Relaxation in BV Versus Quasiconvexification in $W^{1,p}$; a Model for the Interaction Between Fracture and Damage

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1 Introductory remarks

1.1 Setting of the problem:

In recent years damage and fracture have acquired a formidable status in the battlefields of solid mechanics. Perniciously weakening the stiffest solid samples damage has become the everpresent material tyrant, resisting all rationalizing attempts and defying an expanding taxonomy. Fracture, an older sibling, still delivers deadly slashes to even the most respectable materials and at the most unexpected angles.

At the same time the conceptual distance between them has been steadily shrinking: many a damaging material observed at a fine scale exhibits a vast array of tiny cracks while crack propagation strongly depends on the distributed damage in the so-called "fracture-process zone" (see e.g. [Chudnovsky Wu 90]).

A thorough understanding of the relationship between fracture and damage should therefore encompass the entire range of the damaging process, from its distributed onset to its climatic localization in the form of a "crack". This will *not* be achieved in the present study which operates under the simplistic premise of scale separation. Indeed it is assumed thereafter that the weakening micromechanisms that preside over the damaging process are taking place at a scale which is far smaller than that at which fracture takes place. It is also assumed that the configurational force needed to break atomic bonds and to promote crack propagation is not affected by the damaging process. Specifically, following [Francfort Marigo 93] our study will focus on a material that experiences brutal partial damage: the material is only allowed to brutally drop from its healthy state to its damaged state, the latter retaining some positive definite stiffness (hence the partial character of the damage). The mechanisms at the origin of that specific pattern of damage are not part of this investigation. Of course we readily concede that the presupposed independence of the configurational force upon such mechanisms might be construed as a fatal flaw by a stern observer of the lattice.

This material is further allowed to experience fracture, i.e., to develop material discontinuities at a *macroscopic* level. The *quasistatic* evolution of both damage and fracture is governed by a yield criterion, in accordance with the presupposed brittleness of the material. Furthermore the criterion is energetic: it compares the decrease in potential energy due to either damage (in which case it is a local decrease) or fracture (in which case it is a global decrease) to the resulting increment of energy dissipated through either process. Such a criterion is well known in fracture mechanics after the work of GRIFFITH (cf. e.g. [GDOUTOS 90], Chapter 4) and it may be shown to be the only thermodynamically compatible criterion in brutal damage (see [MARIGO 89]).

Both processes are further assumed to be irreversible. In other words, self-healing is absent from both the damaged part of the material and the cracks through that material.

The adopted model results in a time indexed sequence of partial minimization problems; the Lagrangian density depends on the deformation field and on the characteristic function of the damaged area and it must be such that, at the solution(s), the resulting potential energy is separately minimal at each time among all admissible field variations.

Unfortunately the search for stationary points of the potential energy is a delicate one from the standpoint of the calculus of variations and may lead to too many solutions in the absence of additional selection criteria (see [FRANCFORT MARIGO 91], Subsection 2.3). We impose as selection criterion the global stability of the solution(s). In other words we postulate that the material will try to minimize its potential energy among all admissible field variations. This is of course a drastic restriction and its impact on the physical appropriateness of the proposed model could well be devastating. The reader is kindly invited not too judge such a step too harshly; to reject global stability is also to forfeit all hope for a detailed mathematical analysis of the problem.

The potential energy to be minimized is generically of the form

$$\int_{\text{body}} W(\nabla u)dx + \lambda H^{N-1}(S(u)) - \int_{\text{body}} f \cdot u dx, \tag{1.1}$$

where u is the deformation field, $W(\xi)$ is an "elastic type" energy density, λ is a dissipation rate and f represents the body loadings. Two distinctive features lie at the crux of the mathematical difficulty: the functional space where u should live, namely BV – a convenient space so as to lend a meaning to $S(u)$, the jump set of u – and the *non* quasiconvex character of W which forces relaxation even in the absence of jumps of u. In other words the mathematical stake is the relaxation in the strong topology of $L^1(\Omega;\mathbb{R}^k)$ –

Ω is the domain occupied by the body – of a nonconvex functional of the form (1.1) (with appropriate growth, coercivity and regularity properties) over the space $BV(\Omega;\mathbb{R}^k)$.

Unfortunately the problem at hand is not well formulated because of a troublesome pathology of the space BV. The distributional gradient of a function u in $BV(\Omega;\mathbb{R}^k)$ – a finite Radon measure on Ω – may be decomposed as

$$Du = \nabla u dx + (u^+ - u^-) \otimes \nu_u H^{N-1} \lfloor S(u) + C(u),$$

where $\nabla u(x)$ is a $L^1(\Omega;\mathbb{R}^k)$-function (the density of the absolutely continuous part of the measure Du), $S(u)$, the jump set of u, is an $N-1$ rectifiable hypersurface with normal vector ν_u, u^+ and u^-, the traces of u on each side of $S(u)$, are such that, for H^{N-1}-a.e. x_0 in $S(u)$,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^N} \int_{\{y \in B(x_0,\epsilon);(y-x_0)\cdot\nu_{x_0} > (<) 0\}} |u(y) - u^+(u^-)(x_0)|^{N/(N-1)} dy = 0,$$

and $C(u)$, the Cantor part of the measure Du satisfies, for any Borel subset B of Ω,

$$H^{N-1}(B) < +\infty \Longrightarrow |C(u)|(B) = 0.$$

See e.g. [Federer 69], Thm. 4.5.9, [Vol'pert 69], [Federer Zimer 72], [Ambrosio 89b, 93 b].

The following result holds true (cf. [Ambrosio 89b]):
Any u in $L^1(\Omega;\mathbb{R}^k)$ may be approximated in the strong topology of $L^1(\Omega;\mathbb{R}^k)$ by a sequence u_n in $BV(\Omega;\mathbb{R}^k)$ such that

$$Du_n = C(u_n),$$

and consequently such that

$$\begin{cases} \nabla u_n = 0 \text{ a.e. in } \Omega, \\ H^{N-1}(S(u_n)) = 0. \end{cases}$$

Therefore, if the energy density W in (1.1) is such that $W \geq 0$, $W(0) = 0$, the relaxation of

$$\int_\Omega W(\nabla u) dx + \lambda H^{N-1}(S(u))$$

will be identically 0 for any u in $BV(\Omega;\mathbb{R}^k)$.

We are thus forced to restrict the relaxation to sequences u_n in $BV(\Omega;\mathbb{R}^k)$ such that

$$|C(u_n)| = 0,$$

i.e., to sequences u_n in $SBV(\Omega;\mathbb{R}^k)$ for which such a pathology will not occur. The space $SBV(\Omega;\mathbb{R}^k)$ was firstly introduced in [De Giorgi Ambrosio 88].

1.2 Outline:

Section 2 is devoted to the mathematical analysis of the strong-L^1 relaxation of functionals of the form

$$\int_\Omega W(\nabla u)dx + \lambda H^{N-1}(S(u)),\ \ \lambda > 0, \tag{1.2}$$

where W has p-growth ($p > 1$) and satisfies a local Lipschitz condition (see (2.3)). The result is that the relaxation of (1.2) is

$$\int_\Omega W^*(\nabla u)dx + \lambda H^{N-1}(S(u)),$$

where W^* is the quasiconvexification of W (see Theorem 2.1). The analysis relies heavily on the blow-up method (see [Fonseca Müller 92, 93]) and on Ambrosio's lower semi-continuity result in $SBV(\Omega;\mathbb{R}^k)$ for quasiconvex Carathéodory integrands with superlinear growth (see [Ambrosio 93a]).

Section 3 addresses the problem of the evolution of damage and fracture briefly described above. The quasistatic evolution is investigated at discretized times and the resulting sequence of minimization problems is obtained in Problem 3.4. A first subsection examines the first time step t_1 and demonstrates the existence of a minimizing deformation field u_1 for the relaxed problem (Proposition 3.7). From a mechanical standpoint the quasiconvexification of the energy density amounts to the formation of fine mixtures of the healthy and damaged phases at each point of the uncracked part Ω_1 of the body. To this pointwise mixture corresponds a local volume fraction of the damaged phase $\theta_1(x)$. Its existence is guaranteed through Proposition 3.11.

A second subsection investigates the following time steps. Because the relaxation at the first time step is only capable of producing a local volume fraction $\theta_1(x)$ of the damaged material the irreversibility constraint at later time steps—namely the monotonically increasing character of the characteristic function of the damaged material—has to be weakened: the volume fraction $\theta(x)$ of damaged material is constrained to monotonically increase from its value $\theta_1(x)$ at subsequent time steps. We do not know as of yet how to operate a bona fide relaxation of the problem over all time steps and we thus postulate the form of the relaxed problem for the subsequent time steps (Problem 3.13). This forces us to directly examine the weak lower semicontinuity of the resulting functional at those time steps. In fact it suffices to prove that the bulk energy density (that associated to the density $\nabla u(x)$) is quasiconvex. This problem is shown to be equivalent to a homogenization conjecture (see Conjecture 3.15) pertaining to the canonical character of periodic homogenization as far as the energy density associated to mixtures of two phases is concerned. If such a conjecture holds true – and it is known to be in some useful cases (see Remark 3.16)—then the time indexed sequence of "relaxed" problems has solutions and the evolution of the damage and fracture may proceed (Proposition 3.17).

It should be emphasized, at the close of this introduction, that the proposed model is but a tentative step in the direction of a mathematical theory of the quasistatic evolution of either damage or fracture, or both. As far as damage is concerned, it permits to avoid too much phenomenology in the choice of a damage variable: the damage variable, i.e., the local volume fraction of damaged material, appears as a byproduct of the search for a stable evolution (see [Francfort Marigo 93] for a more detailed insight into the structure of the resulting model when each phase is linearly elastic). As far as fracture is concerned, it avoids all reference to a particular crack shape, and furthermore does not a priori require the classical notch-type setting of fracture mechanics at the inception of the crack. It is however plagued by at least two deficiencies that could prove fatal in the long run: it operates, as already discussed, under the premise of global stability and it may generate a continuum of bifurcating solutions in its time-undiscretized version.

2 Relaxation in SBV versus relaxation in $W^{1,p}$

Our goal in this section is to explore the connection between the $W^{1,p}$–quasiconvexification of a functional of the gradient of a vector field and the relaxation in SBV of that same functional, whenever the associated energy is penalized through the introduction of a term of surface energy proportional to the measure of the "jump set" of the trial functions.

Specifically, let $W(\xi)$, $\xi \in \mathbb{R}^N \times \mathbb{R}^k$, be a real-valued function. Its quasiconvexification $W^*(\xi)$ is defined as

$$W^*(\xi) := \inf_{\varphi \in \mathcal{C}_0^\infty(Q;R^k)} \int_Q W(\xi + D\varphi(y))dy,$$

where Q is the unit cube centered at 0.
Define, for any open subset A of an open set Ω of $\mathbb{R}^N$ and any u in $BV(\Omega;\mathbb{R}^k)$

$$I(u,A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \left[\int_A W(\nabla u_n)dx + H^{N-1}(S(u_n)\cap A) \right] \right| \\ u_n \in SBV(A;\mathbb{R}^k), u_n \to u \text{ strongly in } L^1(A;\mathbb{R}^k) \right\}, \quad (2.1)$$

$$J(u,A) := \begin{cases} \int_A W^*(\nabla u)dx + H^{N-1}(S(u)\cap A) & \text{if } u \in SBV(A;\mathbb{R}^k) \\ +\infty & \text{otherwise} \end{cases} \quad (2.2)$$

We propose to prove the following theorem:

Theorem 2.1 *Assume that $W(\xi)$ satisfies*

$$\alpha|\xi|^p \le W(\xi) \le \beta(\delta + |\xi|^p),\ \xi \in \mathbb{R}^N \times \mathbb{R}^k, \quad (2.3a)$$

$$|W(\xi) - W(\eta)| \le \gamma(1 + |\eta|^{p-1} + |\xi|^{p-1})|\xi - \eta|,\ \xi,\eta \in \mathbb{R}^N \times \mathbb{R}^k, \quad (2.3b)$$

where $\alpha,\beta,\gamma > 0$, $\delta \ge 0$ and $1 < p < +\infty$. Then

$$J(u,\Omega) \le I(u,\Omega). \quad (2.4)$$

Further, under the following additional assumption:

$$\delta = 0 \text{ if } \Omega \text{ is unbounded}, \quad (2.3c)$$

$$J(u,\Omega) = I(u,\Omega).$$

Remark 2.2 In view of (2.3a), $W^*(\xi)$ also satisfies

$$\alpha|\xi|^p \le W^*(\xi) \le \beta(\delta + |\xi|^p), \quad (2.5)$$

and as W is continuous, W^* is also continuous (see [Dacorogna 89]).

Remark 2.3 If $u \in SBV(\Omega; \mathbb{R}^k)$, then Theorem 2.1 is a statement about the relaxation of

$$E(u, \Omega) := \int_\Omega W(\nabla u) dx + H^{N-1}(S(u)) \tag{2.6}$$

in $SBV(\Omega; \mathbb{R}^k)$ for the $L^1(\Omega; \mathbb{R}^k)$-topology. In particular,

$$J(u, \Omega) \text{ is } L^1(\Omega; \mathbb{R}^k)\text{-sequentially lower semi-continuous in } SBV(\Omega; \mathbb{R}^k). \tag{2.7}$$

Note that, in the light of Remark 2.2, a direct application of Theorem 4.5 in [Ambrosio 93a] to $J(u, \Omega)$ would yield the latter result (2.7), at least whenever Ω is bounded.

Remark 2.4 As pointed out in Remark 4.7 of [Ambrosio 93a], the existence of minimizers for $J(u, \Omega)$, or $I(u, \Omega)$, on $SBV(\Omega; \mathbb{R}^k)$ is not guaranteed through the direct method of the Calculus of Variations because of the absence of L^1-compactness of the minimizing sequences. Note, however, that if a uniform L^∞-bound is assumed on the minimizing sequence (for example if the infimum is taken over $SBV(\Omega; K)$, where K is a compact subset of $\mathbb{R}^k$), then the minimizing sequence is immediately seen to be bounded in $BV(\Omega; \mathbb{R}^k)$ (hence compact in $L^1(\Omega; \mathbb{R}^k)$) because the singular part $D^s u$ of the measure Du of a function u in $SBV(\Omega; \mathbb{R}^k)$ has the following total variation:

$$|D^s u|(\Omega) = \int_{S(u)} |u^+ - u^-|(x) dH^{N-1}(x),$$

hence

$$|D^s u|(\Omega) \leq 2\|u\|_{L^\infty(\Omega)} H^{N-1}(S(u)).$$

Proof of Theorem 2.1. The proof is divided into two steps. The first step proves that $J(u, \Omega) \geq I(u, \Omega)$ while the second step proves that $I(u, \Omega) \geq J(u, \Omega)$.

