Convergence of critical points of Ambrosio-Tortorelli to critical points of Mumford-Shah: the 1D case

with N. Le & S. Serfaty

Brittle Fracture revisited

• crack evolution in a quasi-static setting $\equiv$ time parameterised set of constrained global, or local minimization pbs, or even pbs with critical points.

• at the computational level, usually introduction of time discrete evolutions:

$$\begin{cases} t_0 = 0 < t_1 < \ldots < t_n = T \\ t_{i+1} - t_i = \Delta t \end{cases}$$

• at each discrete time, functional $\approx$ Mumford-Shah type functional, i.e., in the anti-plane shear setting and for global min.

$$\min_u \mu/2 \int_\Omega |\nabla u|^2 dx + k\mathcal{H}^1(S(u) \setminus \Gamma_{i-1})$$

with $\begin{cases} u = u_i \text{ on } \mathbb{R}^2 \setminus \overline{\Omega} \\ u \in SBV(\mathbb{R}^2). \end{cases}$

$$\Downarrow$$

$$\min_u \left\{ \underbrace{\mu/2 \int_\Omega |\nabla u|^2 dx + k\mathcal{H}^1(S(u))}_{:= \mathcal{F}(u)} : u = g \text{ on } \mathbb{R}^2 \setminus \overline{\Omega} \right\}$$

Approximation of $\mathcal{F}$

- Ambrosio-Tortorelli: for $\Omega' \supset \overline{\Omega}$

$$\mathcal{F}_\varepsilon(u,v) := \mu/2 \int_{\Omega'} \{(v^2+\eta_\varepsilon)|\nabla u|^2 + k(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{\varepsilon})\}dx$$

with $u \equiv g$ outside $\overline{\Omega}$ and $\eta_\varepsilon << \varepsilon$

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- Thm: $\mathcal{F}_\varepsilon \xrightarrow{\Gamma(L^2 \times L^2)} \begin{cases} \mathcal{F} + \int_{\Omega' \setminus \overline{\Omega}} |\nabla g|^2 dx \text{ if } v \equiv 1 \\ \infty \text{ else} \end{cases}$ $\square$

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start with $v_0 \in [0,1] \longrightarrow u_1 : \min_u \int_\Omega (\eta_\varepsilon + v_0)^2 |\nabla u|^2 +$ b.c.

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- Pb.? $\mathcal{F}_\varepsilon$ convex in u, in v but not in (u,v). No guarantee method converges to min! Only to critical points of $\mathcal{F}$ (Bourdin).
- Good from mechanical standpoint, since not clear min. no too restrictive, provided that critical points of AT converge to critical points of MS

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Similar results for Ginzburg-Landau (Sandier-Serfaty). None for Ambrosio-Tortorelli.

1d setting

• $(u_\varepsilon, v_\varepsilon)$ crit. pt. of
$\mathcal{F}_\varepsilon(u,v) = \int_0^\ell \{(\eta_\varepsilon + v^2)(u')^2 + \varepsilon|v'|^2 + \frac{(1-v)^2}{\varepsilon}\}dx$:

$$\begin{cases} ((\eta_\varepsilon + v_\varepsilon^2)u_\varepsilon')' = 0 \\ -\varepsilon v_\varepsilon'' + v_\varepsilon(u_\varepsilon')^2 + (v_\varepsilon - 1)/\varepsilon = 0 \\ u_\varepsilon(0) = 0, u_\varepsilon(\ell) = a \\ v_\varepsilon'(0) = v_\varepsilon'(\ell) = 0 \end{cases}$$

Rk: $\mathcal{F}_\varepsilon \longrightarrow \mathcal{F} := \int_0^\ell (u')^2 + 2\#(S(u) \cap (0,\ell) + \#(S(u) \cap \{0,\ell)\}$

