^{\mathcal{R}})^{\alpha-1} \int_A \Psi([\![u]\!]) dx_\alpha.$$

Consider an attainment sequence $\{v_{\varepsilon^{\mathcal{R}}}\}$ such that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;C) = \lim_{\varepsilon^{\mathcal{R}}} E_\varepsilon^{\mathcal{R}}(v_{\varepsilon^{\mathcal{R}}};C).$$

The argument hinges on the choice of adequate cut–off functions $\varphi_{\varepsilon^{\mathcal{R}}} : \omega \to (0,1)$, independent of the transverse variable x_3, such that the energy associated to the sequence

$\{w_{\varepsilon^{\mathcal{R}}}\}$, with $w_{\varepsilon^{\mathcal{R}}} := \varphi v_{\varepsilon^{\mathcal{R}}} + (1-\varphi)u$, is asymptotically the same as the original one. Consequently, we have to compare $(\varepsilon^{\mathcal{R}})^{\alpha-1}\int_A \Psi([\![\varphi_{\varepsilon^{\mathcal{R}}} v_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha$ with $(\varepsilon^{\mathcal{R}})^{\alpha-1}\int_A \Psi([\![v_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha$. But, in view of (H4),

$$(\varepsilon^{\mathcal{R}})^{\alpha-1}\int_A \Psi([\![\varphi_{\varepsilon^{\mathcal{R}}} v_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha \le (\varepsilon^{\mathcal{R}})^{\alpha-1}\int_A \Psi([\![v_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha,$$

which permits to carry the proof of Lemma 2.6 of [6] over to the current setting. □

Step 2. The proof is identical to that of a similar claim in Theorem 2.5 of [6].

Step 3. For any $u \in W^{1,p}(\omega;\mathbb{R}^3)$, the inner regularity of $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ is, as in the proof of Theorem 2.6 in [6], implicit in Step 2. The subadditivity of $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ follows closely that in Step 3 of the proof of Theorem 2.6 in [6], but for the addition of the surface energy term. In the notation of that proof, we have to control from above a term of the form

$$(\varepsilon^{\mathcal{R}})^{\alpha-1}\int_{L^\delta_\zeta} \Psi([\![(\varphi^\delta_\zeta v^{D^\delta}_{\varepsilon^{\mathcal{R}}} + (1-\varphi^\beta_\zeta v^{B^\delta}_{\varepsilon^{\mathcal{R}}})]\!])dx_\alpha.$$

This latter task is immediately accomplished upon noting that, with the help of (H3–H4),

$$\begin{aligned}(\varepsilon^{\mathcal{R}})^{\alpha-1}&\int_{L^\delta_\zeta} \Psi([\![(\varphi^\delta_\zeta v^{D^\delta}_{\varepsilon^{\mathcal{R}}} + (1-\varphi^\beta_\zeta v^{B^\delta}_{\varepsilon^{\mathcal{R}}})]\!])dx_\alpha\\ &\le (\varepsilon^{\mathcal{R}})^{\alpha-1} C\left(\int_{L^\delta_\zeta}\Psi([\![v^{D^\delta}_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha + \int_{L^\delta_\zeta}\Psi([\![v^{B^\delta}_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha\right).\end{aligned}$$

Thus $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ is inner regular, subadditive. Let $\{w_{\varepsilon^{\mathcal{R}}}\}$ be an attainment sequence for $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\omega)$. Upon extracting a suitable subsequence $\{\overline{\varepsilon}\}$ of $\{\varepsilon^{\mathcal{R}}\}$, we may find a Radon measure μ on ω as the weak–* limit of

$$\begin{aligned}W^-(D_\alpha w_{\overline{\varepsilon}}|1/\overline{\varepsilon}D_3 w_{\overline{\varepsilon}})\mathcal{L}^3\lfloor\omega\times(-1,0) + W^+(D_\alpha w_{\overline{\varepsilon}}|1/\overline{\varepsilon}D_3 w_{\overline{\varepsilon}})\mathcal{L}^3\lfloor\omega\times(0,1)&\\ +\varepsilon^{\alpha-1}\Psi([\![w_{\overline{\varepsilon}}]\!])\mathcal{H}^2\lfloor\omega\times\{0\}&.\end{aligned}$$

Setting, for any open subset $A\subset\omega$,

$$\hat{\mu}(A) := \mu(A\times[-1,1]),$$

it is immediately checked that

$$\begin{gathered}J_{\{\varepsilon^{\mathcal{R}}\}}(u;\omega) \ge \hat{\mu}(\mathbb{R}^2),\\ J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \le \hat{\mu}(\overline{A}),\ A \text{ open} \subset\omega,\end{gathered}$$

hence, according to De Giorgi–Letta's criterion [8] — see also Lemma 7.3 in [6] —, $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ is a finite Radon measure. Further, the sequence $\{u_n\}$ with $u_n := u$ is a valid test sequence in the definition of $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$; thus, according to (H1), $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ is absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$.

Step 4. Theorem 4.3.2 in [7] immediately applies in the present setting, and it yields the desired integral representation. □

Our goal is to identify $W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F})$ for any $\overline{F}\in\mathbb{R}^{3\times2}$.

Here and in the remainder of the paper, for any Borel measurable function $Z:\mathbb{R}^N\to\mathbb{R}^P$, $Q_{N,P}Z$ stands for the *quasiconvexification* of Z, that is, for any $F\in\mathbb{R}^{P\times N}$,

$$Q_{N,P}Z(F) := \inf_\varphi\left\{\int_Q Z(F+D_\beta\varphi)\,dx_\alpha:\ \varphi\in\mathcal{C}^\infty_0(Q;\mathbb{R}^P)\right\}, \tag{2.2}$$

where $Q := (-1/2,1/2)^N$.

For $\overline{F}\in\mathbb{R}^{3\times2}$, define

$$\overline{W}(\overline{F}) := \inf_{z\in\mathbb{R}^3} W(\overline{F}|z), \tag{2.3}$$

and

$$\begin{aligned}\hat{W}(\overline{F}) \ :=\ \inf_{\varphi,\lambda>0} \ &\Big\{\int_{Q'\times(-1,0)} W^-(\overline{F}+D_\beta\varphi|\lambda D_3\varphi)\ dx_\alpha + \\ &\int_{Q'\times(0,1)} W^+(\overline{F}+D_\beta\varphi|\lambda D_3\varphi)\ dx_\alpha : \\ &\varphi\in\mathcal{C}^\infty(\overline{Q'}\times[-1,1];\mathbb{R}^3),\quad \varphi=0 \text{ on } \partial Q'\times(-1,1)\Big\}, \end{aligned} \tag{2.4}$$

where $Q' := (-1/2, 1/2)^2$.

We begin by proving the following easy bounds:

Lemma 2.4. *Assume that (H1), (H3), (H4) hold true. Then*

$$Q_{2\times3}\overline{W}^+(\overline{F}) + Q_{2\times3}\overline{W}^-(\overline{F}) \le W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F}) \le \hat{W}(\overline{F}),$$

for every $\overline{F}\in\mathbb{R}^{3\times2}$ and a.e. $x_0 = (x_\alpha)\in\omega$.