Step1: $J(u, \Omega) \geq I(u, \Omega)$ if $I(u, \Omega) < +\infty$.

We may as well assume that $J(u, \Omega) < +\infty$, otherwise there is nothing to prove. But then $I(u, \Omega) < +\infty$. Indeed $u \in SBV(\Omega; \mathbb{R}^k)$ since $J(u, \Omega) < +\infty$ and u is a test function for $I(\cdot, \Omega)$. By virtue of (2.5), $\nabla u \in L^p(\Omega)$ and $H^{N-1}(S(u)) < +\infty$ and, in view of (2.3a), we conclude that

$$I(u, \Omega) \leq \int_\Omega W(\nabla u) dx + H^{N-1}(S(u)) < +\infty.$$

Actually we will prove Lemma 2.5 below which only requires a weakened form of hypothesis (2.3) (namely $1 \le p < +\infty$ if $u \in SBV(\Omega;\mathbb{R}^k)$).

Lemma 2.5 *If (2.3a), (2.3b) are replaced by the following weakened hypotheses:*

$$\begin{cases} 1 \le p < +\infty & \text{in (2.3a), (2.3b) when } u \in SBV(\Omega;\mathbb{R}^k), \\ 1 < p < +\infty & \text{otherwise,} \end{cases} \tag{2.3$_w$}$$

and if (2.3c) holds true, then $I(u,\Omega) \le J(u,\Omega)$ *whenever* $I(u,\Omega) < +\infty$.

Remark 2.6 It will be proved in Lemma 2.14 below that if $u \in BV(\Omega,\mathbb{R}^k)$ and $I(u,\Omega) < +\infty$, then $u \in SBV(\Omega;\mathbb{R}^k)$. Thus Lemma 2.5 actually states that $I(u,\Omega) \le J(u,\Omega)$ whenever $I(u,\Omega) < +\infty$ and $1 \le p < +\infty$.

Proof of Lemma 2.5. The proof requires five substeps. The analysis is restricted to any bounded, open subset A of Ω and the last substep (Step 1-5) breaks free from this limitation. The first substep (Step 1-1) reduces the study to the case where $u \in BV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$ with $I(u,\Omega) < +\infty$. The second substep (Step 1-2) operates a reduction of $I(u,A)$ to a more easily handled $I_\infty(u,A)$ (see $(2.1)_\infty$ in Proposition 2.8 below) and further reduces u to be an element of $SBV(A;\mathbb{R}^k) \cap L^\infty(A,\mathbb{R}^k)$ (see Remark 2.9 below). The third substep (Step 1-3) establishes that, for such u's, $I_\infty(u,\cdot)$ is a Borel measure on A which is absolutely continuous with respect to the sum of the Lebesgue measure on A and of the restriction of H^{N-1} to $S(u)$. In other words $I_\infty(u,A) = \int_A h\,dx + \int_{S(u)\cap A} g\,dH^{N-1}$, where h and g are the associated densities. In the fourth substep (Step 1-4), h and g are proved to be less than or equal to $W^*(\nabla u)$ and 1, respectively, which establishes that $I_\infty(v,A) \le J(u,A)$, and the last substep permits to conclude.

Step 1-1: The following proposition holds true:

Proposition 2.7 *If A is a bounded, open subset of Ω and if, for every u in $BV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$ with $I(u,A) < +\infty$, $I(u,A) \le J(u,A)$, then*

$$I(u,A) \le J(u,A)$$

whenever $u \in BV(A;\mathbb{R}^k)$ and $I(u,A) < +\infty$.

Proof of Proposition 2.7. We may assume that $J(u,A) < +\infty$, otherwise there is nothing to prove. Since $I(u,A) < +\infty$, choose $u_n \in SBV(A;\mathbb{R}^k)$ with

$$\begin{cases} u_n \to u \text{ strongly in } L^1(A;\mathbb{R}^k), \\ \displaystyle\int_A W(\nabla u_n)\,dx + H^{N-1}(S(u_n)\cap A) < +\infty. \end{cases} \tag{2.8}$$

Note that the latter inequality, together with the first inequality in (2.3a), implies that $\nabla u_n \in L^p(A;\mathbb{R}^k)$.

Define $\varphi_q \in W_0^{1,\infty}(\mathbb{R}^k \times \mathbb{R}^k)$ as

$$\varphi_q(x) := \begin{cases} x & \text{if } |x| \le \mathrm{e}^q, \\ 0 & \text{if } |x| \ge \mathrm{e}^{q+1}, \end{cases}$$

with $|\nabla\varphi_q(x)| \le 1$. Then, according to [Vol'pert 69], $\varphi_q(u_n)$(resp. $\varphi_q(u)$) belong to SBV(resp. BV)$(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k)$, and

$$\begin{cases} \|\varphi_q(u_n)\|_{L^\infty(A)} \le \mathrm{e}^q, \\ S(\varphi_q(u_n))\cap A \subset S(u_n)\cap A, \\ \nabla(\varphi_q(u_n))(x) = \nabla\varphi_q(u_n(x))\circ\nabla u_n(x), \text{ for a.e. } x \text{ in } \Omega. \end{cases} \tag{2.9}$$

Furthermore,

$$\varphi_q(u_n) \to \varphi_q(u) \text{ strongly in } L^1(A;\mathbb{R}^k). \tag{2.10}$$

But, according to (2.8), (2.9), and upon recalling the second inequality in (2.3a),

$$\int_A W(\nabla\varphi_q(u_n)) + H^{N-1}(S(\varphi_q(u_n))\cap A)$$
$$\le \beta(\delta\mathrm{meas}(A) + \int_A |\nabla u_n|^p dx) + H^{N-1}(S(u_n)\cap A) < +\infty. \tag{2.11}$$

By virtue of (2.9) and (2.10), $\{\varphi_q(u_n)\}$ is an admissible sequence in the definition of $I(\varphi_q(u);A)$ and, in view of (2.11), $I(\varphi_q(u);A) < +\infty$. Thus, by hypothesis

$$I(\varphi_q(u);A) \le J(\varphi_q(u);A). \tag{2.12}$$

Now, as q tends to ∞,

$$\varphi_q(u) \to u \text{ strongly in } L^1(A;\mathbb{R}^k),$$

while, as we have already seen, $\varphi_q(u)$ belongs to $BV(A;\mathbb{R}^k)$ and $I(\varphi_q(u);A) < +\infty$. Since, by its very definition, $I(\cdot,A)$ is sequentially L^1- lower semicontinuous on BV (see Remark 2.3 for a remark along these lines), we conclude that

$$I(u,A) \leq \liminf_{q\to+\infty} I(\varphi_q(u);A).$$

In view of (2.12), it remains to show that

$$\liminf_{q\to+\infty} J(\varphi_q(u);A) \leq J(u,A). \tag{2.13}$$

But

$$H^{N-1}(S(\varphi_q(u))\cap A) \leq H^{N-1}(S(u)\cap A), \tag{2.14}$$

and, appealing to (2.3) and to Remark 2.2

$$\begin{aligned}\int_A W^*(\nabla\varphi_q(u))dx &= \int_{|u(x)|\leq e^q} W^*(\nabla u)dx + \int_{|u(x)|>e^q} W^*(\nabla\varphi_q(u))dx \\ &\leq \int_A W^*(\nabla u)dx + \beta\int_{|u(x)|>e^q}(\delta+|\nabla u|^p)dx. \end{aligned}\tag{2.15}$$

Since ∇u belongs to $L^p(A;\mathbb{R}^k)$ ($J(u,A)<+\infty$) and u belongs to $L^1(A;\mathbb{R}^k)$,

$$\lim_{q\to+\infty}\int_{|u(x)|>e^q}(\delta+|\nabla u|^p)dx = 0,$$

which, in view of (2.14), (2.15), proves (2.13) and completes the proof of Proposition 2.7. ■

Proposition 2.7 enables us to limit our investigation to the case where u belongs to $BV(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k)$ with $I(u,A)<+\infty$. For such u's a convenient reduction of $I(u,A)$ may be performed. This is the object of Step 1-2.

Step 1-2. The following proposition holds true:

Proposition 2.8 *If* $u \in BV(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k)$ *and if* $I(u,A) < +\infty$, *where* A *is a bounded, open subset of* Ω, *then*

$$I(u,A) = I_\infty(u,A)$$

with

$$I_\infty(u,A) \;:=\; \inf_{\{u_n\}} \Bigg\{ \liminf_{n\to+\infty} \Bigg[\int_A W(\nabla u_n)dx + H^{N-1}(S(u_n)\cap A) \Bigg] \Bigg| \; u_n \in SBV(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k), u_n \to u \text{ strongly in } L^1(A;\mathbb{R}^k), \sup_n \|u_n\|_{L^\infty(A)} \le \mathcal{C} < +\infty \Bigg\}. \quad (2.1)_\infty$$

Proof of Proposition 2.8. Obviously $I_\infty(u,A) \ge I(u,A)$. The proof that $I(u,A) \ge I_\infty(u,A)$ follows that of a related result in [CELADA DAL MASO 93] or [BARROSO BOUCHITTÉ BUTTAZZO FONSECA 93]; see Lemma 3.7 of the latter reference. It essentially consists in truncating a minimizing sequence $\{u_n\}$ for $I(u,A)$ at a given distance from the origin. Specifically, introduce the following radial truncations $\varphi_i \in C_0^1(\mathbb{R}^k \times \mathbb{R}^k)$ defined as

$$\varphi_i(x) := \begin{cases} x & \text{if } |x| < \mathrm{e}^i, \\ 0 & \text{if } |x| \ge \mathrm{e}^{i+1}, \end{cases}$$

and $|\nabla\varphi_i(x)| \le 1$. Note that φ_i is a Lipschitz function with Lipschitz constant 1. For a fixed $\epsilon > 0$, consider a sequence u_n in $SBV(\Omega;\mathbb{R}^k)$ such that

$$u_n \to u \text{ strongly in } L^1(A;\mathbb{R}^k),$$

$$I(u,A) + \frac{\epsilon}{2} \ge \lim_{n\to\infty} \Bigg[\int_A W(\nabla u_n)dx + H^{N-1}(S(u_n)\cap A) \Bigg]. \quad (2.16)$$

Define

$$w_n^i(x) := \varphi_i(u_n(x)).$$

Then, according to [VOL'PERT 69], w_n^i is in $SBV(A;\mathbb{R}^k)$ and

$$\begin{cases} \|w_n^i\|_{L^\infty(A)} \le \mathrm{e}^i, \\ \\ S(w_n^i)\cap A \subset S(u_n)\cap A, \\ \nabla w_n^i(x) = \nabla\varphi_i(u_n(x)) \circ \nabla u_n(x), \text{ for a.e. } x \text{ in } \Omega. \end{cases} \quad (2.17)$$

If i is large enough, then $\|u\|_{L^\infty(A)} \le \mathrm{e}^i$, thus $u = \varphi_i(u)$. Then

$$\begin{aligned} \|w_n^i - u\|_{L^1(A)} &= ||\varphi_i(w_n) - \varphi_i(u)||_{L^1(A)} \\ &\le \|u_n - u\|_{L^1(A)}. \end{aligned} \quad (2.18)$$

The sequence $\{\varphi_i\}$ is rearranged so that the above inequality holds true for every $i \geq 1$. Furthermore, in view of (2.17),

$$\int_A W(\nabla w_n^i)dx = \int_{|u_n(x)|\leq \mathrm{e}^i} W(\nabla u_n)dx + \int_{\mathrm{e}^i<|u_n(x)|\leq \mathrm{e}^{i+1}} W(\nabla\varphi_i \circ \nabla u_n)dx + \int_{|u_n(x)|>\mathrm{e}^{i+1}} W(0)dx.$$

But, by virtue of the second inequality in (2.3a),

$$\begin{aligned}
&\int_{\mathrm{e}^i<|u_n(x)|\leq \mathrm{e}^{i+1}} W(\nabla\varphi_i \circ \nabla u_n)dx + \int_{|u_n(x)|>\mathrm{e}^{i+1}} W(0)dx \\
&\leq \beta\delta \mathrm{meas}\{|u_n(x)| > \mathrm{e}^i\} + \beta \int_{\mathrm{e}^i<|u_n(x)|\leq \mathrm{e}^{i+1}} |\nabla u_n|^p dx \\
&\leq \frac{\beta}{\mathrm{e}^i}\delta \|u_n\|_{L^1(A)} + \beta \int_{\mathrm{e}^i<|u_n(x)|\leq \mathrm{e}^{i+1}} |\nabla u_n|^p dx.
\end{aligned}$$

The above inequalities, together with (2.17), imply that, for any integer $M > 0$,

$$\sum_{i=1}^{M} \Big[\int_A W(\nabla w_n^i)dx + H^{N-1}(S(w_n^i) \cap A)\Big] \leq M\Big[\int_A W(\nabla u_n)dx + H^{N-1}(S(u_n) \cap A)\Big] + \beta\delta\|u_n\|_{L^1(A)} \sum_{i=1}^{M} \frac{1}{\mathrm{e}^i} + \beta \int_A |\nabla u_n|^p dx. \quad (2.19)$$

Note that, since u_n converges (strongly) to u in $L^1(A;\mathbb{R}^k)$,

$$\|u_n\|_{L^1(A)} \leq \mathcal{C} < +\infty,$$

while the first inequality in (2.3a) and (2.16) yield that, for n large enough,

$$\int_A |\nabla u_n|^p dx \leq \mathcal{C} < +\infty.$$

Consequently (2.19) reads as

$$\frac{1}{M}\sum_{i=1}^{M} \Big[\int_A W(\nabla w_n^i)dx + H^{N-1}(S(w_n^i) \cap A)\Big] \leq \int_A W(\nabla u_n)dx + H^{N-1}(S(u_n) \cap A) + \frac{\beta\delta\mathcal{C}}{M} \sum_{i=1}^{M} \frac{1}{\mathrm{e}^i} + \frac{\beta\mathcal{C}}{M}.$$

Upon choosing $M = M(\epsilon)$ large enough (independently of n) we conclude that

$$\frac{1}{M}\sum_{i=1}^{M}\left[\int_A W(\nabla w_n^i)dx + H^{N-1}(S(w_n^i)\cap A)\right] \le \int_A W(\nabla u_n)dx + H^{N-1}(S(u_n)\cap A) + \frac{\epsilon}{2}.$$

Thus there exists $i(n) \in \{1, ..., M\}$ such that

$$\int_A W(\nabla w_n^{i(n)})dx + H^{N-1}(S(w_n^{i(n)})\cap A) \le \int_A W(\nabla u_n)dx + H^{N-1}(S(u_n)\cap A) + \frac{\epsilon}{2}. \tag{2.20}$$

Set $\bar{u}_n := w_n^{i(n)}$ and recall (2.16), (2.17), (2.18) and (2.20); the sequence $\bar{u}_n$ satisfies

$$\begin{cases} \bar{u}_n \to u \text{ strongly in } L^1(A;\mathbb{R}^k), \\ \|\bar{u}_n\|_{L^\infty(A)} \le \mathrm{e}^{M(\epsilon)}, \\ I(u,A) + \epsilon \ge \limsup_{n\to+\infty} \int_A W(\nabla \bar{u}_n)dx + H^{N-1}(S(\bar{u}_n)\cap A), \end{cases}$$

which was the sought result. The proof of Proposition 2.8 is complete. ■

Now, by virtue of Proposition 2.8, if u is an element of $BV(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k)$ and $I(u,A) < +\infty$, there exists a sequence $\{u_n\}$ in $SBV(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k)$ such that

$$\begin{cases} \sup_n \|u_n\|_{L^\infty(A)} \le \mathcal{C} < +\infty, \\ u_n \to u \text{ strongly in } L^1(A;\mathbb{R}^k), \\ \sup_n H^{N-1}(S(u_n)\cap A) \le \mathcal{C} < +\infty, \\ \sup_n \|\nabla u_n\|_{L^p(A)} \le \mathcal{C} < +\infty. \end{cases} \tag{2.21}$$

Thus, if $p > 1$, a direct application of Theorem 2.1 in [Ambrosio 89a] implies that $u \in SBV(A;\mathbb{R}^k)$, while, if $p = 1$, $u \in SBV(A;\mathbb{R}^k)$ in view of hypothesis $(2.3)_w$.