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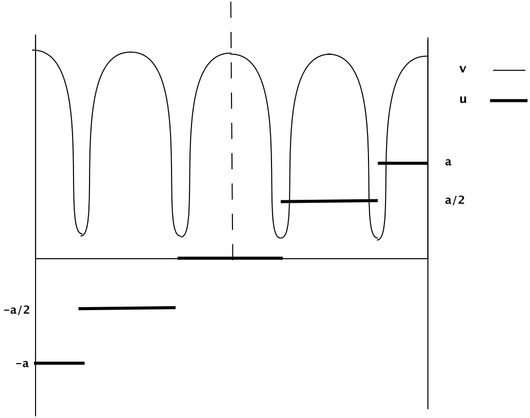
• Additional hyp. : $\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \mathcal{C}$
Hyp. satisfied by numerical scheme.

Results

- Thm. 1: Critical points $(u_\varepsilon, v_\varepsilon) \xrightarrow{\text{a.e.}} (u, 1)$ where $u = ax/\ell$ or u has equidistributed jumps. Conversely, all such critical points can be attained. □



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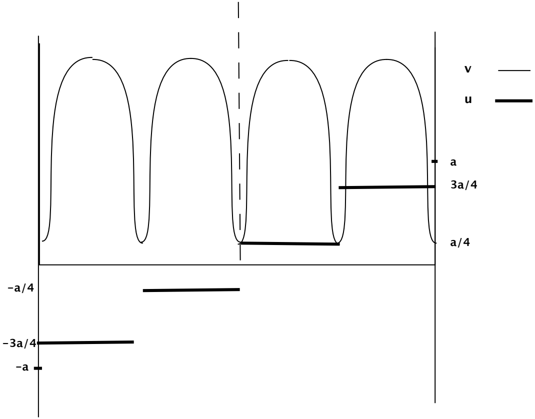

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• Thm. 2 : $\begin{cases} (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 dx \rightharpoonup (u')^2 dx \\ \varepsilon((v_\varepsilon)'^2 dx \rightharpoonup \text{ finite } \sum \text{ Dirac masses } \leftharpoondown \frac{(1-v_\varepsilon)^2}{\varepsilon} \end{cases}$.

If $u \neq ax/\ell$, then the sum is precisely

$$\#\lfloor(S(u) \cap (0,\ell)) + \frac{1}{2}\#\lfloor(S(u) \cap \{0,\ell\}).$$

Elementary estimates

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• 4/ d_ε (the discrepancy) $:= \dfrac{(1 - v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u'_\varepsilon)^2 - \varepsilon (v'_\varepsilon)^2$ is constant; further $d_\varepsilon \leq \mathcal{C}$.

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$\varepsilon (v'_\varepsilon)^2 \leq 1/\varepsilon + \mathcal{C} \Rightarrow \|v'_\varepsilon\|_{L^\infty} \leq \mathcal{C}/\varepsilon$

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Then since $(\eta_\varepsilon + v_\varepsilon^2) = c_\varepsilon / u'_\varepsilon \geq c_0/2\varepsilon \Rightarrow v_\varepsilon \geq \mathcal{C}\sqrt{\varepsilon}$ if $c_0 \neq 0$.

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- 5/ const. dis. $\Rightarrow$ if x_ε crit. pt. for v_ε, hence by sym. a min. or a max.,

$(\eta_\varepsilon + v_\varepsilon^2(x_\varepsilon))(1 - v_\varepsilon(x_\varepsilon))^2 \le \varepsilon \mathcal{C} \Rightarrow v_\varepsilon(x_\varepsilon) > 1 - \mathcal{C}\sqrt{\varepsilon}$ or $v_\varepsilon(x_\varepsilon) < \mathcal{C}\sqrt{\varepsilon} \Rightarrow$

If $m_\varepsilon := \min v_\varepsilon$, $M_\varepsilon := \max v_\varepsilon$, then $\begin{cases} m_\varepsilon > 1 - \mathcal{C}\sqrt{\varepsilon} \text{ or} \\ m_\varepsilon < \mathcal{C}\sqrt{\varepsilon}, M_\varepsilon > 1 - \mathcal{C}\sqrt{\varepsilon} \end{cases}$

v-Jumps

- A v-jump is a point x_ε s.t. $v'_\varepsilon(x_\varepsilon) = 0$ and $v_\varepsilon(x_\varepsilon) \leq \mathcal{C}\sqrt{\varepsilon}$, or equivalently, for ε small enough s.t. $v_\varepsilon(x_\varepsilon) \leq \alpha < 1$.