Proof. Let $x_0\in\omega$ be a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$. We first prove the lower bound for $W_{\{\varepsilon^{\mathcal{R}}\}}((x_0)_\alpha;\overline{F})$. Consider, for δ small enough, a sequence $\{v^\delta_{\varepsilon^{\mathcal{R}}}\}\subset X(Q'(x_0,\delta))$ such that $v^\delta_{\varepsilon^{\mathcal{R}}}\to\overline{F}\cdot x$ strongly in $L^p(Q'(x_0,\delta);\mathbb{R}^3)$ and

$$J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}\cdot x; Q'(x_0,\delta)) = \lim_{\varepsilon^{\mathcal{R}}} E_{\{\varepsilon^{\mathcal{R}}\}}(v^\delta_{\varepsilon^{\mathcal{R}}}; Q'(x_0,\delta)),$$

where $Q'(x_0,\delta) := x_0+\delta Q'$. This is possible according to Theorem 2.2. Then, in view of (2.3),

$$\begin{aligned}\int_{Q'(x_0,\delta)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F})\ dx_\alpha \ &\ge\ \liminf_{\varepsilon^{\mathcal{R}}}\Big\{\int_{Q'(x_0,\delta)\times(-1,0)}\overline{W}^-(D_\beta v^\delta_{\varepsilon^{\mathcal{R}}})\ dx \\ &\qquad + \int_{Q'(x_0,\delta)\times(0,1)}\overline{W}^+(D_\beta v^\delta_{\varepsilon^{\mathcal{R}}})\ dx\Big\} \\ &\ge\ \liminf_{\varepsilon^{\mathcal{R}}}\Big\{\int_{Q'(x_0,\delta)\times(-1,0)}\overline{W}^-(D_\beta v^\delta_{\varepsilon^{\mathcal{R}}})\ dx\Big\} \\ &\qquad + \liminf_{\varepsilon^{\mathcal{R}}}\Big\{\int_{Q'(x_0,\delta)\times(0,1)}\overline{W}^+(D_\beta v^\delta_{\varepsilon^{\mathcal{R}}})\ dx\Big\} \\ &\ge\ \delta^2\Big(Q_{3\times3}\overline{W}^-(\overline{F}) + Q_{3\times3}\overline{W}^+(\overline{F})\Big) \\ &\ge\ \delta^2\Big(Q_{2\times3}\overline{W}^-(\overline{F}) + Q_{2\times3}\overline{W}^+(\overline{F})\Big),\end{aligned}$$

where the third inequality holds since the quasiconvexification of $\overline{W}^{+-}$ is the lower semi–continuous envelope of $\overline{W}^{+-}$ (cf. e.g. [1]), and the last one results from a straightforward application of Fubini's theorem in definition (2.2). Dividing the above inequality by δ^2 and passing to the limit as $\delta\searrow0$ proves the lower bound on $W_{\{\varepsilon^{\mathcal{R}}\}}((x_0)_\alpha;\overline{F})$ since x_0 is a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$.

Next we establish the upper bound. Given $\eta>0$, there exist $\lambda_\eta>0, \varphi_\eta\in\mathcal{C}^\infty(\overline{Q'}\times[-1,1];\mathbb{R}^3)$, with $\varphi=0$ on $\partial Q'\times(-1,1)$, such that

$$\begin{aligned}\hat{W}(\overline{F})+\eta \ \ge\ &\Big\{\int_{Q'\times(-1,0)} W^-(\overline{F}+D_\beta\varphi_\eta|\lambda_\eta D_3\varphi_\eta)\ dx + \\ &\int_{Q'\times(0,1)} W^+(\overline{F}+D_\beta\varphi_\eta|\lambda_\eta D_3\varphi_\eta)\ dx\Big\}.\end{aligned}$$

Extend φ_η by x_α–periodicity to $\mathbb{R}^2 \times (-1,1)$ and set

$$v^\eta_{\varepsilon^{\mathcal{R}}}(x_\alpha, x_3) := \overline{F}\cdot x + \lambda_\eta \varepsilon^{\mathcal{R}} \varphi_\eta\Big(\frac{x_\alpha}{\lambda_\eta \varepsilon^{\mathcal{R}}}; x_3\Big).$$

Then, as $\varepsilon^{\mathcal{R}} \searrow 0$,

$$v^\eta_{\varepsilon^{\mathcal{R}}} \to \overline{F}\cdot x \text{ strongly in } L^p_{\rm loc}(\mathbb{R}^3;\mathbb{R}^3),$$

hence, by the very definition of $J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}\cdot x; Q'(x_0,\delta))$,

$$\begin{aligned}
&\int_{Q'(x_0,\delta)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F})\, dx_\alpha = J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}\cdot x; Q'(x_0,\delta))\\
\le\;& \liminf_{\varepsilon^{\mathcal{R}}} \Bigg\{ \int_{Q'(x_0,\delta)\times(-1,0)} W^-(D_\beta v^\eta_{\varepsilon^{\mathcal{R}}} | 1/\varepsilon^{\mathcal{R}} D_3 v^\eta_{\varepsilon^{\mathcal{R}}})\, dx\\
&+ \int_{Q'(x_0,\delta)\times(0,1)} W^+(D_\beta v^\eta_{\varepsilon^{\mathcal{R}}} | 1/\varepsilon^{\mathcal{R}} D_3 v^\eta_{\varepsilon^{\mathcal{R}}})\, dx \Bigg\}\\
=\;& \lim_{\varepsilon^{\mathcal{R}}} \Bigg\{ \int_{Q'(x_0,\delta)\times(-1,0)} W^-\Big(\overline{F} + D_\beta\varphi_\eta\Big(\frac{x_\beta}{\lambda_\eta\varepsilon^{\mathcal{R}}}, x_3\Big) \Big| \lambda_\eta D_3\varphi_\eta\Big(\frac{x_\beta}{\lambda_\eta\varepsilon^{\mathcal{R}}}, x_3\Big)\Big)\, dx\\
&+ \int_{Q'(x_0,\delta)\times(0,1)} W^+\Big(\overline{F} + D_\beta\varphi_\eta\Big(\frac{x_\beta}{\lambda_\eta\varepsilon^{\mathcal{R}}}, x_3\Big) \Big| \lambda_\eta D_3\varphi_\eta\Big(\frac{x_\beta}{\lambda_\eta\varepsilon^{\mathcal{R}}}, x_3\Big)\Big)\, dx \Bigg\}\\
=\;& \delta^2 \Bigg\{ \int_{Q'\times(-1,0)} W^-(\overline{F} + D_\beta\varphi_\eta | \lambda_\eta D_3\varphi_\eta)\, dx\\
&\quad + \int_{Q'\times(0,1)} W^+(\overline{F} + D_\beta\varphi_\eta | \lambda_\eta D_3\varphi_\eta)\, dx \Bigg\}\\
\le\;& \delta^2 \hat{W}(\overline{F}) + \eta.
\end{aligned}$$

In the previous string of (in)equalities there is no surface energy term because the constructed sequence is smooth across the interface $x_3 = 0$, while the last equality holds true because a rescaled periodic function converges weakly to its mean as the rescaling goes to 0. Letting η tend to 0, dividing by δ^2 and letting δ tend to 0 yields the result because x_0 is a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$. □

3. Cohesive interface

In this section, we consider an interfacial energy density which in the case where $\alpha < 1$ corresponds to a cohesive interface where the surface energy increases continuously from zero with the jump in the deformation across the interface. We assume that

(H6) $\Psi(t) \le C|t|^\gamma,\ \gamma > 1-\alpha.$

Under assumption (H6), we show that the limit energy density coincides with the lower bound established in Lemma 2.4. This agrees with our mechanical intuition, as explained in the Introduction. Specifically, we obtain the following theorem.