Remark 2.9 In conclusion of this step, we have thus established that, if A is a bounded, open subset of Ω, if $u \in BV(A;\mathbb{R}^k)$ with $I(u,A) < +\infty$ and if $(2.3\text{a})_w$ holds true, then $I(u,A) \le J(u,A)$, provided that $I_\infty(v,A) \le J(v,A)$ whenever $v \in SBV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$(with $I(v,A) < +\infty$).

Step 1-3. The following proposition holds true:

Proposition 2.10 *Assume that $u \in SBV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$ and that $E(u,A) < +\infty$, where A is a bounded, open subset of Ω(recall (2.6) for a definition of E). Then $I_\infty(u,\cdot)$ extends to a nonnegative Radon measure on A which is absolutely continuous with respect to $\mathcal{L}^N + H^{N-1}\lfloor S(u)$, where $\mathcal{L}^N$ stands for the Lebesgue measure on A and $H^{N-1}\lfloor S(u)$ denotes the restriction of the $(N-1)-$Hausdorff measure to $S(u)$.*

Remark 2.11 In the context of Proposition 2.10, if $E(u,A) < +\infty$, (2.3a) implies that $J(u,A) < +\infty$ and, because $u \in SBV(A;\mathbb{R}^k)$, $I(u,A) < +\infty$. For the present purpose, we could have as well assumed that $J(u,A) < +\infty$.

Proof of Proposition 2.10. Note that, in view of the second inequality in (2.3) (or rather in $(2.3)_w$), for every open subset B of A,

$$I_\infty(u,B) \le \beta \int_B (\delta + |\nabla u|^p)dx + H^{N-1}\lfloor S(u)(B).$$

We now propose to prove that $I_\infty(u,\cdot)$ is the trace on $\mathcal{A} := \{B \text{ open } | B \subset A\}$ of a Borel regular measure on A. To this effect, De Giorgi–Letta's criterion [De Giorgi Letta 77] is applied; four conditions must be fulfilled for any elements B, C of $\mathcal{A}$, namely,

$$\begin{aligned}
&\text{a)} \quad I_\infty(u,B) \le I_\infty(u,C), \ \text{ if } B \subset C,\\
&\text{b)} \quad I_\infty(u,B\cup C) = I_\infty(u,B) + I_\infty(u,C), \ \text{ if } B \cap C = \emptyset,\\
&\text{c)} \quad I_\infty(u,B\cup C) \le I_\infty(u,B) + I_\infty(u,C),\\
&\text{d)} \quad I_\infty(u,B) = \sup\{I_\infty(u,C) | C \subset\subset B\}.
\end{aligned}$$

Conditions a) and b) are trivially satisfied by $I_\infty(u,\cdot)$. To establish c) and d), we follow the method indicated in [Ambrosio Mortola Tortorelli 91].

First note that if $C \in \mathcal{A}$, then for any fixed $\epsilon > 0$, there exists $C_\epsilon \subset\subset C$ such that

$$I_\infty(u, C\backslash\bar{C}_\epsilon) \le E(u, C\backslash\bar{C}_\epsilon) \le \epsilon. \tag{2.22}$$

Indeed we choose C_ϵ such that

$$E(u, C\backslash\bar{C}_\epsilon) = \int_{C\backslash\bar{C}_\epsilon} W(\nabla u)dx + H^{N-1}(S(u)\cap(C\backslash\bar{C}_\epsilon)) \leq \epsilon,$$

which is always possible since $W(\nabla u) \in L^1(C;\mathbb{R}^k)$ and $H^{N-1}(S(u)\cap C) < +\infty$; (2.22) follows immediately by taking $u_n = u$ in the definition of $I_\infty(u, C\backslash\bar{C}_\epsilon)$ and recalling that $u \in SBV(A;\mathbb{R}^k)$. If c) holds true, then d) holds true because, by virtue of (2.22), for any $\epsilon > 0$, we are at liberty to choose $B_\epsilon \subset\subset C_\epsilon \subset\subset B$ such that

$$I_\infty(u, B\backslash\bar{B}_\epsilon) \leq \epsilon.$$

Then, since

$$B \subset (B\backslash\bar{B}_\epsilon)\cup C_\epsilon,$$

condition c) implies that

$$I_\infty(u, B) \leq I_\infty(u, B\backslash\bar{B}_\epsilon) + I_\infty(u, C_\epsilon) \leq I_\infty(u, C_\epsilon) + \epsilon.$$

Letting ϵ tend to 0 yields condition d).
It remains to establish c). To this effect, $B\cup C$ is decomposed, for any $t \in (0,1)$, into

$$\begin{aligned} B_t &:= \{x \in B\cup C|\, t\ \mathrm{dist}(x, B\backslash C) < (1-t)\ \mathrm{dist}(x, C\backslash B)\}, \\ C_t &:= \{x \in B\cup C|\, t\ \mathrm{dist}(x, B\backslash C) > (1-t)\ \mathrm{dist}(x, C\backslash B)\}, \\ S_t &:= (B\cup C)\backslash(B_t\cup C_t). \end{aligned}$$

Since $\cup_{t\in(0,1)} S_t \subset B\cup C$ and $I(u, B\cup C) < +\infty$, we have

$$\mathcal{L}_N(\bigcup_{t\in(0,1)} S_t) + H^{N-1}(S(u)\cap(\bigcup_{t\in(0,1)} S_t)) < +\infty. \tag{2.23}$$

The sets S_t are pairwise disjoint; thus (2.23) implies that

$$\sum_{t\in(0,1)} [\mathcal{L}_N(S_t) + H^{N-1}(S(u)\cap S_t)] < +\infty,$$

from which we infer the existence of $t_0 \in (0,1)$ such that

$$\mathcal{L}_N(S_{t_0}) + H^{N-1}(S(u)\cap S_{t_0}) = 0. \tag{2.24}$$

For a given $\epsilon > 0$, (2.22) permits to consider $B_\epsilon \subset\subset B'_\epsilon \subset\subset B_{t_0}$ and $C_\epsilon \subset\subset C'_\epsilon \subset\subset C_{t_0}$ such that

$$E(u, B_{t_0} \backslash \bar{B}_\epsilon) \leq \frac{\epsilon}{2},\ E(u, C_{t_0} \backslash \bar{C}_\epsilon) \leq \frac{\epsilon}{2}.$$

Then, by virtue of (2.24), and also because $B_{t_0} \cap C_{t_0} = \emptyset$, we obtain

$$\begin{aligned} E(u, (B \cup C) \backslash (\bar{B}_\epsilon \cup \bar{C}_\epsilon)) &= \int_{B_{t_0} \backslash \bar{B}_\epsilon} W(\nabla u) dx \\ &+ \int_{C_{t_0} \backslash \bar{C}_\epsilon} W(\nabla u) dx + H^{N-1}(S(u) \cap (B_{t_0} \backslash \bar{B}_\epsilon)) \\ &+ H^{N-1}(S(u) \cap (C_{t_0} \backslash \bar{C}_\epsilon)) \\ &= E(u, B_{t_0} \backslash \bar{B}_\epsilon) + E(u, C_{t_0} \backslash \bar{C}_\epsilon) \leq \epsilon, \end{aligned}$$

from which we conclude that

$$I_\infty(u, (B \cup C) \backslash (\bar{B}_\epsilon \cup \bar{C}_\epsilon)) \leq \epsilon. \tag{2.25}$$

Assume that

$$I_\infty(u, C) \leq I_\infty(u, D) + I_\infty(u, C \backslash \bar{E}), \tag{2.26}$$

whenever C, D, E are elements of $\mathcal{A}$ such that $E \subset\subset D \subset\subset C$. Then

$$I_\infty(u, B \cup C) \leq I_\infty(u, B'_\epsilon \cup C'_\epsilon) + I_\infty(u, (B \cup C) \backslash (\bar{B}_\epsilon \cup \bar{C}_\epsilon)).$$

Since $B'_\epsilon \cap C'_\epsilon = \emptyset$, conditions a), b) together with (2.25) yield

$$I_\infty(u, B \cup C) \leq I_\infty(u, B) + I_\infty(u, C) + \epsilon$$

and c) is obtained upon letting ϵ tend to zero.

It remains to prove (2.26). Note that $I(u, A) < +\infty$ because $E(u, A) < +\infty$ and $u \in SBV(A; \mathbb{R}^k)$, thus is a valid test function in $(2.1)_\infty$. For a given $\epsilon > 0$ there exists, according to Proposition 2.8, a sequence $\{u_n\}$ in $SBV(C \backslash \bar{E}; \mathbb{R}^k) \cap L^\infty(C \backslash \bar{E}; \mathbb{R}^k)$ satisfying

$$\begin{cases} u_n \to u \text{ strongly in } L^1(C \backslash \bar{E}; \mathbb{R}^k), \\ \sup_n \|u_n\|_{L^\infty(C \backslash \bar{E})} \leq \mathcal{C} \leq +\infty, \\ I_\infty(u, C \backslash \bar{E}) + \epsilon \geq \lim_{n \to +\infty} \Big[\int_{C \backslash \bar{E}} W(\nabla u_n) dx + H^{N-1}(S(u_n) \cap (C \backslash \bar{E})) \Big], \end{cases}$$

as well as a sequence $\{v_n\}$ in $SBV(D;\mathbb{R}^k)\cap L^\infty(D;\mathbb{R}^k)$ such that

$$\begin{cases} v_n \to u \text{ strongly in } L^1(D;\mathbb{R}^k), \\ \sup_n \|v_n\|_{L^\infty(D)} \le \mathcal{C}, \\ I_\infty(u,D)+\epsilon \ge \lim_{n\to+\infty}[\int_D W(\nabla u_n)dx + H^{N-1}(S(v_n)\cap D)]. \end{cases}$$

The L^1- convergence and the uniform $L^\infty-$ estimate on both u_n and v_n actually imply that

$$u_n - v_n \to 0 \text{ strongly in } L^p(D\backslash\bar{E};\mathbb{R}^k). \tag{2.27}$$

We propose to construct a sequence w_n over the whole domain C by connecting u_n to v_n across $D\backslash\bar{E}$. To this end, fix $\rho>0$ and $D_\rho\subset D$ open such that $\text{meas}(D_\rho\setminus E)<\rho$ and the set $D_\rho\backslash\bar{E}$ is partitioned into two layers $S_1^{(i)},\ i=1,2$, defined as

$$S_1^{(1)} := \{x\in D_\rho\backslash\bar{E}|\ 0<\text{dist}(x,\partial E)\le \frac{1}{2}\text{dist}(\partial E,\mathbb{R}^N\setminus D_\rho)\}$$
$$S_1^{(2)} := \{x\in D_\rho\backslash\bar{E}|\ \frac{1}{2}\text{dist}(\partial E,\mathbb{R}^N\setminus D_\rho)<\text{dist}(x,\partial E)\le \text{dist}(\partial E,\mathbb{R}^N\setminus D_\rho)\}.$$

Then, for every n, either $S_1=S_1^{(1)}$ or $S_1=S_1^{(2)}$ is such that

$$\begin{aligned} &\int_{S_1} W(\nabla u_n)dx + H^{N-1}(S(u_n)\cap S_1) \\ &+\int_{S_1} W(\nabla v_n)dx + H^{N-1}(S(v_n)\cap S_1) \quad \le \quad \frac{M}{2} \end{aligned} \tag{2.28}$$

where

$$+\infty > M = \sup_n \Big[\int_{D\backslash\bar{E}} (W(\nabla u_n)+W(\nabla v_n))dx + H^{N-1}(S(u_n)\cap(D\backslash\bar{E})) \\ +H^{N-1}(S(v_n)\cap(D\backslash\bar{E}))\Big].$$

This is obvious by contradiction. Consequently one of the layers, denoted by S_1, must satisty (2.28) for a subsequence $\{u_n^{(1)},v_n^{(1)}\}$ of $\{u_n,v_n\}$. Furthermore, by virtue of (2.27), $\{u_n^{(1)},v_n^{(1)}\}$ may be chosen such that

$$\frac{1}{|\beta_1-\alpha_1|^p}\int_{S_1}|u_n^{(1)}-v_n^{(1)}|^p dx < \frac{1}{2}.$$

This procedure is repeated recursively and it yields a sequence of layers

$$S_j := \{x \in D_\rho \backslash \bar{E} |\ 0 < \alpha_j < \mathrm{dist}(x, \partial E) < \beta_j\}$$

with $|\beta_j - \alpha_j| \searrow 0$ and

$$\left\{ \begin{array}{l} \displaystyle\int_{S_j} W(\nabla u_n^{(j)})dx + H^{N-1}(S(u_n^{(j)}) \cap S_j) + \\ \\ \displaystyle\int_{S_j} W(\nabla v_n^{(j)})dx + H^{N-1}(S(v_n^{(j)}) \cap S_j) \leq \frac{M}{j+1}, \\ \\ \displaystyle\frac{1}{|\beta_j - \alpha_j|^p} \int_{S_j} |u_n^{(j)} - v_n^{(j)}|^p dx < \frac{1}{j+1}. \end{array} \right. \tag{2.29}$$