- No v-jump $\Rightarrow (u_\varepsilon, v_\varepsilon) = (ax/\ell, cst_\varepsilon \to 1)$ is the only crit.pt.

- Thus if $c_0 < a/\ell$, $\exists v$-jump. Assume v-jump exists from now on.

Symmetry

- $$\begin{cases} -v_\varepsilon'' + \dfrac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \dfrac{v_\varepsilon - 1}{\varepsilon} = 0 \\ v_\varepsilon'(0) = v_\varepsilon'(L) = 0. \end{cases} \overset{\text{symmetr.}}{\Longrightarrow} n_\varepsilon \text{ identical graphs on } [-\ell, \ell]$$

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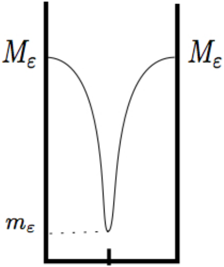

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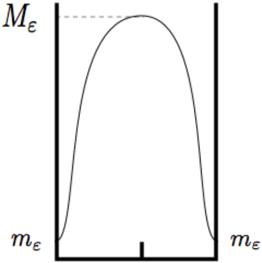

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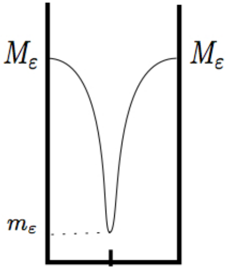

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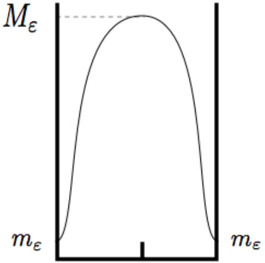

bell

- energetic cost of a well or bell: $v_\varepsilon(s_\varepsilon) = .1, v_\varepsilon(t_\varepsilon) = .9$

$$\int_{s_\varepsilon}^{t_\varepsilon} \varepsilon |v_\varepsilon'|^2 + \frac{(1-v_\varepsilon)^2}{\varepsilon} dx \geq \int_{s_\varepsilon}^{t_\varepsilon} 2v_\varepsilon'(1 - v_\varepsilon)dx \geq [(2v_\varepsilon - v_\varepsilon^2)]_{s_\varepsilon}^{t_\varepsilon} \geq .5 \Rightarrow$$

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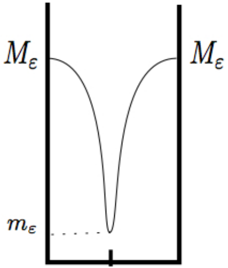

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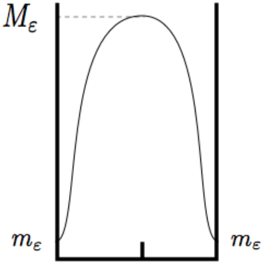

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n_ε bounded $\Rightarrow$ n_ε fixed indtly of ε.