Theorem 3.1. *Assume that (H1), (H3), (H4), (H6) hold true. Then, for a.e. $x \in \omega$,*

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F}) = Q_{2\times3}\overline{W}^+(\overline{F}) + Q_{2\times3}\overline{W}^-(\overline{F}).$$

Proof. Fix $\eta > 0$. In view of (H1), of the density of $\mathcal{C}^\infty_0(Q';\mathbb{R}^3)$ into $L^p(Q';\mathbb{R}^3)$, and using a measurability selection criterion, we may find pairs $\varphi^{+,-}_\eta, \xi^{+,-}_\eta \in \mathcal{C}^\infty_0(Q';\mathbb{R}^3)$ such that

$$\begin{aligned}
Q_{2\times3}\overline{W}^+(\overline{F}) + Q_{2\times3}\overline{W}^-(\overline{F}) + \eta \quad &\ge \quad \int_{Q'} \big(W^-(\overline{F} + D_\beta\varphi^- | \xi^-) \;+\\
&\qquad W^+(\overline{F} + D_\beta\varphi^+ | \xi^+)\big)\; dx_\alpha.
\end{aligned} \tag{3.1}$$

Extend $\varphi_\eta^{+,-}, \xi_\eta^{+,-}$ by x_α–periodicity to $\mathbb{R}^2$ and set

$$v_{\varepsilon^{\mathcal{R}}}(x_\alpha, x_3) := \begin{cases} \overline{F}\cdot x + (\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}(\varphi^- + \varepsilon^{\mathcal{R}} x_3 \xi^-)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right) & \text{in } \omega\times(-1,0),\\ \overline{F}\cdot x + (\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}(\varphi^+ + \varepsilon^{\mathcal{R}} x_3 \xi^+)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right) & \text{in } \omega\times(0,1),\end{cases}$$

with $\sigma > 0$ to be chosen later. Clearly, $v_{\varepsilon^{\mathcal{R}}} \to \overline{F}\cdot x$ strongly in $L^p(\omega\times(-1,1);\mathbb{R}^3)$. Now, if $x_0\in\omega$ is a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$ and δ is small enough,

$$\begin{aligned}\int_{Q'(x_0,\delta)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_\beta;\overline{F})\,dx_\alpha &= J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}\cdot x; Q'(x_0,\delta))\\ &\le \liminf_{\varepsilon^{\mathcal{R}}}\Big\{\int_{Q'(x_0,\delta)\times(-1,0)} W^-(D_\alpha v_{\varepsilon^{\mathcal{R}}}|1/\varepsilon^{\mathcal{R}} D_3 v_{\varepsilon^{\mathcal{R}}})\,dx\\ &\qquad + \int_{Q'(x_0,\delta)\times(0,1)} W^+(D_\alpha v_{\varepsilon^{\mathcal{R}}}|1/\varepsilon^{\mathcal{R}} D_3 v_{\varepsilon^{\mathcal{R}}})\,dx\\ &\qquad + (\varepsilon^{\mathcal{R}})^{\alpha-1}\int_{Q'(x_0,\delta)}\Psi([\![v_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha\Big\}\\ \le \liminf_{\varepsilon^{\mathcal{R}}}\Big\{\int_{Q'(x_0,\delta)\times(-1,0)} & W^-(\overline{F}+D_\alpha\varphi^- + \varepsilon^{\mathcal{R}}x_3 D_\alpha\xi^-|\xi^-)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right)dx\\ + \int_{Q'(x_0,\delta)\times(0,1)} & W^+(\overline{F}+D_\alpha\varphi^+ + \varepsilon^{\mathcal{R}}x_3 D_\alpha\xi^+|\xi^+)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right)dx\\ &\qquad + (\varepsilon^{\mathcal{R}})^{\alpha-1}\int_{Q'(x_0,\delta)}\Psi([\![v_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha\Big\}.\end{aligned}$$

The functions W^-, W^+ are continuous, hence uniformly continuous on compact sets, and, since $\overline{F}+D_\alpha\varphi^-, \xi^-, x_3 D_\alpha\xi^-$ (resp. $\overline{F}+D_\alpha\varphi^+, \xi^+, x_3 D_\alpha\xi^+$) take their value in compact sets of $\mathbb{R}^{3\times 2}$ and $\mathbb{R}^3$, the last term in the previous string of inequalities reduces to

$$\begin{aligned}&\liminf_{\varepsilon^{\mathcal{R}}}\Big\{\int_{Q'(x_0,\delta)\times(-1,0)} W^-(\overline{F}+D_\alpha\varphi^-|\xi^-)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right)dx\\ &\qquad + \int_{Q'(x_0,\delta)\times(0,1)} W^+(\overline{F}+D_\alpha\varphi^+|\xi^+)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right)dx\\ &\qquad + (\varepsilon^{\mathcal{R}})^{\alpha-1}\int_{Q'(x_0,\delta)}\Psi([\![v_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha\Big\}\\ &\le \liminf_{\varepsilon^{\mathcal{R}}}\Big\{\int_{Q'(x_0,\delta)\times(-1,0)} W^-(\overline{F}+D_\alpha\varphi^-|\xi^-)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right)dx_\alpha\\ &\qquad + \int_{Q'(x_0,\delta)\times(0,1)} W^+(\overline{F}+D_\alpha\varphi^+|\xi^+)\left(\frac{x_\alpha}{(\varepsilon^{\mathcal{R}})^{\frac{1}{1+\sigma}}}\right)dx_\alpha\\ &\qquad + \mathcal{C}C(\varepsilon^{\mathcal{R}})^{\alpha-1}\delta^2(\varepsilon^{\mathcal{R}})^{\frac{\gamma}{1+\sigma}}\Big\},\end{aligned}$$

where $\mathcal{C}$ is a constant that only depends on $||\varphi^+||_{L_\infty} + ||\varphi^-||_{L_\infty} + ||\xi^+||_{L_\infty} + ||\xi^-||_{L_\infty}$. But a rescaled periodic function weakly converges to its mean so that, in view of (3.1),

$$\begin{aligned}\int_{Q'(x_0,\delta)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_\beta;\overline{F})\,dx_\alpha &\le \delta^2\left\{Q_{2\times3}\overline{W}^+(\overline{F}) + Q_{2\times3}\overline{W}^-(\overline{F}) + \eta\right\}\\ &+ \mathcal{C}C\delta^2\limsup_{\varepsilon^{\mathcal{R}}}(\varepsilon^{\mathcal{R}})^{\alpha-1+\frac{\gamma}{1+\sigma}}.\end{aligned}$$

We now choose $\sigma > 0$ so that $\gamma > (1+\sigma)(1-\alpha)$, which is possible by virtue of (H6). Then, the last term in the above inequality is 0 and the desired result is obtained upon letting η tend to 0, dividing by δ^2, letting δ tend to 0 and recalling that x_0 is a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$. □

Remark 3.2. Note that the result is independent of the value of α as long as hypothesis (H6) holds. Of course, for $\alpha > 1$, $J_{\{\varepsilon\}}(u;\omega)$ may be finite for $u \in L^p(\omega \times (-1,1);\mathbb{R}^3) \setminus W^{1,p}(\omega;\mathbb{R}^3)$ (see Remark 2.1 and Section 5).