Take $\varphi_j \in \mathcal{C}_0^\infty(C)$ with $0 \leq \varphi_j \leq 1$, $\|\nabla \varphi_j\|_{L^\infty(D_\rho \backslash \bar{E})} \leq \frac{1}{|\beta_j - \alpha_j|}$ and

$$\begin{array}{l} \varphi_j(x) \equiv 1 \quad \text{if} \quad x \in E \text{ or } \mathrm{dist}(x, \partial E) \leq \alpha_j, \\ \varphi_j(x) \equiv 0 \quad \text{if} \quad x \notin E \text{ and } \mathrm{dist}(x, \partial E) \geq \beta_j, \end{array}$$

and set

$$w_j(x) := (1 - \varphi_j(x)) u_j^{(j)}(x) + \varphi_j(x) v_j^{(j)}(x).$$

Certainly

$$\|w_j - u\|_{L^1(C; R^k)} \leq \int_{C \backslash \bar{E}} |u_j^{(j)} - u| dx + \int_D |v_j^{(j)} - u| dx,$$

thus,

$$w_j \to u \text{ strongly in } \text{Ł}^1(C, \mathbb{R}^k). \tag{2.30}$$

Furthermore,

$$H^{N-1}(S(w_j) \cap C) \leq H^{N-1}(S(u_j^{(j)}) \cap (C \backslash \bar{E})) + H^{N-1}(S(v_j^{(j)}) \cap D), \tag{2.31}$$

while, since

$$\nabla w_j = \nabla u_j^{(j)}(1 - \varphi_j) + \nabla v_j^{(j)} \varphi_j + (v_j^{(j)} - u_j^{(j)}) \otimes \nabla \varphi_j,$$

we obtain, upon recalling the second inequality in (2.3a) (or rather $(2.3\text{a})_w$)

$$\begin{array}{rcl} \displaystyle\int_C W(\nabla w_j) dx & \leq & \displaystyle\int_{C \backslash \bar{E}} W(\nabla u_j^{(j)}) dx + \int_D W(\nabla v_j^{(j)}) dx \\ & + & \displaystyle\beta' \int_{S_j} \{\delta + |\nabla u_j^{(j)}|^p + |\nabla v_j^{(j)}|^p\} dx \\ & + & \displaystyle\beta' \int_{S_j} \{\delta + \frac{1}{|\beta_j - \alpha_j|^p} |v_j^{(j)} - u_j^{(j)}|^p\} dx \end{array}$$

for some $\beta' > 0$. Thus, using (2.29) and appealing this time to the first inequality in $(2.3a)_w$,

$$\begin{aligned}\int_C W(\nabla w_j)dx &\leq \int_{C\setminus\bar{E}} W(\nabla u_j^{(j)})dx + \int_D W(\nabla v_j^{(j)})dx \\ &+\beta'\left\{2\delta|S_j| + \frac{1}{j+1} + \frac{2M}{j+1}\right\}. \qquad (2.32)\end{aligned}$$

Collecting (2.30), (2.31), (2.32) and passing to the limit in j yields

$$\begin{aligned}I_\infty(u,C) &\leq I_\infty(u,C\setminus\bar{E}) + I_\infty(u,D) + 2\epsilon \\ &+ \beta' \limsup_{j\to+\infty}\left\{2\delta\rho + \frac{1}{j+1}(2M+1)\right\} \\ &= I_\infty(u,C\setminus\bar{E}) + I_\infty(u,D) + 2\epsilon + 2\beta'\delta\rho\end{aligned}$$

and (2.26) is obtained upon letting ρ and ϵ tend to 0. Thus $I_\infty(u,\cdot)$ extends to a nonnegative finite Radon measure on A.

Since $E(u,A) < +\infty$ then $H^{N-1}(S(u) \cap A)$ is finite. Thus $H^{N-1}\lfloor S(u)$ is a Radon measure on A. Taking u as a test function for $I_\infty(u,A)$, we have $I_\infty(u,A) \leq E(u,A)$ which implies that $I_\infty(u,\cdot)$ is absolutely continuous with respect to $\mathcal{L}^N + H^{N-1}\lfloor S(u)$. The Lebesgue decomposition theorem guarantees the existence of two densities h and g which are, respectively, Lebesgue measurable on A and $H^{N-1}\lfloor S(u)-$measurable on A, such that

$$I_\infty(u,\cdot) = h\mathcal{L}^N + gH^{N-1}\lfloor S(u). \qquad (2.33)$$

Furthermore h and g are the Radon-Nikodym derivative of $I_\infty(u,\cdot)$ with respect to $\mathcal{L}^N$ and $H^{N-1}\lfloor S(u)$, respectively, i.e., if $Q(x_0,\epsilon)$ denotes, for a given x_0 in A, the cube of side ϵ centered at x_0, then

$$h(x_0) = \lim_{\epsilon\to 0^+} \frac{I_\infty(u,Q(x_0,\epsilon))}{\epsilon^N}, \text{ for } \mathcal{L}^N\text{−a.e. } x_0 \text{ in } A, \qquad (2.34)$$

$$g(x_0) = \lim_{\epsilon\to 0^+} \frac{I_\infty(u,Q(x_0,\epsilon))}{H^{N-1}(S(u)\cap Q(x_0,\epsilon))} \qquad (2.35)$$
$$\text{for } H^{N-1}\text{−a.e. } x_0 \text{ in } S(u)\cap A$$

and the proof of Proposition 2.10 is complete. (See e.g. [Evans Gariepy 92] Subsection 1.6.2., Theorem 2.3. Note that in that reference the results are expressed in terms of small balls instead of small cubes.) ■

We assume now that $u \in SBV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$ *and that* $J(u,A) < +\infty$, otherwise there is nothing to prove. Then, according to Remark 2.11, Proposition 2.10 still holds true. We propose to show in the next substep that

$$\begin{cases} h(x) \leq W^*(\nabla u(x)), \ \mathcal{L}^N - \text{a.e. in } A, \\ \\ g(x) \leq 1, \quad H^{N-1} - \text{a.e. in} A, \end{cases}$$

from which it will be immediately concluded that

$$I_\infty(u,A) \leq J(u,A).$$

Step 1-4. The proof that $g(x) \leq 1$, H^{N-1}–a.e. in $S(u) \cap A$ is straightforward. Indeed, since $u \in SBV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$,

$$I_\infty(u,B) \leq \int_B W(\nabla u)dx + H^{N-1}(S(u) \cap B),$$

where B is an arbitrary open subset of A, and consequently for all Borel subsets B. It suffices to consider B to be an arbitrary Borel subset of $S(u)$ to conclude that $g(x) \leq 1$ H^{N-1} – a.e. in$A$.

Remark 2.12 The mutually singular character of $\mathcal{L}^N$ and $H^{N-1}\lfloor S(u)$ also implies that

$$\lim_{\epsilon \to 0^+} \frac{H^{N-1}(S(u) \cap Q(x_0,\epsilon))}{\epsilon^N} = 0, \mathcal{L}^N - \text{a.e. in } A. \tag{2.36}$$

This relation will be used thereafter.

It remains to show that $h(x) \leq W^*(\nabla u(x))$, $\mathcal{L}^N$–a.e. in A. Since $J(u,A) < +\infty$, $\nabla u \in L^p(A;\mathbb{R}^k)$ and for $\mathcal{L}^N$–a.e. x_0 in A, (2.34), (2.36) hold true together with

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^N} \int_{Q(x_0,\epsilon)} |\nabla u(x) - \nabla u(x_0)|^p dx = 0 \tag{2.37}$$

Note that (2.37) is merely a statement about the Lebesgue points of ∇u. The argument uses a blow up technique similar to that in [Fonseca Müller 92]. Fix a suitable x_0 in A, satisfying (2.34), (2.36) and (2.37). For any positive integer n, there exists, by the very definition of W^*, an element $\varphi_n \in \mathcal{C}_0^\infty(Q, R^k)$ such that

$$W^*(\nabla u(x_0)) + \frac{1}{n} \geq \int_Q W(\nabla u(x_0) + \nabla \varphi_n(x))dx.$$

Remark that, by virtue of the first inequality in (2.3) (or rather $(2.3)_w$)

$$\|\nabla \varphi_n\|_{L^p(A)} \leq \mathcal{C} < +\infty.$$

Extend φ_n by $Q-$periodicity to $\mathcal{C}^\infty(\mathbb{R}^N;\mathbb{R}^k)$ and set

$$u_n^m(x) := \nabla u(x_0)x + \frac{1}{m}\varphi_n(mx).$$

Then,

$$\begin{aligned}\int_Q W(\nabla u_n^m)dx &= \frac{1}{m^N}\int_{mQ} W(\nabla u(x_0) + \nabla\varphi_n(x))dx \\ &= \int_Q W(\nabla u(x_0) + \nabla\varphi_n(x))dx \\ &\leq W^*(\nabla u(x_0)) + \frac{1}{n}.\end{aligned}$$

Further, as m tends to $+\infty$,

$$u_n^m \to \nabla u(x_0)x \text{ strongly in } L^1(Q;\mathbb{R}^k),$$

and a straightforward diagonalization process yields a sequence v_n $(= u_n^{m(n)})$ of smooth functions such that

$$\begin{cases} \sup_n \|v_n\|_{L^\infty(Q)} \leq \mathcal{C}, \\ \sup_n \|\nabla v_n\|_{L^p(Q)} \leq \mathcal{C}, \\ v_n \to \nabla u(x_0)x \text{ strongly in } L^1(Q;R^k), \\ \int_Q W(\nabla v_n)dx \to W^*(\nabla u(x_0)). \end{cases} \tag{2.38}$$

For a given $\epsilon > 0$, set

$$u_n^\epsilon(x) := u(x) + \epsilon\Big[v_n(\frac{x-x_0}{\epsilon}) - \nabla u(x_0)(\frac{x-x_0}{\epsilon})\Big].$$

Then, because u belongs to $L^\infty(A;\mathbb{R}^k)$ and by virtue of (2.38),

$$\begin{cases} \sup_n \|u_n^\epsilon\|_{L^\infty(Q(x_0,\epsilon))} \leq \mathcal{C}, \\ u_n^\epsilon \to u \text{ strongly in } L^1(Q(x_0,\epsilon);\mathbb{R}^k), \\ H^{N-1}(S(u_n^\epsilon)\cap Q(x_0,\epsilon)) = H^{N-1}(S(u)\cap Q(x_0,\epsilon)). \end{cases}$$

The sequence u_n^ϵ is a valid sequence of test function for $I_\infty(u, Q(x_0,\epsilon))$ in $(2.1)_\infty$ and we obtain

$$\begin{aligned}\frac{1}{\epsilon^N} I_\infty(u, Q(x_0,\epsilon)) &\leq \liminf_{n\to+\infty}\left\{\frac{1}{\epsilon^N}\int_{Q(x_0,\epsilon)} W(\nabla u_n^\epsilon)dx + \frac{1}{\epsilon^N} H^{N-1}(S(u)\cap Q(x_0,\epsilon))\right\} \\ &= \liminf_{n\to+\infty}\int_Q W(\nabla u(x_0+\epsilon x) - \nabla u(x_0) + \nabla v_n(x))dx \\ &+ \frac{1}{\epsilon^N} H^{N-1}(S(u)\cap Q(x_0,\epsilon)).\end{aligned}$$

The limit (2.36) in Remark (2.12) is recalled and we obtain

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^N} I_\infty(u, Q(x_0,\epsilon)) \\ &\leq \limsup_{\epsilon\to 0^+}\liminf_{n\to+\infty}\int_Q W(\nabla u(x_0+\epsilon x) - \nabla u(x_0) + \nabla v_n(x))dx. \quad (2.39)\end{aligned}$$

We now appeal, for the first time in the proof of Lemma 2.5, to (2.3b) (or rather $(2.3\text{b})_w$). Then

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^N} I_\infty(u, Q(x_0,\epsilon)) &\leq \limsup_{\epsilon\to 0^+}\liminf_{n\to+\infty}\Bigg[\int_Q W(\nabla v_n)dx + \\ \gamma\int_Q (1 + |\nabla v_n(x)|^{p-1} &+ |\nabla u(x_0+\epsilon x) - \nabla u(x_0)|^{p-1})|\nabla u(x_0+\epsilon x) - \nabla u(x_0)|dx\Bigg] \\ &\leq \limsup_{\epsilon\to 0^+}\liminf_{n\to+\infty}\Bigg[\int_Q W(\nabla v_n)dx + \mathcal{C}\Big(\int_Q |\nabla u(x_0+\epsilon x) - \nabla u(x_0)|^p dx\Big)^{1/p} \\ &\Big(1 + \|\nabla v_n\|^{p-1}_{L^p(Q)} + |\nabla u(x_0)|^{p-1} + \|\nabla u\|^{p-1}_{L^p(Q(x_0,\epsilon))}\Big)\Bigg].\end{aligned}$$

We have used Hölder's inequality to pass from the second to the third inequality. Recalling (2.38), we conclude that

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^N} I_\infty(u, Q(x_0,\epsilon)) \leq \\ &W^*(\nabla u(x_0)) + C\limsup_{\epsilon\to 0^+}\Big(\int_Q |\nabla u(x_0+\epsilon x) - \nabla u(x_0)|^p dx\Big)^{1/p} \leq \\ &W^*(\nabla u(x_0)) + C\limsup_{\epsilon\to 0^+}\Big(\frac{1}{\epsilon^N}\int_{Q(x_0,\epsilon)} |\nabla u(x) - \nabla u(x_0)|^p dx\Big)^{1/p}\end{aligned}$$

which, in view of (2.34), (2.37), finally yields

$$h(x_0) \leq W^*(\nabla u(x_0)),$$

which was the desired result. We have thus proved that

$$I_\infty(u, A) \leq J(u, A),$$

for all u's in $SBV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$ (if $J(u, A) = +\infty$ there is nothing to prove) and, according to Remark 2.9, this implies that, for any v in $BV(A;\mathbb{R}^k)$, $I(v, A) \leq J(v, A)$ whenever $(2.3)_w$ holds true, and $I(v, A) < +\infty$.