- Also u_ε is also extended by sym. to $[\ell, \ell]$: $u_\varepsilon(-\ell) = -a$

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$\Rightarrow \mathcal{C} \geq \left|\int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} \varepsilon v_\varepsilon''\right| \geq \int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} \mathcal{C}/v_\varepsilon^3 - |\{v_\varepsilon \leq M\sqrt{\varepsilon}\}|/\varepsilon$

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$a = \int_0^\ell u_\varepsilon' =$
$\int_{\{v_\varepsilon \leq M\sqrt{\varepsilon}\}} u_\varepsilon' + \int_{\{M\sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}} c_\varepsilon/(\eta_\varepsilon + v_\varepsilon^2) + \int_{\{v_\varepsilon \geq 1/2\}} u_\varepsilon'$

$\leq \mathcal{C}M^3\sqrt{\varepsilon}$ el.est. $+ \frac{c_0}{M^2\varepsilon}|\{v_\varepsilon \leq 1/2\}| + (\rightarrow c_0\ell) \overset{\varepsilon \searrow 0}{\Rightarrow} a \leq c_0/M^2 + c_0\ell$

$\uparrow \leq \mathcal{C}\varepsilon$ by energy est.

Contradiction letting $M \nearrow \infty$: $c_0 \geq a/\ell$.

Form of u and proof of first theorem

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- example well case: n=1. Then, $v_\varepsilon \to 1$ uniformly on $[\ell, \ell] \setminus (-\delta, \delta)$ since set where $v_\varepsilon \leq 1 - \varepsilon^{1/4}$ is centered around 0 and has measure $\leq \sqrt{\varepsilon}$ from energy bound $\Rightarrow u'_\varepsilon \to c_0$ uniformly there $\Rightarrow u_\varepsilon(x) = u_\varepsilon(\ell) - \int_x^\ell u'_\varepsilon \to a - c_0(\ell - x)$.

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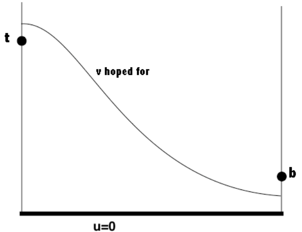
• example well case: n=1. Then, $v_\varepsilon \to 1$ uniformly on $[\ell,\ell] \setminus (-\delta,\delta)$ since set where $v_\varepsilon \leq 1 - \varepsilon^{1/4}$ is centered around 0 and has measure $\leq \sqrt{\varepsilon}$ from energy bound $\Rightarrow u'_\varepsilon \to c_0$ uniformly there $\Rightarrow u_\varepsilon(x) = u_\varepsilon(\ell) - \int_x^\ell u'_\varepsilon \to a - c_0(\ell - x)$.

• Conversely: enough to consider 1 jump. look for minimizers of $\mathcal{F}_\varepsilon$ in $\mathcal{D} := \{(u,v) : u(0) = 0, u(\ell) = a, v(0) \geq t, v(\ell) \leq b\}$, with $0 < b < t < 1$



$(u_\varepsilon, v_\varepsilon)$ minimizer is st
$\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq 1 + o(1)$
Must show that $v_\varepsilon(0) > t$ and $v_\varepsilon(\ell) < b$ to get $v'_\varepsilon(0) = v'_\varepsilon(\ell) = 0$, hence v_ε is a critical pt.

Proof of second theorem

- $\int_0^\ell (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 \varphi dx = \int_0^\ell c_\varepsilon u_\varepsilon' \varphi dx = -\int_0^\ell c_\varepsilon u_\varepsilon \varphi' dx$
$\to -c_0 \int_0^\ell u\varphi' dx = \int_0^\ell u'^2 \varphi dx$ since, if $u' = 0, c_0 = 0$.

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- $d_\varepsilon + c_\varepsilon u'_\varepsilon = (v_\varepsilon - 1)^2/\varepsilon - \varepsilon(v'_\varepsilon)^2$: integrate it away from set where $v_\varepsilon \leq 1 - \varepsilon^{1/4}$ (centered around v-jumps $\Rightarrow$ tends to 0, hence $d_0 + c_0^2 = 0$.

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So, $(1 - v_\varepsilon)^2/\varepsilon dx$ and $\varepsilon(v'_\varepsilon)^2 dx \to \sum$ of weighted Dirac masses at limits of v-jumps.

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- For each Dirac, the weight is 1 (or $\frac{1}{2}$ at both ends).
- Open pb.: Are there v-jumps for $c_0 \neq 0$?