4. Brittle interface

We now consider a brittle interface where the surface energy is of Griffith's type, i.e., it is constant except when the deformation is continuous. Assume that

(H7) $\Psi(0) = 0$ and $\Psi(t) = 1,\ t \neq 0$.

To proceed further in this case, we need an additional assumption about the nature of the two elastic energies, namely,

(H8) W^+ (resp. W^-)$(F_\alpha|F_3) = W^+$ (resp.W^-)$(F_\alpha|-F_3)$.

We believe that this additional assumption is unnecessary and is simply an artifact of our strategy of proof. One should be able to remove it, but we confess to our inability at doing away with it. We note however that this hypothesis is consistent with a wide class of materials, including any material which has the film plane to be a plane of reflection symmetry.

We show that, under hypotheses (H7) and (H8), the limit energy density is in fact equal to the upper bound established in Lemma 2.4. Specifically,

Theorem 4.1. *Assume that (H1), (H3), (H4), (H5), (H7) and (H8) hold true. Then, for a.e. $x \in \omega$,*

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F}) = \hat{W}(\overline{F}).$$

Proof. Let x_0 be a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$. Then, for q large enough so that $Q'(x_0,1/q) \subset \omega$, we have

$$\int_{Q'\left(x_0,\frac{1}{q}\right)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_\beta;\overline{F})\,dx_\alpha = J_{\{\varepsilon^{\mathcal{R}}\}}\left(\overline{F}\cdot x; Q'\left(x_0,\frac{1}{q})\right)\right). \tag{4.1}$$

According to Theorem 2.2, there exists a sequence $\{w^q_{\varepsilon^{\mathcal{R}}}\} \subset X\left(Q'\left(x_0,\frac{1}{q}\right)\right)$ such that $w^q_{\varepsilon^{\mathcal{R}}} \to 0$, $\varepsilon^{\mathcal{R}} \searrow 0$ strongly in $L^p\left(Q'(x_0,\frac{1}{q}\right) \times (-1,1);\mathbb{R}^3\right)$ and

(4.2)

$$\begin{aligned} J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}\cdot x; Q'\left(x_0,\frac{1}{q})\right) = \lim_{\varepsilon^{\mathcal{R}}} \Bigg\{ &\int_{Q'\left(x_0,\frac{1}{q}\right)\times(-1,0)} W^-(\overline{F} + D_\alpha w^q_{\varepsilon^{\mathcal{R}}}|1/\varepsilon^{\mathcal{R}} D_3 w^q_{\varepsilon^{\mathcal{R}}})\,dx \\ &+ \int_{Q'\left(x_0,\frac{1}{q}\right)\times(0,1)} W^+(\overline{F} + D_\alpha w^q_{\varepsilon^{\mathcal{R}}}|1/\varepsilon^{\mathcal{R}} D_3 w^q_{\varepsilon^{\mathcal{R}}})\,dx \\ &+ (\varepsilon^{\mathcal{R}})^{\alpha-1} \int_{Q'\left(x_0,\frac{1}{q}\right)} \Psi([\![w^q_{\varepsilon^{\mathcal{R}}}]\!])dx_\alpha \Bigg\}. \end{aligned}$$

Set

$$w_{q,\varepsilon^{\mathcal{R}}} := q w^q_{\varepsilon^{\mathcal{R}}}\left(x_0 + \frac{1}{q}x_\alpha, x_3\right),\ x_\alpha \in Q'.$$

Then, according to (4.1), (4.2), and since Ψ is homogeneous of degree 0,

(4.3)
$$\begin{aligned} W_{\{\varepsilon^{\mathcal{R}}\}}((x_0)_\alpha;\overline{F}) = \quad \lim_q \lim_{\varepsilon^{\mathcal{R}}} \Bigg\{ \int_{Q'\times(-1,0)} & W^-\Big(\overline{F} + D_\alpha w_{q,\varepsilon^{\mathcal{R}}} | \frac{1}{q\varepsilon^{\mathcal{R}}} D_3 w_{q,\varepsilon^{\mathcal{R}}}\Big)\, dx \\ + \int_{Q'\times(0,1)} & W^+\Big(\overline{F} + D_\alpha w_{q,\varepsilon^{\mathcal{R}}} | \frac{1}{q\varepsilon^{\mathcal{R}}} D_3 w_{q,\varepsilon^{\mathcal{R}}}\Big)\, dx \\ + (\varepsilon^{\mathcal{R}})^{\alpha-1} \int_{Q'} & \Psi([\![w_{q,\varepsilon^{\mathcal{R}}}]\!]) dx_\alpha \Bigg\}. \end{aligned}$$

Thus, in view of (4.3), (H5), (H7), and by a straightforward diagonalization argument, there exists a sequence $\{v_q := w_{q,\varepsilon^{\mathcal{R}}(q)}\}$ with $v_q \to 0$ strongly in $L^p(Q'\times(-1,1);\mathbb{R}^3)$, $q \nearrow \infty$, such that

$$\frac{1}{q\varepsilon^{\mathcal{R}}(q)} \ge q, \tag{4.4}$$

$$\lim_q \mathcal{L}^2(\{x_\alpha \in \omega : [\![v_q]\!] \neq 0\}) = 0, \tag{4.5}$$

while

$$\begin{aligned} W_{\{\varepsilon^{\mathcal{R}}\}}((x_0)_\alpha;\overline{F}) \ge \quad \limsup_q \Bigg\{ \int_{Q'\times(-1,0)} & W^-\Big(\overline{F} + D_\alpha v_q | \frac{1}{q\varepsilon^{\mathcal{R}}(q)} D_3 v_q\Big)\, dx \\ + \int_{Q'\times(0,1)} & W^+\Big(\overline{F} + D_\alpha v_q | \frac{1}{q\varepsilon^{\mathcal{R}}(q)} D_3 v_q\Big)\, dx \Bigg\}. \end{aligned} \tag{4.6}$$

Recalling definition (2.4) for $\hat{W}$, the proof would be complete if $v_q \in W^{1,p}(\omega \times (-1,1);\mathbb{R}^3)$ and $v_q = 0$ on $\partial Q'(0,1) \times (-1,1)$. Unfortunately, such may not be the case and we must modify v_q accordingly. In contrast to the setting of [6], a slicing argument does not work due to a lack of control of the energy on the transverse strips $\{x \in \omega : [\![v_q]\!] \neq 0\} \times (-1,1)$. We thus appeal to the following lemma applied to the sequences $\{u_q := \overline{F}\cdot x + v_q\}$ and $\{\varepsilon(q) := q\varepsilon^{\mathcal{R}}(q)\}$:

Lemma 4.2. *Let A be an open subset of ω, let $\{\varepsilon(q)\}$ be a sequence of nonnegative real numbers such that $\varepsilon(q) \underset{q\to\infty}{\longrightarrow} 0$, and let $\{u_q\}$ be a sequence in $W^{1,p}(A\times(-1,0);\mathbb{R}^3) \cap W^{1,p}(A\times(0,1);\mathbb{R}^3)$, $p > 1$, such that*