Step 1-5. If Ω is bounded, then we can take $A = \Omega$ and Lemma 2.5 is proved. If not we assume firstly that u belongs to $L^p(\Omega;\mathbb{R}^k)$ with $I(u, \Omega) < +\infty$ and $J(u, \Omega) < +\infty$. Consider a sequence Ω_n of compactly embedded bounded open sets with $\cup\Omega_n = \Omega$, and an associated sequence φ_n of elements of $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^k)$ with $\varphi_n \equiv 1$ on Ω_n, $0 \leq \varphi_n \leq 1$ and $|\nabla\varphi_n(x)| \leq 1$. Set

$$u_n := \varphi_n u.$$

Then $u_n \in BV(\Omega;\mathbb{R}^k)$.
Further $\text{supp}\{u_n\} \subset A_n$ bounded open subset of Ω. Thus, obviously,

$$\begin{cases} I(u_n, \Omega) = I(u_n, A_n), \\ J(u_n, \Omega) = J(u_n, A_n). \end{cases}$$

Note that $I(u_n, \Omega)$ is easily checked to be finite since $u \in L^p(\Omega;\mathbb{R}^k)$. Then, according to the previous step,

$$I(u_n, A_n) \leq J(u_n, A_n),$$

hence

$$I(u_n, \Omega) \leq J(u_n, \Omega).$$

But, on the other hand, $\nabla u \in L^p(\Omega;\mathbb{R}^k)$ in view of $(2.3\text{a})_w$, so that, by virtue of (2.3c),

$$\begin{aligned} J(u_n, \Omega) &\leq J(u, \Omega) + \beta \int_{\{\varphi_n(x)<1\}} |\nabla(\varphi_n u)|^p dx \\ &\leq J(u, \Omega) + \mathcal{C} \int_{\{\varphi_n(x)<1\}} (|u|^p + |\nabla u|^p) dx. \end{aligned}$$

Since φ_n converges to 1 a.e. in Ω as n tends to $+\infty$, the dominated convergence theorem yields

$$\limsup_{n\to+\infty} J(u_n, \Omega) \leq J(u, \Omega). \tag{2.40}$$

On the other hand, the dominated convergence theorem also implies that

$$u_n \to u \text{ strongly in } L^1(\Omega;\mathbb{R}^k)$$

as $n \to +\infty$. Thus the very definition of $I(\cdot,\Omega)$ as a relaxed functional, in the strong $L^1(\Omega;\mathbb{R}^k)-$topology, of $E(\cdot,\Omega)$ implies that

$$I(u,\Omega) \leq \liminf_{n\to+\infty} I(u_n,\Omega),$$

which, together with (2.40), yields the desired inequality.
If u does not belong to $L^p(\Omega;\mathbb{R}^k)$ but $I(u,\Omega) < +\infty$, it is approximated by $\varphi_q(u)$, with φ_q as in the proof of Proposition 2.7. The proof that $I(u,\Omega) \leq J(u,\Omega)$ is then identical to that of Proposition 2.7 upon replacing A by Ω and dropping the term $\beta\delta$ meas$A$ in (2.11) and $\beta\int_{\{|u(x)>q\}}\delta dx$ in (2.15), because $\delta = 0$ when Ω is unbounded.
The proof of Lemma 2.5 is complete. ■

Step 2. We will prove the following

Lemma 2.13 *Under the only hypothesis (2.3), for any* $u \in BV(A;\mathbb{R}^k)$

$$I(u,\Omega) \geq J(u,\Omega).$$

Proof of Lemma 2.13. We are at liberty to assume that $I(u,\Omega) < +\infty$ otherwise there is nothing to prove.

Let $u_n \in SBV(\Omega;\mathbb{R}^k)$ be such that, as n tends to $+\infty$,

$$\begin{aligned} &u_n \to u \text{ strongly in } L^1(\Omega;\mathbb{R}^k),\\ &E(u_n,\Omega) = \int_\Omega W(\nabla u_n)dx + H^{N-1}(S(u_n)\cap\Omega) \leq \mathcal{C} < +\infty,\\ &I := \lim_{n\to+\infty} E(u_n,\Omega).\end{aligned}$$

Our goal is to prove that

$$I \geq J(u,\Omega).$$

The result announced in Lemma 2.13 is a direct consequence of Lemma 2.14 below and of a lower semi-continuity result of Ambrosio [Ambrosio 93a].

Lemma 2.14 *If* $I(u,\Omega) < +\infty$ *then* $u \in SBV(\Omega;\mathbb{R}^k)$.

Proof. Consider the family of cut-off functions φ_i introduced in the proof of Proposition 2.8. By (2.17) and (2.3a) we have

$$\sup_n \int_\Omega |\nabla(\varphi_i(u_n))|^p dx + H^{N-1}(S(\varphi_i(u_n)) \cap \Omega) < +\infty$$

and using the compactness theorem in SBV (see [AMBROSIO 89a]) we extract a subsequence

$$\varphi_i(u_{n_j}) \to v \text{ in } L^1(\Omega, \mathbb{R}^k),$$

where $v \in SBV(\Omega; \mathbb{R}^k)$. On the other hand, as $u_{n_j} \to u$ strongly in $L^1(\Omega, \mathbb{R}^k)$, we have

$$v = \varphi_i(u) \in SBV(\Omega; \mathbb{R}^k) \text{ for all positive integer } i.$$

Using the chain rule for distributional derivatives (see [AMBROSIO DAL MASO 90])

$$0 = C(\varphi_i(u)) = \nabla\varphi_i(\tilde{u})C(u) \text{ in} \Omega \setminus S(u), \tag{2.41}$$

where, for $x \notin S(u)$, the approximate limit $\tilde{u}(x)$ of u at x is the common value of $u^+(x), u^-(x)$. As $\tilde{u}(x)$ is a Borel function (see [EVANS GARIEPY 92], Lemma 1, Section 5.9), the sets

$$E_m := \{x \in \Omega \,|\, |\tilde{u}(x)| < m\}$$

are Borel sets and $|C(u)| = 0$ if and only if

$$|C(u)|(E_m) = 0 \text{ for all positive integer } m.$$

Fix an integer number m and let $i > m$. By (2.42)

$$\begin{aligned} 0 &= |\nabla\varphi_i(\tilde{u})C(u)(E_m)| \\ &= |\mathbf{I}C(u)(E_m)| \\ &= |C(u)|(E_m), \end{aligned}$$

where $\mathbf{I}$ is the identity matrix in $\mathbb{R}^k \times \mathbb{R}^k$. ■

Finally, since $u \in SBV(\Omega; \mathbb{R}^k)$, Ambrosio's lower semicontinuity theorem (see [AMBROSIO 93a]) yields

$$\begin{aligned} I &= \lim_{n\to+\infty} \int_\Omega W(\nabla u_n)dx + H^{N-1}(S(u_n) \cap \Omega) \\ &\geq \liminf_{n\to+\infty} \int_\Omega W^*(\nabla u_n)dx + H^{N-1}(S(u_n) \cap \Omega) \\ &\geq \int_\Omega W^*(\nabla u)dx + H^{N-1}(S(u) \cap \Omega) \\ &= J(u, \Omega) \end{aligned}$$

which completes the proof of Lemma 2.13.

Lemma 2.5 and 2.13 coalesce into Theorem 2.1.

3 Stable damage and fracture evolution in a brittle elastic continuum

This section is devoted to the investigation of a model of evolution for a continuum that undergoes both damage and fracture. The proposed model results in a time indexed sequence of minimization problems, the energy functionals of which fit squarely within the class of functionals addressed in the previous section.

Specifically the model is a generalization of that introduced in [Francfort Marigo 93]. An elastic body occupying the open connected domain Ω of $\mathbb{R}^N$, $1 \le N \le 3$, is considered, and its evolution is monitored for discrete times

$$0 = t_0 \le t_1 \le \cdots \le t_I = t.$$

At time $t_0 = 0$ the body is assumed to be undamaged and crack free and a loading process is imposed upon it over the time interval $[0, t]$. For the sake of simplicity we assume that the loading is a body loading, in other words the entire loading process is characterized by a sequence $\{f_i;\ 1 \le i \le I\}$ of body forces whose precise regularity will be given below.

In the absence of self-healing the cracks will grow with time; thus the crack free part of the body is assumed to be a decreasing sequence $\{\Omega_i;\ 0 \le i \le I\}$ of bounded, open subsets of Ω with $\Omega_0 = \Omega$. The body forces f_i will be assumed to belong to $L^\infty(\Omega_{i-1}; \mathbb{R}^N)$. At each (discretized) time t_i, $i \ge 1$, and at each point x in Ω_i, the elastic energy density can take two values $W_u(\xi)$, or $W_d(\xi)$, with

$$\beta(1 + |\xi|^p) \ge W_u(\xi) \ge W_d(\xi) \ge \alpha|\xi|^p,\ \xi \in \mathbb{R}^{N^2}, \tag{3.1a}$$

$$W_u \text{ and } W_d \text{ are quasiconvex,} \tag{3.1b}$$

with $1 < p < +\infty$ and $0 < \alpha \le \beta < +\infty$. In view of (3.1b), W_u and W_d are continuous ([Dacorogna 89, Fonseca 88]). Moreover, it was proven in [Marcellini 85] that (3.1a) and (3.1c) imply that

$$\begin{aligned} &|W_u(\xi) - W_u(\eta)| (\text{resp. } |W_d(\xi) - W_d(\eta)|) \le \\ &\beta(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|,\ (\xi, \eta) \in \mathbb{R}^{N^2}. \end{aligned} \tag{3.1c}$$

The density W_u is that of the undamaged material while W_d is that of the damaged material. In other words, if $\chi_i(x)$ denotes the characteristic function of the damaged part of Ω_{i-1}, the elastic energy density of the material occupying Ω_{i-1} is

$$W_i(x, \xi) := (1 - \chi_i(x))W_u(\xi) + \chi_i(x)W_d(\xi) \tag{3.2}$$

at the time t_i.

Remark 3.1 A mechanically inclined reader might challenge the form of the (un)damaged elastic energies, and most notably the growth condition which excludes energies that would blow up as $\det\xi$ goes to 0^+, a nonlinear elastic must. As is usual in the literature pertaining to equilibrium problems for multiple integrals, we firstly address the case where the energy densities are finite. The results for finite quasiconvex integrands are often not trivial to obtain and so we must deal with these first, in the hope that the analysis will give us some insight into the more general problem.

We remark that our hypotheses on the form of the elastic energies also exclude the case of linearized elasticity because pointwise coercivity in the sense of (3.1a) is never satisfied by even the most innocuous linearly elastic materials. Our framework is not to be construed as extending to the case where the energies are functions of the symmetrized gradient $e = \frac{1}{2}(\xi + \xi^T)$ because the correct functional space is not BV anymore, nor its offspring SBV, but BD – the space of bounded deformation – for which very little is known at this juncture.

In conclusion a mechanically rigid reader will most certainly be dissatisfied with the model as it stands while a more lenient one will merely interject that it is indeed a weak generalization of the model proposed in [Francfort Marigo 93] since it does not even encompass the original model. We are fairly confident that a better understanding of spaces like BD would provide the missing ingredient, although such a statement is a mathematical syllogism which in plainer language should be labelled a leap of faith.

The evolution of the damage process is described through the evolution of $\chi_i(x)$. Because damage is irreversible, $\chi_j(x) = 1$ whenever $\chi_i(x) = 1$, $j \geq i$ and $x \in \Omega_{j-1}$. Furthermore the following yield criterion governs the process: for $x \in \Omega_{i-1}$, $\chi_i(x) = 0$ provided that the gradient of the transformation at that part and up to that time—namely $\{\nabla u_k(x),\ k \leq i\}$ where $u_k(x)$ is the deformation field on Ω_{k-1} at time t_k—has never wandered outside an open subset $\mathcal{R}$ of $\mathbb{R}^{N^2}$ defined as

$$\mathcal{R} := \{\xi \in \mathbb{R}^{N^2} | \, W_u(\xi) - W_d(\xi) < K\}. \tag{3.3}$$

In (3.3), K is a characteristic constant of the material and it represents the rate of released energy when passing from an undamaged to a damaged configuration.

The evolution law for $\chi_i(x)$ becomes, for $x \in \Omega_{i-1}$,

$$\chi_i(x) = \begin{cases} 0 \text{ if } \chi_{i-1}(x) = 0 \text{ and } \nabla u_i(x) \in \mathcal{R}, \\ \\ 1 \text{ if } \chi_{i-1}(x) = 1 \text{ or } \nabla u_i(x) \notin \mathcal{R}, \end{cases} \tag{3.4}$$

$2 \leq i \leq I$, which is meaningful because the monotone character of the sequence Ω_i implies that if $x \in \Omega_{i-1}$, then $x \in \Omega_{i-2}$.

The globally dissipated energy D_i from the start up time to the time t_i is given by

$$D_i := \int_\Omega K\chi_{i+1}(x)dx.$$

The modeling of the fracturing process is conceptually similar to that of the damage process. The crack free domain Ω_i is the set complement in Ω_{i-1} of the closure of the set on which the deformation solution field $u_i(x)$ experiences jump discontinuities (the "crack" at the time t_i). Specifically,

$$u_i \in SBV(\Omega_{i-1}; \mathbb{R}^N), \tag{3.5}$$

while

$$\Omega_i = \Omega_{i-1} \backslash \overline{S(u_i)}. \tag{3.6}$$

Note that (3.6) does define a monotonically decreasing sequence of domains $\Omega = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_I$, which is a natural way of imposing the irreversibility of the fracturing process.

Remark 3.2 In the light of (3.5), $\nabla u_i(x)$ in (3.4) is density of the absolutely continuous part of the weak derivative $Du_i(x)$.

The crack evolution is governed by the usual Griffith criterion (see e.g. [Gdoutos 90], Ch. IV): for the crack to propagate the energy released through an infinitesimal virtual "extension" must exceed a critical threshold, $\lambda > 0$, the critical energy release rate. Put otherwise a crack will not propagate if, for any possible "extension" of that crack, the resulting decrease (if any) in potential energy cannot offset the energy dissipated through that "extension". Define, for any $v \in SBV(\Omega_{i-1};\mathbb{R}^N)$, the potential energy $\mathcal{P}_i(v)$ to be

$$\mathcal{P}_i(v) := \int_{\Omega_{i-1}} W_i(x,\nabla v)dx - \int_{\Omega_{i-1}} f_i \cdot v dx.$$

Then the evolution law for the crack becomes

$$\mathcal{P}_i(v) - \mathcal{P}_i(u_i) + \lambda[H^{N-1}(S(v)\cap\Omega_{i-1}) - H^{N-1}(S(u_i)\cap\Omega_{i-1})] \geq 0 \quad (3.7)$$

for any admissible v's.

The evolution of damage and fracture as described above may be reformulated as a two field partial minimization problem. Specifically we set

$$\begin{aligned} \mathcal{L}_i(u,\chi) &:= \int_{\Omega_{i-1}} [(1-\chi(x))W_u(\nabla u(x)) + \chi(x)W_d(\nabla u(x))]dx \\ &+ K\int_{\Omega_{i-1}} \chi dx + \lambda H^{N-1}(S(u)\cap\Omega_{i-1}) - \int_{\Omega_{i-1}} f_i\cdot u dx, \end{aligned} \quad (3.8)$$

where Ω_{i-1} is defined in (3.6).

In view of (3.4) we also define

$$S_i := SBV(\Omega_{i-1};\mathbb{R}^N) \quad (3.9)$$

$$X_i := \{\chi \in L^\infty(\Omega_{i-1};\{0,1\}) \,|\, \chi(x) \geq \chi_{i-1}(x) \text{ a.e. on } \Omega_{i-1}\}. \quad (3.10)$$

Then χ_i, u_i satisfy (3.4), (3.7) if and only if

$$\begin{cases} \mathcal{L}_i(u_i,\chi_i) \leq \mathcal{L}_i(u,\chi_i), \ u \in S, \\ \mathcal{L}_i(u_i,\chi_i) \leq \mathcal{L}_i(u_i,\chi), \ \chi \in X_i. \end{cases} \quad (3.11)$$

In the setting of pure damage it was observed in [Francfort Marigo 91] that a principle of the kind (3.11) generates too many solutions and that an additional selection criterion is desirable. A natural candidate is a global stability principle, which forces (u_i,χ_i) to be a global minimizer of $\mathcal{L}_i$ defined in (3.8) over all admissible pairs (u,χ) in $S \times X_i$. In other words the (discretized) evolution of the interaction between damage and fracture is described through the following

Problem 3.3 *For $i \in \{1, ..., I\}$, find (u_i, χ_i) that minimizes $\mathcal{L}_i$ over $S_i \times X_i$ with $\Omega_{-1} = \Omega_0 = \Omega,\ \chi_0 = 0, u_0 = \mathrm{id}$, and $\Omega_{i-1} = \Omega_{i-2} \setminus \overline{S(u_{i-1})},\ 1 \le i \le I+1$.*

In a min–min problem the order in which the minimization is carried out is unimportant. Minimizing in χ then in u, we define, for $i \in \{1, ..., I\},\ x \in \Omega_{i-1}$ and $\xi \in \mathbb{R}^{N^2}$,

$$\psi_i(x,\xi) := \begin{cases} W_d(\xi) + K \text{ if} & \chi_{i-1}(x) = 1, \\ \\ \min\{W_u(\xi), W_d(\xi) + K\} & \text{if } \chi_{i-1}(x) = 0, \end{cases} \tag{3.12}$$

and, for u in S_i,

$$\begin{aligned} \Phi_i(u) \ &:= \int_{\Omega_{i-1}} \psi_i(x, \nabla u) dx \\ &\quad + \lambda H^{N-1}(S(u) \cap \Omega_{i-1}) - \int_{\Omega_{i-1}} f_i \cdot u dx. \end{aligned} \tag{3.13}$$

Then, Problem 3.3 is easily seen to be equivalent to the following single field minimization problem:

Problem 3.4 *For $i \in \{1, ..., I\}$, find u_i that minimizes Φ_i over S_i.*

Remark 3.5 Although χ_i has seemingly disappeared from the formulation of Problem 3.4, its presence is felt through the expression (3.12) for ψ_i.

Remark 3.6 The energy density $\psi_i(x, \xi)$ is a Carathéodory function because W_u and W_d are continuous. In view of (3.1a), it satisfies

$$\alpha|\xi|^p \le \psi_i(x,\xi) \le \beta'(1 + |\xi|^p). \tag{3.14}$$

Furthermore

$$\begin{aligned} &|\psi_i(x,\xi) - \psi_i(x,\eta)| \\ &\le \max\{|W_u(\xi) - W_u(\eta)|, |W_d(\xi) - W_d(\eta)|\}, \end{aligned}$$

so that, by virtue of estimate (3.1c), we obtain

$$|\psi_i(x,\xi) - \psi_i(x,\eta)| \le \beta(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|. \tag{3.15}$$

Let us focus for the time being on the first time step t_1.

3.1 The first time step:

Since $\chi_0 \equiv 0$, the energy density ψ_1 does not depend upon x, i.e. $\psi_1 = \min\{W_u, W_d + K\}$. In view of (3.14) and (3.15), it satisfies hypotheses (2.3a) and (2.3b) and we conclude that Theorem 2.1 applies. Denote by ψ_1^* the $W^{1,p}-$quasiconvexification of ψ_1, i.e.

$$\psi_1^*(\xi) := \inf_{\Phi \in \mathcal{C}_0^\infty(Q;\mathbb{R}^N)} \int_Q \psi_1(\xi + D\Phi(y))dy, \tag{3.16}$$

where Q is a unit cube centered at 0, and set

$$\Phi_1^*(u) := \int_\Omega \psi_1^*(\nabla u)dx + \lambda H^{N-1}(S(u)) - \int_\Omega f_1 \cdot u dx. \tag{3.17}$$

Then Theorem 2.1 implies that Φ_1^* is the lower semi-continuous envelope of Φ_1 defined in (3.13) for the strong topology of $L^1(\Omega;\mathbb{R}^N)$.

The above result is not entirely satisfactory because it fails to guarantee the existence of a minimizer for (3.17) over $S_1 = BV(\Omega;\mathbb{R}^N)$. Of course, if such a minimizer exists, the resulting value of Φ_1^* is the infimum of Φ_1 over S_1. The missing ingredient is the compactness in S_1 of a minimizing sequence for Problem 3.4. Indeed the functional Φ_1^* (or Φ_1) is not coercive over $BV(\Omega;\mathbb{R}^N)$. At the present time we do not know how to remove this obstacle without additional assumptions on the admissible test fields . The simplest such assumption is to impose on the fields to take their values in a compact set M of $\mathbb{R}^N$. Such a restriction is physically reasonable because the model certainly implicitly precludes very large displacement other than rigid body ones, since those would provoke the onset of e.g. plasticity which is beyond the framework of this study; it is however an admittedly unusual restriction in a problem of elasticity.

We thus assume that S_i defined in (3.9) is replaced by

$$S_i^\infty := SBV(\Omega_{i-1}; M). \tag{3.18}$$

Then Problem 3.4 at time t_1 becomes

Problem 3.4∞ *Find u_1 that minimizes Φ_1 over S_1^∞.*

The first inequality in (3.14) together with the definition (3.18) of S_1^∞ imply that a minimizing sequence u_n for Φ_1 satisfies

$$\begin{cases} \sup_n \|u_n\|_{L^\infty(\Omega)} \leq \mathcal{C} < +\infty, \\ \sup_n H^{N-1}(S(u_n) \cap \Omega) \leq \mathcal{C} < +\infty, \\ \|\nabla u_n\|_{L^p(\Omega)} \leq \mathcal{C} < +\infty, \end{cases} \tag{3.19}$$

from which it is deduced that $\{u_n\}$ is uniformly bounded in $SBV(\Omega; \mathbb{R}^N)$ hence that, at the possible expense of extracting a subsequence still labelled u_n,

$$u_n \to u \text{ strongly in } L^1(\Omega; \mathbb{R}^N). \tag{3.20}$$

By virtue of (3.18), (3.19) and because $p > 1$ a direct application of Theorem 2.1 in [Ambrosio 89a] implies that $u \in SBV(\Omega; \mathbb{R}^N)$ while the previous consideration permit to assert that u minimizes Φ_1^* over S_1^∞.

We have thus proved the following

Proposition 3.7 *The infimum of Φ_1, defined in (3.13), over S_1^∞, defined in (3.18), is the value of Φ_1^*, defined in (3.17), at any of its minimizers over S_1^∞. Such minimizers exist.*

The specific form of W_1—see (3.2)—permits to be somewhat more precise in the description of ψ_1^*. To this end we recall another expression for ψ_1^* which holds true because of (3.1a) (cf. [Ball Murat 84], Theorem 3.1, Corollary 3.2 and Conjecture 3.7 (2); cf. also [Kohn 91], equation (2.12) and Lemma 2.2 in the case of an energy that depends on the linearized strain). Specifically we define, for any $\chi \in L^\infty(Q; \{0, 1\})$ (Q a unit cube),

$$\begin{aligned} W_\chi(x, \xi) &:= (1 - \chi(x))W_u(\xi) + \chi(x)W_d(\xi),\ \xi \in \mathbb{R}^{N^2}, x \in Q, \\ W_\chi^*(\xi) &:= \inf_\varphi \Big\{ \int_Q W_\chi(x, \xi + \nabla\varphi(x))dx | \varphi \in W^{1,p}(Q; \mathbb{R}^N), \\ &\qquad \varphi \text{ is } Q-\text{periodic}\}, \end{aligned} \tag{3.21}$$

$$W^*(\theta, \xi) := \inf_\chi \{W_\chi^*(\xi) | \, \chi \in L^\infty(Q; \{0, 1\}), \int_Q \chi dx = \theta\}. \tag{3.22}$$

Then

$$\psi_1^*(\xi) = \inf_{0 \leq \theta \leq 1} [W^*(\theta, \xi) + K\theta]. \tag{3.23}$$

Remark 3.8 If W_u and W_d are convex in ξ, then (3.21) may be identified with the energy density associated to the $\Gamma-$limit I^0_χ of the following functionals defined on $W^{1,p}(\Omega;\mathbb{R}^N)$:

$$I^\epsilon_\chi(u) := \int_\Omega W_\chi(\frac{x}{\epsilon}, \nabla u)dx,$$

where $W_\chi(\cdot,\xi)$ has been $Q-$periodically extended to the whole of $\mathbb{R}^N$ (see e.g. [MARCELLINI 78]) and an interpretation (and/or alternative derivation) of the quasiconvexification of ψ_1 may be proposed using homogenization techniques (see [FRANCFORT MARIGO 93]).

When W_u or W_d are not convex the $\Gamma-$ limit I^0_χ of I^ϵ_χ does not admit W^*_χ as energy density but W^0_χ defined as

$$\begin{aligned} W^0_\chi(\xi) \quad := \quad & \inf_{k\in\mathbb{Z}^+} \inf_\varphi \left\{ \frac{1}{k^N}\int_{kQ} W_\chi(x,\xi+\nabla\varphi)dx | \varphi \in W^{1,p}(kQ;\mathbb{R}^N), \right. \\ & \left. \varphi \text{ is } \ kQ - \text{periodic} \right\}, \end{aligned} \tag{3.24}$$

and W^0_χ can be shown not to necessarily coincide with W^*_χ ([MÜLLER 87]), although certainly $W^0_\chi \le W^*_\chi$. It is however worth pointing out that, defining

$$W^0(\theta,\xi) := \inf_\chi \left\{ W^0_\chi(\xi) |\, \chi \in L^\infty(Q;\{0,1\}),\, \int_Q \chi(x)dx = \theta \right\}, \tag{3.25}$$

the following result holds true

$$W^0(\theta,\xi) = W^*(\theta,\xi), \tag{3.26}$$

as easily checked upon performing, for a fixed χ, k and φ, the change of variables $x = ky$, setting

$$\chi_k(y) := \chi(ky),\ \varphi_k(y) := \frac{1}{k}\varphi(ky)$$

and noting that, for any $kQ-$ periodic φ,

$$\frac{1}{k^N}\int_{kQ} W_\chi(\xi + \nabla\varphi(x))dx = \int_Q W_{\chi_k}(\xi+\nabla\varphi_k(y))dy.$$

Note that the resulting φ_k is Q-periodic, while

$$\int_Q \chi_k(y)dy = \frac{1}{k^N}\int_{kQ}\chi(x)dx = \int_Q \chi(x)dx = \theta.$$

Thus, by (3.23), ψ_1^* may reexpressed as

$$\psi_1^*(\xi) = \inf_{0\le\theta\le 1}[W^0(\theta,\xi) + K\theta].$$

This remark, which seems to be new, states in essence that optimal energy bounds on the periodic mixtures at fixed volume fraction of two arbitrary energies can be obtained by consideration of a single period notwithstanding convexity.

The following lemma whose proof follows that of a similar result for the quadratic case ([Allaire Kohn 93], Proposition 8.1) holds true:

Lemma 3.9 *W^* is locally Hölder continuous over $[0,1]\times\mathbb{R}^{N^2}$. It further satisfies, for any $0\le\theta\le 1$,*

$$\alpha|\xi|^p \le W^*(\theta,\xi) \le \beta(1+|\xi|^p),\ \xi\in\mathbb{R}^{N^2}, \tag{3.27a}$$

$$|W^*(\theta,\xi) - W^*(\theta,\eta)| \le \mathcal{C}(1+|\xi|^{p-1}+|\eta|^{p-1})|\xi-\eta|, \tag{3.27b}$$

with ξ, $\eta\in\mathbb{R}^{N^2}$, $0\le\mathcal{C}\le+\infty$.

Proof of Lemma 3.9. By virtue of (3.1a) and of Jensen's inequality applied to $|\xi|^p$, it is immediately seen that (3.27a) holds true.

The locally Lipschitz character of W^* in ξ is straightforward. For any $1>\epsilon>0$, there exists an admissible pair (χ_ξ,φ_ξ) of test functions such that

$$\int_Q [(1-\chi_\xi)W_u(\xi+\nabla\varphi_\xi) + \chi_\xi W_d(\xi+\nabla\varphi_\xi)]dx \le W^*(\theta,\xi)+\epsilon. \tag{3.28}$$

Because of (3.1a) inequality (3.28) implies in turn that

$$\alpha\|\xi+\nabla\varphi_\xi\|^p_{L^p(Q)} \le \beta(1+|\xi|^p)+1,$$

or still that

$$\|\nabla\varphi_\xi\|_{L^p(Q)} \le \mathcal{C}(1+|\xi|).$$

But, for any η in $\mathbb{R}^{N^2}$, we obtain, by virtue of (3.1c),

$$\begin{aligned} W^*(\theta,\eta) &\le \int_Q [(1-\chi_\xi)W_u(\eta+\nabla\varphi_\xi) + \chi_\xi W_d(\eta+\nabla\varphi_\xi)]dx \\ &\le \int_Q [(1-\chi_\xi)W_u(\xi+\nabla\varphi_\xi) + \chi_\xi W_d(\xi+\nabla\varphi_\xi)]dx \\ &\quad +C\int_Q (1+|\xi|^{p-1}+|\eta|^{p-1}+|\nabla\varphi_\xi|^{p-1})|\xi-\eta|dx. \end{aligned}$$

Thus (3.28), together with the estimate on $\|\nabla\varphi_\xi\|_{L^p(Q)}$ yield

$$W^*(\theta,\eta) \leq W^*(\theta,\xi) + \mathcal{C}(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta| + \epsilon.$$

Letting ϵ tend to zero and exchanging the role of ξ and η permits to conclude and to establish (3.27b).