(4.7)
$$\begin{cases} u_q \underset{q\to\infty}{\longrightarrow} u \text{ strongly in } L^p(A\times(-1,1);\mathbb{R}^3), \text{ with } u \in L^\infty(A;\mathbb{R}^3), \\[2ex] \int_{A\times(-1,0)} \Big(|D_\alpha u_q|^p + \frac{1}{\varepsilon(q)^p}|D_3 u_q|^p\Big)\, dx + \int_{A\times(0,1)} \Big(|D_\alpha u_q|^p + \frac{1}{\varepsilon(q)^p}|D_3 u_q|^p\Big)\, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \le \mathcal{C} < \infty, \\[2ex] \mathcal{L}^2\Big(\{[\![u_q]\!] \neq 0\}\Big) \underset{q\to\infty}{\longrightarrow} 0. \end{cases}$$

Then, under assumption (H8), for any $\eta > 0$, there exists a sequence $\{\overline{u}_q^\eta\}$ in $W^{1,p}(A \times (-1,1);\mathbb{R}^3)$ such that

$$\begin{cases} \overline{u}_q^\eta \underset{q\to\infty}{\longrightarrow} u \text{ strongly in } L^p(A\times(-1,1);\mathbb{R}^3), \\[2ex] \liminf_{q\to\infty} \Big\{ \int_{A\times(-1,0)} W^-\Big(D_\alpha u_q | \frac{1}{\varepsilon(q)} D_3 u_q\Big)\, dx + \int_{A\times(0,1)} W^+\Big(D_\alpha u_q | \frac{1}{\varepsilon(q)} D_3 u_q\Big)\, dx \Big\} \\ \quad \ge \liminf_{q\to\infty} \Big\{ \int_{A\times(-1,0)} W^-(D_\alpha \overline{u}_q^\eta | D_3 \overline{u}_q^\eta)\, dx + \int_{A\times(0,1)} W^+(D_\alpha \overline{u}_q^\eta | D_3 \overline{u}_q^\eta)\, dx \Big\} - \eta. \end{cases}$$

Let us postpone for now the proof of Lemma 4.2 and complete the proof of the theorem. In view of (4.4), (4.5), (4.6) and (H1), Lemma 4.2 is applicable. Denote by $\{\overline{F}\cdot x+\overline{v}_q^\eta\}$ the sequence produced by Lemma 4.2. Then, by virtue of (4.6),

$$W_{\{\varepsilon^{\mathcal{R}}\}}((x_0)_\alpha;\overline{F}) \geq \liminf_q \left\{ \int_{Q'\times(-1,0)} W^-(\overline{F}+D_\alpha\overline{v}_q|D_3\overline{v}_q^\eta)\,dx + \int_{Q'\times(0,1)} W^+(\overline{F}+D_\alpha\overline{v}_q|D_3\overline{v}_q^\eta)\,dx \right\} - \eta. \tag{4.8}$$

We still have to cure the potential failure of $\overline{v}_q^\eta$ in taking the value 0 on $\partial Q'\times(-1,1)$. This latter task is achieved by appropriately cutting the values of $\overline{v}_q^\eta$ near $\partial Q'\times(-1,1)$ with a smooth cut–off. We do not go into the details of that process since it is verbatim that described in the proof of inequality (3.8) in the proof of Theorem 3.1 in [6]. Thus,

$$\hat{W}(\overline{F}) \leq \liminf_q \left\{ \int_{Q'\times(-1,0)} W^-(\overline{F}+D_\alpha\overline{v}_q^\eta|D_3\overline{v}_q^\eta)\,dx + \int_{Q'\times(0,1)} W^+(\overline{F}+D_\alpha\overline{v}_q^\eta|D_3\overline{v}_q^\eta)\,dx \right\}. \tag{4.9}$$

Recalling (4.8), (4.9) and letting η tend to 0^+, we finally conclude that

$$W_{\{\varepsilon^{\mathcal{R}}\}}((x_0)_\alpha;\overline{F}) \geq \hat{W}(\overline{F}).$$

The proof of the theorem is complete, provided that Lemma 4.2 holds true. This is the object of the proof below. □

Proof. *(Lemma 4.2)* Assume that (4.7) holds. Set

$$S_i^q := \begin{cases} A\times((i-1)\varepsilon(q), i\varepsilon(q)),\ N(q)\geq i\geq 1,\\ A\times(i\varepsilon(q),(i+1)\varepsilon(q)),\ -N(q)\leq i\leq -1,\end{cases}$$

where the number $N(q)$ of layers is

$$N(q) = \begin{cases} \dfrac{1}{\varepsilon(q)}, & \text{if } \frac{1}{\varepsilon(q)} \text{ is an integer},\\[2ex] \left[\dfrac{1}{\varepsilon(q)}\right]+1, & \text{otherwise}.\end{cases}$$

Further define, with $\mathrm{sgn}(t)$ denoting the sign of $t\in\mathbb{R}$,

$$\begin{cases} F_i^q : S_i^q & \rightarrow \quad S^q_{sgn(i)},\\[1ex] (x_\alpha,x_3) & \mapsto \quad \Big(x_\alpha, (-1)^{i+1}x_3+(-1)^i\mathrm{sgn}(i)\left[\frac{|i|}{2}\right]2\varepsilon(q)\Big),\end{cases}$$

and

$$F^q(x_\alpha,x_3) := \begin{cases} F_+^q(x_\alpha,x_3),\ x_3\geq 0,\\ F_-^q(x_\alpha,x_3),\ x_3\leq 0,\end{cases}$$

where

$$F_+^q : A\times(0,1)\to S_1^q,\quad F_+^q := \sum_1^{N(q)} \chi_{S_i^q\cap(A\times(0,1))}F_i^q,$$

and

$$F_-^q : A\times(-1,0)\to S_{-1}^q,\quad F_-^q := \sum_{-N(q)}^{-1} \chi_{S_i^q\cap(A\times(-1,0))}F_i^q.$$

Also set

$$\hat{u}_q := \begin{cases} \hat{u}_q^+(x_\alpha, x_3) := u_q\Big(x_\alpha, \frac{1}{\varepsilon(q)}x_3\Big), \ (x_\alpha, x_3) \in S_1^q, \\ \\ \hat{u}_q^-(x_\alpha, x_3) := u_q\Big(x_\alpha, \frac{1}{\varepsilon(q)}x_3\Big), \ (x_\alpha, x_3) \in S_{-1}^q. \end{cases}$$

Note that, in view of the bound on the gradients in (4.7), a trivial $\frac{1}{\varepsilon(q)}$–dilation yields

$$(4.10) \qquad \begin{cases} \frac{1}{\varepsilon(q)} \int_{S_1^q} |D\hat{u}_q^+|^p \, dx \le \mathcal{C}, \\ \\ \frac{1}{\varepsilon(q)} \int_{S_{-1}^q} |D\hat{u}_q^-|^p \, dx \le \mathcal{C}. \end{cases}$$

Also note that

$$(4.11) \qquad [\![\hat{u}_q]\!] = [\![u_q]\!] \text{ across } x_3 = 0.$$

Finally define

$$(4.12) \qquad U_q(x_\alpha, x_3) := \hat{u}_q(F^q(x_\alpha, x_3)), \ \text{ for } (x_\alpha, x_3) \in Q' \times (-1, 1).$$

Then, $U_q \in W^{1,p}(A \times (-1, 0); \mathbb{R}^3) \cap W^{1,p}(A \times (0, 1); \mathbb{R}^3)$ and, in view of (4.11),

$$(4.13) \qquad [\![U_q]\!] = [\![u_q]\!] \text{ across } x_3 = 0.$$

Furthermore, by virtue of (4.10),

$$(4.14) \qquad \begin{aligned} \int_{A\times(0,1)} |DU_q|^p \, dx \quad &= \int_{A\times(0,1)} |D\hat{u}_q^+(F_+^q) DF_+^q|^p \, dx \\ &= \sum_1^{N(q)} \int_{S_i^q \cap (A\times(0,1))} |D\hat{u}_q^+(F_+^q)|^p \, dx \le \sum_1^{N(q)} \int_{S_1^q} |D\hat{u}_q^+(y)|^p \, dy \\ &= N(q) \int_{S_1^q} |D\hat{u}_q^+(y)|^p \, dy \le N(q) \tfrac{1}{\varepsilon(q)} \mathcal{C} \le 2\mathcal{C}, \end{aligned}$$

where the second equality holds because

$$(4.15) \qquad DF_+^q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

A similar estimate holds true in $A \times (-1, 0)$. Finally, remark that the target u does not depend upon x_3, thus

$$\begin{aligned} \int_{A\times(0,1)} |U_q - u|^p \, dx \quad &= \quad \int_{A\times(0,1)} \Big| u_q\Big(x_\alpha, \frac{F_+^q(x)_3}{\varepsilon(q)}\Big) - u(x_\alpha) \Big|^p \, dx \\ &= \quad \sum_1^{N(q)} \int_{S_i^q} \Big| u_q\Big(x_\alpha, \frac{F_i^q(x)_3}{\varepsilon(q)}\Big) - u(x_\alpha) \Big|^p \, dx \\ &= \quad \sum_1^{N(q)} \int_{S_1^q} \Big| u_q\Big(x_\alpha, \frac{x_3}{\varepsilon(q)}\Big) - u(x_\alpha) \Big|^p \, dx \\ &= \quad N(q)\varepsilon(q) \int_{A\times(0,1)} |u_q(x_\alpha, x_3) - u(x_\alpha)|^p \, dx \\ &\le \quad 2\mathcal{C} \int_{A\times(0,1)} |u_q(x_\alpha, x_3) - u(x_\alpha)|^p \, dx, \end{aligned}$$

so that, since a similar computation could be performed on $A \times (-1, 0)$,

$$(4.16) \qquad U_q \to u, \ \text{ strongly in } L^p(A \times (-1, 1).$$

Recalling (4.13), (4.14), (4.16), and the second limit in (4.7), we conclude that $U_q \in SBV(A \times (-1,1);\mathbb{R}^3)$, that $S(U_q) \subset \{[\![u_q]\!] \neq 0\} \cup E_q$, with $\mathcal{H}^2(E_q) = 0$, and that

$$\begin{cases} U_q \longrightarrow u \text{ strongly in } L^p(A \times (-1,1);\mathbb{R}^3), \\ \mathcal{H}^2(S(U_q)) \longrightarrow 0, \\ \int_{A\times(-1,1)} |DU_q|^p \, dx \le \mathcal{C} < \infty. \end{cases} \tag{4.17}$$

Note that, in view of (H8), (4.15), together with the definition (4.12) of U_q,

$$\begin{aligned} &\int_{A\times(-1,0)} W^-\Big(D_\alpha u_q|\frac{1}{\varepsilon(q)}D_3 u_q\Big)\,dx + \int_{A\times(0,1)} W^+\Big(D_\alpha u_q|\frac{1}{\varepsilon(q)}D_3 u_q\Big)\,dx \\ &= \frac{1}{\varepsilon(q)N(q)}\Big\{\int_{A\times(-1,0)} W^-(D_\alpha U_q|D_3U_q)\,dx + \int_{A\times(0,1)} W^+(D_\alpha U_q|D_3U_q)\,dx\Big\}. \end{aligned} \tag{4.18}$$

Fix $\eta > 0$. Since $u \in L^\infty(A;\mathbb{R}^3)$, we may replace $\{U_q\}$ by $\{W_q^\eta\}$ with $W_q^\eta := T_{k(q,\eta)}(U_q)$, where $\{k(q,\eta)\}$ is a well–chosen subsequence of $\mathbb{Z}^+$ such that $k(q,\eta) \le M(\eta)$ (a positive constant *independent of* q) and where, for each $k \in \mathbb{Z}^+$, $T_k \in W_0^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)$ is such that

$$T_k(x) := \begin{cases} x, & |x| \le e^k, \\ 0, & |x| > e^{k+1}, \end{cases}$$

while

$$||\nabla T_k(x)|| \le 1.$$

A proper choice of $M(\eta)$ and $\{k(q,\eta)\}$ (see, for example, the proof of Proposition 2.8 in [10]) permits to ensure that $\{W_q^\eta\}$ still satisfies (4.17), while

$$\begin{aligned} &\liminf_q \Big\{\int_{A\times(-1,0)} W^-\Big(D_\alpha W_q^\eta|\frac{1}{\varepsilon(q)}D_3 W_q^\eta\Big)\,dx + \int_{A\times(0,1)} W^+\Big(D_\alpha W_q^\eta|\frac{1}{\varepsilon(q)}D_3 W_q^\eta\Big)\,dx\Big\} \\ &\le \liminf_q \Big\{\int_{A\times(-1,0)} W^-(D_\alpha U_q|D_3U_q)\,dx + \int_{A\times(0,1)} W^+(D_\alpha U_q|D_3U_q)\,dx\Big\} + \eta. \end{aligned} \tag{4.19}$$

Now $||W_q^\eta||_{L^\infty(A\times(-1,0))} \le e^{M(\eta)}$ while, in view of the second limit in (4.17) applied to W_q^η, $\mathcal{H}^2(S(W_q^\eta)) \longrightarrow 0$, thus the singular part of the measure DW_q^η is such that

$$|D_s W_q^\eta|(A \times (-1,1) \underset{q}{\longrightarrow} 0. \tag{4.20}$$

By virtue of (4.17) (for W_q^η), (4.20), we are in a position to apply an equiintegrability lemma (see Lemma 2.1 of [15]; see also [2]) to conclude to the existence of a sequence $\{\overline{u}_q\}$ in $W^{1,p}(A \times (-1,1);\mathbb{R}^3)$ such that

$$\begin{cases} \{\overline{u}_q\} \text{ is bounded in } W^{1,p}(A \times (-1,1);\mathbb{R}^3), \\ \{|D\overline{u}_q|^p\} \text{ is equiintegrable}, \\ \mathcal{L}^3\Big(\{W_q^\eta \neq \overline{u}_q\} \cup \{\nabla W_q^\eta \neq D\overline{u}_q\}\Big) \to 0, \end{cases} \tag{4.21}$$

where $\nabla W_q^\eta = DW_q^\eta|_{\{(x_\alpha,x_3):x_3\neq 0\}}$.