The locally Hölder continuous character of W^* in θ is more involved because it requires application of a MEYER' type regularity result. Specifically, for any $\epsilon > 0$ choose χ_ξ, an admissible test function in (3.22), such that

$$W^*_{\chi_\xi}(\xi) \leq W^*(\theta,\xi) + \epsilon.$$

At this point we use for the first time the hypothesis (3.1c) that W_u and W_d are quasiconvex, hence that $W_{\chi_\xi}(x,\cdot)$ is too. Then, by virtue of (3.1a) the infimum value $W^*_{\chi_\xi}(\xi)$ is attained for a $Q-$periodic φ_ξ in $W^{1,p}(Q;\mathbb{R}^N)$ in (3.21) (see e.g. [ACERBI FUSCO 86], Theorem II.4). Thus

$$\int_Q W_{\chi_\xi}(x, \xi + \nabla\varphi_\xi(x))dx \leq W^*(\theta,\xi) + \epsilon. \tag{3.29}$$

Then according to Theorem 3.1 in Section V of [GIAQUINTA 83], $\varphi_\xi \in W^{1,m}_{\text{loc}}(Q;\mathbb{R}^N)$ for some $m > p$ and the following estimate holds true, for $Q' \subset\subset Q$,

$$\|\nabla\varphi_\xi\|_{L^m(Q')} \leq \mathcal{C}_{Q'}, \tag{3.30}$$

where $\mathcal{C}_{Q'}$ denotes throughout a constant that depends upon Q', α, β, Ω and ξ only. Note that (3.1a) and (3.27a) have been implicitly used in deriving inequality (3.30). Choose $\chi' \in L^\infty(Q;\{0,1\})$ with $Q' := \text{supp}(\chi' - \chi_\xi) \subset\subset Q$ and such that, setting

$$\int_Q \chi' dx = \theta',$$

then

$$\int_Q |\chi' - \chi|(x)dx = |\theta' - \theta|.$$

We obtain

$$\begin{aligned} W^*(\theta',\xi) &\leq \int_Q W_{\chi'}(\xi + \nabla\varphi_\xi)dx \\ &= \int_Q W_{\chi_\xi}(\xi + \nabla\varphi_\xi)dx \end{aligned}$$

$$
\begin{aligned}
&+ \int_Q (\chi' - \chi_\xi)(W_d - W_u)(\xi + \nabla\varphi_\xi)dx \\
&\leq \int_Q W_{\chi_\xi}(\xi + \nabla\varphi_\xi)dx \\
&+ \mathcal{C}\int_Q \psi|\chi' - \chi_\xi|(1 + |\xi|^p + |\nabla\varphi_\xi|^p)dx,
\end{aligned}
$$

where $\psi \in \mathcal{C}_0^\infty(Q)$ with $\psi \equiv 1$ on Q' and where we have used (3.1a). Inequality (3.29) is recalled and Hölder's inequality is applied to the last term of the last inequality in the above string of inequalities. We obtain

$$
\begin{aligned}
W^*(\theta', \xi) \leq\ & W^*(\theta, \xi) + \epsilon \\
&+\mathcal{C}(\int_Q |\chi' - \chi_\xi|^{m/(m-p)})^{(m-p)/m}(1 + |\xi|^p + \|\nabla\varphi_\xi\|^p_{L^m(\mathrm{supp}\psi)})
\end{aligned}
$$

or still, upon invoking (3.30),

$$
W^*(\theta', \xi) \leq W^*(\theta, \xi) + \mathcal{C}_{Q'}|\theta' - \theta|^{(m-p)/m} + \epsilon.
$$

Letting ϵ tend to zero permits once again to conclude. Note that, provided that θ' is close enough to θ, there will always exist a χ' with $\mathrm{supp}(\chi' - \chi_\xi) \subset\subset Q'$, $\int_Q \chi' dx = \theta'$ and $\int_Q |\chi' - \chi|(x)dx = |\theta' - \theta|$.

Remark 3.10 We denote by $\theta(\xi)$ the minimum of all minimizers of $\{W^*(\theta, \xi) + K\theta,\ 0 \leq \theta \leq 1\}$. By virtue of (3.23)

$$
\psi_1^*(\xi) = W^*(\theta(\xi), \xi) + K\theta(\xi),\ \xi \in \mathbb{R}^{N^2}.
$$

If $\xi(x), x \in \Omega$, is a simple function then $\theta(\xi(x)), x \in \Omega$, is also simple, hence measurable.

Let $u_1 \in S^\infty$ be a minimizer for Φ_1^*, the existence of which is guaranteed by Proposition 3.7. Choose a sequence $\{\xi_n(x)\}$ of simple functions that converges pointwise to $\nabla u_1(x)$ and set

$$
\theta_1(x) := \limsup_{n\to+\infty} \theta(\xi_n(x)).
$$

Then θ_1 is a measurable function, and because $W^*(\theta, \xi)$ is continuous over $[0, 1] \times \mathbb{R}^{N^2}$ we have for all $\theta \in [0, 1]$, a.e. $x \in \Omega$ and after extracting a suitable subsequence

$$\begin{aligned} W^*(\theta_1(x), \nabla u_1(x)) + K\theta_1(x) &= \lim_{n\to+\infty} W^*(\theta(\xi_n(x)), \xi_n(x)) + K\theta(\xi_n(x)) \\ &\leq \lim_{n\to+\infty} W^*(\theta, \xi_n(x)) + K\theta \\ &= W^*(\theta, \nabla u_1(x)) + K\theta \end{aligned}$$

and so

$$W^*(\theta_1(x), \nabla u_1(x)) + K\theta_1(x) = \min\{W^*(\theta, \nabla u_1(x)) + K\theta,\ 0 \leq \theta \leq 1\}. \tag{3.31}$$

The function $\theta_1(x)$ should be thought of as the local volume fraction of the damaged material at the first time step. We have thus proved the following

Proposition 3.11 *To each minimizer $u_1(x)$ of Φ_1^* over S^∞, there corresponds a measurable volume fraction $\theta_1(x)$ such that*

$$\begin{aligned} \inf_{u\in S^\infty} \Phi_1(u) &= \int_\Omega W^*(\theta_1(x), \nabla u_1(x))dx + K\int_\Omega \theta_1(x)dx \\ &\quad +\lambda H^{N-1}(S(u_1)) - \int_\Omega f_1 \cdot u_1 dx, \end{aligned}$$

with W^* defined in (3.22).

Remark 3.12 Note that the pair $(u_1, \theta_1) \in S^\infty \times X_1$ may not be unique. The subsequent history of the evolution of the damage/fracture process will depend upon the solution (u_1, θ_1) at time step t_1. The reader may find it convenient to think of the evolution as possibly exhibiting bifurcations at each time step.

3.2 The subsequent time steps

We assume that $(u_1(x), \theta_1(x))$ has been determined and recall Problem 3.3. At the second time step t_2, we must set, according to (3.6) and Problem 3.3,

$$\Omega_1 := \Omega\backslash\overline{S(u_1)},$$

which is the uncracked part of Ω after time t_1. Thus the new domain is unambiguously assigned once u_1 is known. The irreversibility constraint, namely $\chi(x) \geq \chi_1(x)$ a.e. in Ω, is not so easily handled because the relaxation at time t_1 has merely produced a local volume fraction $\theta_1(x)$ of damaged material.

We define, for $1 \leq i \leq I$ and for a.e. x in Ω_{i-1},

$$\psi_i^*(x,\xi) := \min_{\theta_{i-1}(x)\leq\theta\leq 1}[W^*(\theta,\xi) + K\theta], \ \xi \in \mathbb{R}^{N^2}, \tag{3.32}$$

where W^* has been defined in (3.22) and $\theta_0(x) := 0$.

In the spirit of the relaxation performed at the first time step in Subsection 3.1 we propose the following relaxed formulation for the subsequent time steps:

Problem 3.13 *For $i \in \{1, ..., I\}$, find $(u_i(x), \theta_i(x))$ such that u_i minimizes*

$$\Phi_i^*(u) := \int_{\Omega_{i-1}} \psi_i^*(x, \nabla u)dx \quad + \quad \lambda H^{N-1}(S(u) \cap \Omega_{i-1}) - \int_{\Omega_{i-1}} f_i \cdot u dx \tag{3.33}$$

over S_i^∞, where ψ_i^* is defined in (3.31) and S_i^∞ in (3.18). The local volume fraction $\theta_i(x)$ is such that, for a.e. x in Ω_{i-1},

$$\begin{cases} \psi_i^*(x, \nabla u_i(x)) = W^*(\theta_i(x), \nabla u_i(x)) + K\theta_i(x), \\ \theta_i(x) \geq \theta_{i-1}(x). \end{cases} \tag{3.34}$$

That Problem 3.13 admits a solution at time t_1 has been established in Proposition 3.7 and also because (3.34) follows from (3.31). At subsequent time steps t_i, Φ_i^* admits a minimizer *if* the density $\psi_i^*(x,\xi)$ defined in (3.32) is shown to be a quasiconvex Carathéodory function with p growth, i.e., such that, for some $\beta' < +\infty$, and for a.e. x in Ω_{i-1},

$$\alpha|\xi|^p \leq \psi_i^*(x,\xi) \leq \beta'(1 + |\xi|^p), \ \xi \in \mathbb{R}^{N^2}.$$

In such a case the energy density $\psi_i^*(x,\xi)$ will meet all the requirements that permit application of Theorem 4.3 in [Ambrosio 93a]. Since, as already seen in Section 2, $H^{N-1}(S(u) \cap \Omega_{i-1})$ is a jump integral, we conclude that $\Phi_i^*(u)$ defined in (3.33) is strong-$L^1(\Omega_{i-1}; \mathbb{R}^N)$-lower semicontinuous in

$SBV(\Omega_{i-1};\mathbb{R}^N)$. Furthermore, the minimizing sequences for Φ_i^* are bounded in $BV(\Omega_{i-1};\mathbb{R}^N)$ hence compact in $L^1(\Omega_{i-1};\mathbb{R}^N)$, which permits to conclude to the existence of a minimizer u_i for Φ_i^* in Problem 3.13. The following lemma holds true:

Lemma 3.14 *Assume that $2 \le i \le I$. If there exists $(u_{i-1}(x), \theta_{i-1}(x))$ such that u_{i-1} minimizes Φ_{i-1}^* over S_i^∞, then the density $\psi_i^*(x,\xi)$ defined in (3.32) is a Carathéodory function satisfying, for some $\beta' < +\infty$, and a.e. x in Ω_{i-1},*

$$\alpha|\xi|^p \le \psi_i^*(x,\xi) \le \beta'(1+|\xi|^p),\ \xi \in \mathbb{R}^{N^2}, \tag{3.35a}$$

$$|\psi_i^*(x,\xi) - \psi_i^*(x,\eta)| \le \beta'(1+|\xi|^{p-1}+|\eta|^{p-1})|\xi-\eta|,\ \xi,\eta \in \mathbb{R}^{N^2}. \tag{3.35b}$$

Furthermore, if u_i is a minimizer for Φ_i^* over S_i^∞, then there exists a local volume function θ_i satisfying (3.34).

Proof of Lemma 3.14 It has been proved in Lemma 3.9 that W^* is in particular continuous over $[0,1]\times\mathbb{R}^{N^2}$. Thus assume that $\theta_{i-1}(x)$ exists and is measurable. Then, Ψ^* defined as

$$\Psi^*(\tilde\theta,\tilde\xi) := \min_{\tilde\theta\le\theta\le1}[W^*(\theta,\tilde\xi)+K\theta],\ \tilde\xi \in \mathbb{R}^{N^2},\ 0\le\tilde\theta\le1, \tag{3.36}$$

is continuous on $[0,1]\times\mathbb{R}^{N^2}$ because W^* is continuous. We conclude that

$$\psi_i^*(x,\xi) = \Psi^*(\theta_{i-1}(x),\xi),$$

is a Carathéodory function over $\Omega_{i-1}\times\mathbb{R}^{N^2}$.

We now prove that, upon assuming the existence of a minimizer u_i to Φ_i^* defined in (3.33), a local volume fraction $\theta_i(x)$ satisfying (3.34) may be defined. To this effect we denote by $\theta(\tilde\theta,\tilde\xi)$ the minimum of all minimizers in (3.36). If $(\bar\theta(x),\bar\xi(x)),\ x\in\Omega$, is a simple function then the associated $\theta(\bar\theta(x),\bar\xi(x)),\ x\in\Omega$, is also simple, hence measurable. If $(\bar\theta(x),\bar\xi(x)),\ x\in\Omega$ is a measurable pair, then we consider a sequence $\{(\bar\theta_n(x),\bar\xi_n(x))\}$ of simple functions that converges pointwise to $(\bar\theta(x),\bar\xi(x))$. We set, for a.e. x in Ω_{i-1},

$$\Theta(\bar\theta,\bar\xi)(x) := \limsup_{n\to\infty}\theta(\bar\theta_n(x),\bar\xi_n(x)).$$

The function $\Theta(\bar\theta,\bar\xi)(x)$ is measurable. Furthermore, for almost every x in Ω_{i-1} and for every $\theta\ge\bar\theta_n(x)$,

$$W^*(\theta(\bar\theta_n(x),\bar\xi_n(x))) + K\theta(\bar\theta_n(x),\bar\xi_n(x)) \le W^*(\theta,\bar\xi_n(x)) + K\theta,$$

thus, by virtue of the continuous character of W^*

$$W^*(\Theta(\bar{\theta},\bar{\xi})(x),\bar{\xi}(x)) + K\Theta(\bar{\theta},\bar{\xi})(x) \le W^*(\theta,\bar{\xi}(x)) + K\theta, \tag{3.37}$$

for every $\theta > \bar{\theta}(x)$, and hence for every $\theta \ge \bar{\theta}(x)$. But, for a.e. x in Ω_{i-1},

$$\theta(\bar{\theta}_n(x),\bar{\xi}_n(x)) \ge \bar{\theta}_n(x),$$

thus

$$\Theta(\bar{\theta},\bar{\xi})(x) \ge \bar{\theta}(x).$$

We have thus exhibited a measurable function $\Theta(\bar{\theta},\bar{\xi})$ such that, for a.e. x in Ω_{i-1},

$$1 \ge \Theta(\bar{\theta},\bar{\xi})(x) \ge \bar{\theta}(x),$$

while by virtue of (3.37)

$$W^*(\Theta(\bar{\theta},\bar{\xi})(x),\bar{\xi}(x)) + K\Theta(\bar{\theta},\bar{\xi})(x) = \min_{\theta \ge \bar{\theta}(x)} [W^*(\theta,\bar{\xi})(x)) + K\theta]. \tag{3.38}$$

It now suffices to set, for a.e. x in Ω_{i-1},

$$\theta_i(x) := \Theta(\theta_{i-1},\nabla u_i)(x).$$

The proof of (3.35a) is immediate by virtue of (3.27a) in Lemma 3.9. That (3.35b) is also satisfied follows from an argument identical to that led to (3.27b). The proof of Lemma 3.14 is complete. ■

At this point the only missing ingredient is the quasiconvexity of ψ_i^* from which the existence of a minimizer u_i follows. In view of (3.1a), (3.1c), (3.22), (3.26), (3.32), the question is immediately reduced to that of the possible quasiconvex character of a functional defined on $\Omega \times \mathbb{R}^{N^2}$ by

$$\omega(x,\xi) := \inf_{\bar{\theta}(x) \le \theta \le 1} [W^0(\theta,\xi) + k\theta], \tag{3.39}$$

for a.e. x in Ω and every ξ in $\mathbb{R}^{N^2}$, where $\bar{\theta}$ is some element of $L^\infty(\Omega;[0,1])$. In (3.39) W^0 is the energy defined in (3.25) of Remark 3.8.