Then, in view of (4.18), (4.19), together with (H1),

$$\begin{aligned}
&\liminf_{q\to\infty}\Big\{\int_{A\times(-1,0)} W^-\Big(D_\alpha u_q|\frac{1}{\varepsilon(q)}D_3u_q\Big)\,dx+\int_{A\times(0,1)} W^+\Big(D_\alpha u_q|\frac{1}{\varepsilon(q)}D_3u_q\Big)\,dx\Big\}\\
&\geq\Big(\lim_{q\to\infty}\frac{1}{\varepsilon(q)N(q)}\Big)\Big[\liminf_{q\to\infty}\Big\{\int_{A\times(-1,0)} W^-(D_\alpha W_q^\eta|D_3W_q^\eta)\,dx\\
&\qquad+\int_{A\times(0,1)} W^+(D_\alpha W_q^\eta|D_3W_q^\eta)\,dx\Big\}-\eta\Big]\\
&\geq\liminf_{q\to\infty}\Big\{\int_{(A\times(-1,0))\setminus\{\nabla W_q^\eta\neq D\overline{u}_q\}} W^-(D_\alpha\overline{u}_q|D_3\overline{u}_q)\,dx\\
&\qquad+\int_{(A\times(0,1))\setminus\{\nabla W_q^\eta\neq D\overline{u}_q\}} W^+(D_\alpha\overline{u}_q|D_3\overline{u}_q)\,dx\Big\}-\eta\\
&\geq\liminf_{q\to\infty}\Big\{\int_{(A\times(-1,0))} W^-(D_\alpha\overline{u}_q|D_3\overline{u}_q)\,dx+\int_{(A\times(0,1))} W^+(D_\alpha\overline{u}_q|D_3\overline{u}_q)\,dx\Big\}\\
&\qquad-C\limsup_{q\to\infty}\Big\{\int_{\{\nabla W_q^\eta\neq D\overline{u}_q\}}(1+|D_\alpha\overline{u}_q|^p+|D_3\overline{u}_q)|^p)\,dx\Big\}-\eta.
\end{aligned}$$

But the last term in the last inequality above tends to 0 when $q\nearrow\infty$ by virtue of (4.21). Lemma 4.2 is proved. □

Remark 4.3. Equality (4.18) would fail in the absence of (H8) because of the presence of minus signs in front of the x_3–derivatives of U_q on even layers S_i^q when going from the first two integrals to the second two integrals. Also note that this is the only place in the proof of Lemma 4.2 – and in that of Theorem 4.1 – where (H8) is used.

5. Weak interface

In this section we investigate the impact of dropping assumption (H5), namely that $\alpha<1$; hence we assume that $\alpha\geq1$ unless otherwise noted in this section.

Note firstly that, in view of Remark 2.1,

$$J_{\{\varepsilon\}}(u;\omega)<\infty \text{ if and only if } u=\begin{cases} u^+ \text{ in } \omega\times(0,1),\\ u^- \text{ in } \omega\times(-1,0),\end{cases}\tag{5.1}$$

where $u^+,u^-\in W^{1,p}(\omega;\mathbb{R}^3)$.

Theorem 5.1. *Assume that (H1) holds true and that $\alpha>1$. Then (5.1) is satisfied and further, for such u's,*

$$J_{\{\varepsilon\}}(u;\omega)=\int_\omega\Big\{Q_{2\times3}\overline{W}^+(D_\alpha u^+)+Q_{2\times3}\overline{W}^-(D_\alpha u^-)\Big\}\,dx_\alpha.$$

Proof. The lower bound is straightforward; indeed, for some subsequence of $\{\varepsilon\}$ — still denoted by $\{\varepsilon\}$ — there exists a sequence $\{u_\varepsilon\}$ with $u_\varepsilon\to u=\begin{cases} u^+ \text{ in } \omega\times(0,1),\\ u^- \text{ in } \omega\times(-1,0),\end{cases}$

strongly in $L^p(\omega\times(-1,1);\mathbb{R}^3)$, such that

$$\begin{aligned}
J_{\{\varepsilon\}}(u;\omega) &= \lim_{\varepsilon}\Big\{\int_{\omega\times(-1,0)} W^-\Big(D_\alpha u_\varepsilon|\frac{1}{\varepsilon}D_3u_\varepsilon\Big)\,dx+\int_{\omega\times(0,1)} W^+\Big(D_\alpha u_\varepsilon|\frac{1}{\varepsilon}D_3u_\varepsilon\Big)\,dx \\
&\qquad +\varepsilon^{\alpha-1}\int_\omega \Psi([\![u_\varepsilon]\!])\,dx_\alpha\Big\} \\
&\geq \liminf_{\varepsilon}\Big\{\int_{\omega\times(-1,0)} \overline{W}^-(D_\alpha u_\varepsilon)\,dx+\int_{\omega\times(0,1)}\overline{W}^+(D_\alpha u_\varepsilon)\,dx\Big\} \\
&\geq \liminf_{\varepsilon}\Big\{\int_{\omega\times(-1,0)} \overline{W}^-(D_\alpha u_\varepsilon)\,dx\Big\}+\liminf_{\varepsilon}\Big\{\int_{\omega\times(0,1)}\overline{W}^+(D_\alpha u_\varepsilon)\,dx\Big\} \\
&\geq \int_\omega\Big\{Q_{2\times3}\overline{W}^+(D_\alpha u^+)+Q_{2\times3}\overline{W}^-(D_\alpha u^-)\Big\}\,dx_\alpha,
\end{aligned}$$

where the last inequality holds since the quasiconvexification of $\overline{W}^{+-}$ is the lower semi–continuous envelope of $\overline{W}^{+-}$ (see the beginning of the proof of Lemma 2.4 above for a detailed justification of the last inequality).

Conversely, if $u=\begin{cases} u^+ \text{ on } \omega\times(0,1),\\ u^- \text{ on } \omega\times(-1,0),\end{cases}$, then, by virtue of (H1) together with the measurable selection criterion and the density of $\mathcal{C}_0^\infty(\omega;\mathbb{R}^3)$ into $L^p(\omega;\mathbb{R}^3)$, for any $\eta>0$ there exist a pair $\xi^+,\xi^-\in\mathcal{C}_0^\infty(\omega;\mathbb{R}^3)$ such that

$$\int_\omega\{\overline{W}^+(D_\alpha u^+)+\overline{W}^-(D_\alpha u^-)\}dx_\alpha\geq\int_\omega\{W^+(D_\alpha u^+|\xi^+)+W^-(D_\alpha u^-|\xi^-)\}dx_\alpha-\eta.$$

Defining as test sequence for $J_{\{\varepsilon\}}(u;\omega)$

$$u_\varepsilon:=\begin{cases} u^++\varepsilon x_3\xi^+, \text{ in } \omega\times(0,1),\\ u^-+\varepsilon x_3\xi^-, \text{ in } \omega\times(-1,0),\end{cases}$$

and remarking that

$$[\![u_\varepsilon]\!]=[\![u]\!] \text{ across } x_3=0,$$

we conclude, through an easy application of Lebesgue's Dominated Convergence theorem and because $\alpha>1$, that

$$J_{\{\varepsilon\}}(u;\omega)\leq\int_\omega\overline{W}^+(D_\alpha u^+)\,dx_\alpha+\int_\omega\overline{W}^-(D_\alpha u^-)\,dx_\alpha+\eta.$$

Letting $\eta\to0$, we obtain

$$J_{\{\varepsilon\}}(u;\omega)\leq\int_\omega\overline{W}^+(D_\alpha u^+)\,dx_\alpha+\int_\omega\overline{W}^-(D_\alpha u^-)\,dx_\alpha. \tag{5.2}$$

But, since $J_{\{\varepsilon\}}(\cdot;\omega)$ is, in particular, sequentially weakly lower–semicontinuous at u for sequences $\{u_n\}$ of the form

$$u_n:=\begin{cases} u_n^+ \text{ in } \omega\times(0,1), \text{ with } u_n^+\in W^{1,p}(\omega;\mathbb{R}^3) \text{ and } u_n^+\to u^+ \text{ strongly in} L^p(\omega;\mathbb{R}^3),\\ u^- \text{ in } \omega\times(-1,0),\end{cases}$$

(5.2) implies

$$J_{\{\varepsilon\}}(u;\omega)\leq\int_\omega Q_{2\times3}\overline{W}^+(D_\alpha u^+)\,dx_\alpha+\int_\omega\overline{W}^-(D_\alpha u^-)\,dx_\alpha.$$

Repeating the argument for $\overline{W}^-$ finally yields

$$J_{\{\varepsilon\}}(u;\omega)\leq\int_\omega Q_{2\times3}\overline{W}^+(D_\alpha u^+)\,dx_\alpha+\int_\omega Q_{2\times3}\overline{W}^-(D_\alpha u^-)\,dx_\alpha.$$

The proof of the theorem is complete. □

We finally investigate the case $\alpha=1$. In such a case, the surface energy explicitly remains in the expression for $J_{\{\varepsilon\}}(u;\omega)$:

Theorem 5.2. *Assume that (H1) holds true and that $\alpha = 1$. Then (5.1) is satisfied and further, for such u's,*

$$J_{\{\varepsilon\}}(u;\omega) = \int_\omega \left\{ Q_{2\times 3}\overline{W}^+(D_\alpha u^+) + Q_{2\times 3}\overline{W}^-(D_\alpha u^-) \right\} dx_\alpha + \int_\omega \Psi([\![u]\!]) dx_\alpha .$$

Proof. The proof of the lower bound is nearly identical to that in the proof of Theorem 5.1. Upon noting that the liminf of a sum is greater than or equal to the sum of the liminfs, the additional term to control is $\liminf_\varepsilon \int_\omega \Psi([\![u_\varepsilon]\!]) dx_\alpha$ which, by virtue of the lower–semicontinuous character of Ψ and Fatou's lemma, is greater than or equal to $\int_\omega \Psi([\![u]\!]) dx_\alpha$. The proof of the upper bound is exactly that in the proof of Theorem 5.1. Note that the surface energy term is then exactly equal, for all ε's, to $\int_\omega \Psi([\![u]\!]) dx_\alpha$. □

6. Thick substrate

We conclude by discussing briefly the case of a thin film bonded to a thick rigid substrate. The displacement in the thick rigid substrate is identically zero (or equivalently the deformation is identity), hence we need only to consider the elastic energy of the film and the surface energy at the film-substrate interface. Thus, after a suitable change of variables as in the introduction, we obtain the following problem. Let $\Omega = \omega \times (0,1)$. We define, for any open subset A of ω and any $u \in L^p(\Omega;\mathbb{R}^3)$,

$$E_\varepsilon(u;A) := \begin{cases} \int_{A\times(0,1)} W^+(D_\alpha u|1/\varepsilon D_3 u)dx \\ \quad + \varepsilon^{\alpha-1}\int_A \Psi\left(u(x_1,x_2,0) - \{x_1,x_2,0\}\right) dx_\alpha, & u \in W^{1,p}(A\times(0,1);\mathbb{R}^3), \\ \infty, & \text{otherwise.} \end{cases}$$

We define $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$ as before (see Section 3). We obtain the following theorem, the proof of which follows those in the previous sections.

Theorem 6.1. *Assume that (H1), (H3), (H4) hold true. For any open subset $A \subset \omega$, $J_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;A)$ is the $\Gamma(L^p)$–limit of $E_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;A)$.*

If $\alpha < 1$ then

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = \begin{cases} |A| W_{\{\varepsilon^{\mathcal{R}}\}}(I) & u \equiv id, \\ \infty & \textit{otherwise}, \end{cases}$$

where $W_{\{\varepsilon^{\mathcal{R}}\}}(I) \in \mathbb{R}$ satisfies

$$Q\overline{W}^+(I) \le W_{\{\varepsilon^{\mathcal{R}}\}}(I) \le \hat{W}^+(I).$$

Further, if (H6) holds then

$$W_{\{\varepsilon^{\mathcal{R}}\}}(I) = Q\overline{W}^+(I),$$

while if (H7) and (H8) hold then

$$W_{\{\varepsilon^{\mathcal{R}}\}}(I) = \hat{W}^+(I).$$

If $\alpha > 1$ then

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = \begin{cases} \int_\omega Q_{2\times 3}\overline{W}^+(D_\alpha u)dx_\alpha & u \in W^{1,p}(\omega;\mathbb{R}^3), \\ \infty & \textit{otherwise}, \end{cases}$$

while if $\alpha = 1$ then

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = \begin{cases} \int_\omega \left\{ Q_{2\times 3}\overline{W}^+(D_\alpha u) + \Psi(u(x_1,x_2)) \right\} dx_\alpha & u \in W^{1,p}(\omega;\mathbb{R}^3) \\ \infty & \textit{otherwise.} \end{cases} .$$

7. Concluding remarks

Finally, we would like to summarize the results of this study. The reader is referred to the Introduction for a mechanical interpretation. We have assumed throughout that $W^\pm$ are continuous, satisfy (H1) and that Ψ is lower semicontinuous. Then,

1. If $\alpha < 1$, then the only targets that give rise to a finite energy are in $W^{1,p}(\omega;\mathbb{R}^3)$;
2. If $\alpha \geq 1$, then the only targets that give rise to a finite energy are of the form
$$u = \begin{cases} u^- \text{ on } \omega \times (-1,0), \\ u^+ \text{ on } \omega \times (0,1), \end{cases}$$
with $u^\pm \in W^{1,p}(\omega;\mathbb{R}^3)$;
3. If $\alpha < 1$ and (H3), (H4) hold true, then the limit behavior is local in the sense of Theorem 2.2 and the energy bounds of Lemma 2.4 are satisfied;
4. If $\alpha < 1$ and (H3), (H4), (H6) hold true, then we obtain the cohesive model of Theorem 3.1;
5. If $\alpha < 1$ and (H3), (H4), (H7), (H8) hold true, then we obtain the brittle model of Theorem 4.1;
6. If $\alpha > 1$, then we always obtain the decoupled model of Theorem 5.1;
7. If $\alpha = 1$, then we always obtain a model coupled by the surface energy term as in Theorem 5.2;
8. If one of the layers is identified as a rigid substrate, then we obtain all previous results upon replacing the bottom part of the target deformation field, u^-, by id, which will force the entire deformation u to be id in all cases where it is continuous across the interface.

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