Appealing to e.g. Theorem II.2 in [Acerbi Fusco 84] the problem of the quasiconvex character of ω defined in (3.39) may be rephrased in terms of lower semicontinuity. Specifically, the density ω is shown to be Carathéodory and to satisfy, for a.e. x in Ω and every ξ in $\mathbb{R}^{N^2}$,

$$\alpha|\xi|^p \le \omega(x,\xi) \le \beta'(1+|\xi|^p),$$

through an argument which is identical to that used for the ψ_i^*'s. Then quasiconvexity will be established if the functional

$$v \longrightarrow \int_\Omega \omega(x, \nabla v)dx$$

is proved to be sequentially weak-$*$ lower semi-continuous on $W^{1,\infty}(\Omega;\mathbb{R}^N)$.

We thus consider a sequence $\{v_n\}$ in $W^{1,\infty}(\Omega;\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v \text{ weak-} * \text{ in } W^{1,\infty}(\Omega;\mathbb{R}^N),$$

and an associated sequence $\{\theta_n\}$ such that, for a.e. x in Ω,

$$\begin{cases} 1 \geq \theta_n(x) \geq \bar{\theta}(x), \\ \\ \omega(x, \nabla v_n(x)) = W^0(\theta_n(x), \nabla v_n(x)) + K\theta_n(x). \end{cases}$$

Such a sequence exists by an argument identical to that used for the existence of $\theta_i(x)$ satisfying (3.34) in Lemma 3.14; furthermore, at the possible expense of extracting a subsequence still labelled θ_n, we may assume that, as n tends to ∞,

$$\theta_n \rightharpoonup \theta \text{ weak-} * \text{ in } L^\infty(\Omega;[0,1]),$$

with $\theta(x) \geq \bar{\theta}(x)$, a.e. in Ω.

We now assume that the following result holds true:

Conjecture 3.15 *Let W_u and W_d be defined as in (3.1) and W_χ^0 as in (3.24). For any sequence $\{\chi_q\}$ in $L^\infty(\Omega;\{0,1\})$, define*

$$I_{\chi_q}(v) := \int_\Omega [(1-\chi_q(x))W_u(\nabla v(x)) + \chi_q(x)W_d(\nabla v(x))]dx. \tag{3.40}$$

(C1) *Assume that*

$$\begin{cases} \chi_q \rightharpoonup \theta \textit{ weak - } * \textit{ in } L^\infty(\Omega;[0,1]), \\ \\ I_{\chi_q} \xrightarrow{\Gamma} I^0_{\{\chi_q\}} \textit{ in } W^{1,p}(\Omega;\mathbb{R}^N) \end{cases} \tag{3.41}$$

(which is always possible after extraction of a suitable subsequence). Denote by $W^0_{\{\chi_q\}}(x,\xi)$ the energy density associated to $I^0_{\{\chi_q\}}$ (such a density exists

according to [Buttazzo Dal Maso 85]. *Then, for a.e. x in Ω, there exists a sequence $\chi_r(x;\cdot) \in L^\infty(Q;\{0,1\})$ such that*

$$\begin{cases} \theta(x) = \int_Q \chi_r(x;y)dy, \\ W^0_{\{\chi_q\}}(x,\xi) = \lim_{r\to+\infty} W^0_{\chi_r(x;\cdot)}(\xi),\ \xi \in \mathbb{R}^{N^2}, \end{cases}$$

where W^0_χ ($\chi \in L^\infty(Q;\{0,1\})$ has been defined in (3.24).

(C2) *Conversely, if $W(x,\xi)$ is a Carathéodory function such that, for a.e. x in Ω, there exists a sequence $\chi_r(x;\cdot)$ satisfying*

$$\begin{cases} \theta(x) = \int_Q \chi_r(x;y)dy, \\ W(x,\xi) = \lim_{r\to+\infty} W^0_{\chi_r(x;\cdot)}(\xi),\ \xi \in \mathbb{R}^{N^2}, \end{cases}$$

then there exists a sequence $\chi_q \in L^\infty(\Omega;\{0,1\})$ such that

$$\begin{cases} \chi_q \rightharpoonup \theta \ \textit{weak - } * \ \textit{ in } L^\infty(\Omega;[0,1]), \\ I_{\chi_q} \xrightarrow{\Gamma} \int_\Omega W(x,\nabla\cdot)dx \ \textit{ in } W^{1,p}(\Omega;\mathbb{R}^N). \end{cases}$$

Remark 3.16 Conjecture 3.15 says in essence that the energy density associated to the $\Gamma-$limit of any functional of the form (3.40) "coincides" pointwise with that of the $\Gamma-$ limit of a functional of the form (3.40) specialized to sequence $\{\chi_q\}$ of the form

$$\chi_q(x) = \chi(qx)$$

where χ is a characteristic function defined on Q and extended by periodicity to the whole of $\mathbb{R}^N$. For such sequences the $\Gamma-$limit is known to admit W^0_χ as energy density (see Remark 3.8). In other words it asserts the canonical character of periodic homogenization as far as effective energy densities are concerned. That conjecture is true in the quadratic case ([Dal Maso Kohn 94]) and it may be shown to be true in the case where W_u and W_d are convex and satisfy (3.1a) ([Francfort Murat 94]). Whether it is true in the general case where W_u and W_d are arbitrary, or even quasiconvex, is an open question.

We thus assume that Conjecture 3.15 holds true. Then, upon setting $v_\infty \equiv v, \theta_\infty \equiv \theta$,

$$\begin{aligned}\int_\Omega W^0(\theta_n(x), \nabla v_n(x))dx &= \inf\Big\{ \int_\Omega W^0_{\{\chi_q\}}(x, \nabla v_n(x))dx | \chi_q \in L^\infty(\Omega;[0,1]) \\ &\quad \text{with } I_{\chi_q} \overset{\Gamma}{\rightharpoonup} I^0_{\{\chi_q\}} \text{ and} \\ &\quad \chi_q \rightharpoonup \theta_n \text{ weak-}* \text{ in } L^\infty(\Omega;[0,1])\Big\}, \qquad (3.42)\end{aligned}$$

for $n = 1, 2, \ldots, \infty$. Indeed, if $\chi_q \rightharpoonup \theta_n$ weak-$*$ in $L^\infty(\Omega;[0,1])$ and $I_{\chi_q} \overset{\Gamma}{\rightharpoonup} I^0_{\{\chi_q\}}$, then by (C1) there exists a sequence $\chi_r(x, \cdot) \in L^\infty(Q;\{0,1\})$ such that

$$W^0_{\{\chi_q\}}(x, \xi) = \lim_{r\to+\infty} W^0_{\chi_r(x,\cdot)}(\xi)$$

with

$$\int_Q \chi_r(x, y)dy = \theta_n(x),$$

for a.e. x in Ω and every ξ in $\mathbb{R}^{N^2}$; thus

$$\begin{aligned}\int_\Omega W^0_{\{\chi_q\}}(x, \nabla v_n(x))dx &= \int_\Omega \lim_{r\to+\infty} W^0_{\chi_r(x,\cdot)}(\nabla v_n(x))dx \\ &\geq \int_\Omega W^0(\theta_n(x), \nabla v_n(x))dx. \qquad (3.43)\end{aligned}$$

The last inequality in (3.43) above holds true by virtue of (3.25). On the other hand the continuous character of W^* – see Lemma 3.9 – together with (3.26) permit to find a sequence $\{v_{ns}(x)\}, x \in \Omega$, of simple functions on Ω such that

$$W^0(\theta_n(x), v_{ns}(x)) \overset{s\to\infty}{\longrightarrow} W^0(\theta_n(x), \nabla v_n(x)), \text{ a.e. } x \text{ in } \Omega.$$

On a measurable subset Ω_j of Ω $(1 \leq j \leq j(n, s))$ where v_{ns} is constant there exists a sequence of functions $\{\chi^{nsj}_r(x, \cdot)\} \in L^\infty(Q;\{0,1\})$ such that

$$\begin{cases} \displaystyle\int_Q \chi^{nsj}_r(x, y)dy = \theta_n(x), \\ W^0(\theta_n(x), v_{ns}(x)) = \displaystyle\lim_{r\to+\infty} W^0_{\chi^{nsj}_r(x,\cdot)}(v_{ns}(x)), \end{cases}$$

for a.e. x in Ω_j. Through a diagonalization process we conclude to the existence of a sequence $\{\chi^n_r(x, \cdot)\} \in L^\infty(Q;\{0,1\})$ such that

$$\int_Q \chi^n_r(x, y)dy = \theta_n(x).$$

$$W^0(\theta_n(x), \nabla v_n(x)) = \lim_{r\to\infty} W^0_{\chi^n_r(x,\cdot)}(\nabla v_n(x)). \tag{3.44}$$

Note that the continuous character of W^0_χ in ξ and the compactness of $\{\nabla v_n(x)|\, x \in \Omega\}$ in $\mathbb{R}^N$ has been implicitly used in deriving (3.44). Define, for a.e. x in Ω and every ξ in $\mathbb{R}^{N^2}$,

$$\overline{W^0_n}(x,\xi) := \lim_{r\to\infty} W^0_{\chi^n_r(x,\cdot)}(\xi). \tag{3.45}$$

Then $\overline{W^0_n}$ is easily checked to be Carathéodory. Thus, by (C2), there exists a sequence $\chi^n_q \in L^\infty(\Omega;\{0,1\})$ with

$$\chi^n_q \rightharpoonup \theta_n \text{ weak-}* \text{ in } L^\infty(\Omega;[0,1]),$$

and

$$I_{\chi^n_q} \xrightarrow{\Gamma} \int_\Omega \bar{W}^0_n(x,\nabla\cdot)dx.$$

Hence

$$W^0_{\{\chi^n_q\}} = \overline{W^0_n}.$$

But, by virtue of (3.44) and (3.45)

$$\overline{W^0_n}(x,\nabla v_n(x)) = W^0(\theta_n(x),\nabla v_n(x)), \text{ for a.e. } x \text{ in } \Omega.$$

Thus

$$\int_\Omega W^0_{\{\chi^n_q\}}(x,\nabla v_n(x))dx = \int_\Omega W^0(\theta_n(x),\nabla v_n(x))dx,$$

which, together with (3.43), proves (3.42).

In view of (3.42) the very definition of Γ convergence implies the existence of a sequence $\{\chi_{nq}\}$ in $L^\infty(\Omega;[0,1])$ of a sequence $\{v_{nq}\}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$ and of an integer $q(n)$, with $q(n)$ depending on n such that

$$\begin{cases} \chi_{nq} \overset{q\to+\infty}{\rightharpoonup} \theta_n \text{ weak-}* \text{ in } L^\infty(\Omega;[0,1]), \\ v_{nq} \overset{q\to+\infty}{\rightharpoonup} v_n \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N), \\ \dfrac{1}{n} + \displaystyle\int_\Omega W^0(\theta_n(x),\nabla v_n(x))dx \ge I_{\chi_{nq}}(v_{nq}),\ q \ge q(n). \end{cases} \tag{3.46}$$

A diagonalization argument permits to conclude to the existence of sequence $\tilde{\chi}_n = \chi_{nq'(n)}$, $\tilde{v}_n = v_{nq'(n)}$ such that

$$\begin{cases} \tilde{\chi}_n \longrightarrow \theta \text{ weak-} * \text{ in } L^\infty(\Omega;[0,1]), \\ \tilde{v}_n \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N), \\ \liminf_{n\to\infty} I_{\tilde{\chi}_n}(\tilde{v}_n) \leq \liminf_{n\to\infty} \int_\Omega W^0(\theta_n(x), \nabla v_n(x))dx. \end{cases} \tag{3.47}$$

The third inequality of (3.46) and the fact that (a subsequence of) $\tilde{\chi}_n$ satisfies (cf. (3.41))

$$I_{\tilde{\chi}_n} \xrightarrow{\Gamma} I^0_{\{\tilde{\chi}_n\}},$$

imply that

$$\begin{aligned} \liminf_{n\to\infty} \int_\Omega W^0(\theta_n(x), \nabla v_n(x))dx &\geq I^0_{\{\tilde{\chi}_n\}}(v) \\ &= \int_\Omega W^0_{\{\tilde{\chi}_n\}}(x, \nabla v(x))dx \\ &\geq \inf\Big\{ \int_\Omega W^0_{\{\chi_q\}}(x, \nabla v(x))dx | \chi_q \rightharpoonup \theta \\ &\qquad \text{weak-} * \text{ in } L^\infty(\Omega;[0,1])\Big\} \\ &= \int_\Omega W^0(\theta(x), \nabla v(x))dx. \end{aligned}$$

Note that (3.42) applied with $n = \infty$ has been used in deriving the last equality. Thus, recalling (3.39) and the inequality $\theta(x) \geq \bar{\theta}(x)$, a.e. in Ω, we obtain

$$\begin{aligned} \liminf_{n\to\infty} \int_\Omega \omega(x, \nabla v_n(x))dx &\geq \int_\Omega [W^0(\theta(x), \nabla v(x)) + K\theta(x)]dx \\ &\geq \int_\Omega \omega(x, \nabla v(x))dx, \end{aligned}$$

which was the result sought.

We conclude that

Proposition 3.17 *If W_u and W_d defined in (3.1) are such that Conjecture 3.15 holds true (such is the case if W_u and W_d are e.g. convex) then Problem 3.13 admits a solution.*

Remark 3.18 The reader should refrain from drawing the conclusion that Problem 3.13 is a "relaxation" of Problem 3.3 (or 3.4). Indeed although Subsection 3.1 established that Φ_1^* is a bona fide relaxation of Φ_1, the argument breaks down at subsequent time steps because the irreversibility constraint $\chi(x) \geq \chi_{i-1}(x)$ a.e. in Ω_{i-1} for admissible χ's in X_i (see (3.10)) has been relaxed to $\theta(x) \geq \theta_{i-1}(x)$ in the definition of ψ_i^*.

This pathology which has already been encountered in [Francfort Marigo 93] has not as of yet been circumvented. The complete discretized time/space relaxation, which may be different from that hinted at in this subsection, is beyond reach because it would require a better understanding of the minimizing sequences $\chi_n^i(x)$ for a given $\theta^i(x)$ in expressions like (3.22).

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