Heterogeneous Elasto-plasticity

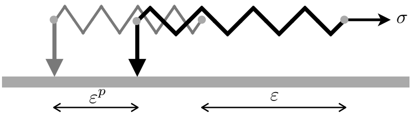
$\pi\lambda\acute{\alpha}\sigma\sigma\varepsilon\iota\nu$

G. F. & Alessandro Giacomini

Small strain elastoplasticity

Small strain elasto-plasticity – the rheology

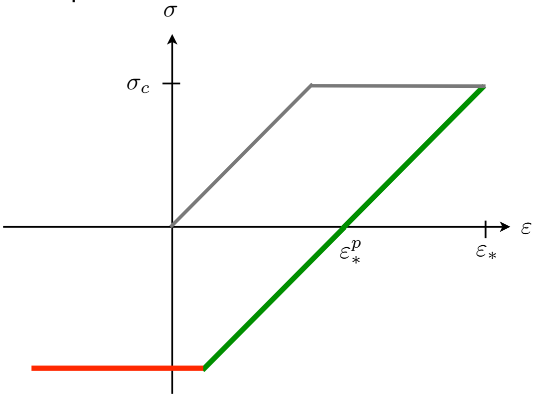
- A model with brake and spring:



$$|\sigma| \le \sigma_c$$

$$\text{with} \begin{cases} \dot{\varepsilon}^p \ge 0 & \sigma = \sigma_c \\ \dot{\varepsilon}^p = 0 & |\sigma| < \sigma_c \\ \dot{\varepsilon}^p \le 0 & \sigma = -\sigma_c \end{cases}$$

- Response:



Small strain elasto-plasticity – the formulation $\varepsilon^p \equiv p$

$$\mathbb{M}^{N\times N}_{dev} := \{\tau \text{ symmetric} : \ \text{tr}\ \tau = 0\}$$

$$\tau = \frac{\text{tr}\ \tau}{N}\mathfrak{i} + \tau_D$$

- $Eu := \dfrac{Du + Du^t}{2} = e + p$

$p \in \mathbb{M}^{N\times N}_{dev}$

$\sigma = Ae;\ \operatorname{div} \sigma = 0$ in Ω

A : Hooke's law

$\sigma \in K := \{\tau : f(\tau_D) \leq 0\}$

with K closed convex

$(f$ conv., $f(0) < 0$, $f \overset{|\tau|\nearrow\infty}{\nearrow} \infty)$

set of admissible stresses

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- Flow rule:

$\dot{p}(t) \in \mathcal{A}\sigma :=$

$$\left\{\tau \in \mathbb{M}^{N\times N}_{dev} : \exists\lambda \geq 0 \text{ s.t. } \tau = \lambda\frac{\partial f}{\partial \tau}(\sigma(t)) \text{ and } \lambda f(\sigma(t)) = 0\right\}$$
$$\Updownarrow$$

$\dot{p}(t) \in N_K(\sigma(t))$, the normal cone to K at $\sigma(t) \in \partial K(t)$

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- Existence of an evolution known under C^2-smoothness for $\partial\Omega+$ C^2-smoothness of $\partial_{\partial\Omega}[\partial_d\Omega]$:
  - by viscoplastic approx. (Suquet 1978)
  - through var. evolutions (Dal Maso-De Simone-Mora 2004)

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$E(u) = e + p$ kin. compatibility
$$\begin{cases} u \in AC(0,T; BD(\Omega)) \\ e \in AC(0,T; L^2(\Omega;\mathbb{R}^N)) \\ p \in AC(0,T; \mathcal{M}_b(\Omega \cup \partial_d\Omega; \mathbb{M}^{N\times N}_{dev})) \end{cases}$$

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b.c. on $\partial_d\Omega$ has been relaxed: $p = [w - u] \odot \nu$, $w - u \perp \nu$

A remark about stress admissibility – Lipschitz domain Ω

- From div $\sigma = 0 \ + \ \sigma \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym} \cap K)$, we get:

$$(\sigma_D \nu)_\tau (\text{the tangential part of } \sigma\nu) \in (K\nu)_\tau$$

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- Since $\dot{p} = [\dot{w}(t) - \dot{u}(t)] \odot \nu$ on $\partial_d \Omega$, with $[\dot{w}(t) - \dot{u}(t)] \perp \nu$, we expect a boundary flow rule:

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$\not\Uparrow$

bulk flow rule

The variational approach to elastoplasticity

Variational evolution in a nutshell

- Define:

– diss. pot. : $H(p) := \sup\{\sigma_D \cdot p : \sigma \in K\}$

– dissipation: $\mathcal{H}(q) := \int_{\Omega \cup \partial_d \Omega} H\left(\frac{q}{|q|}(x)\right) d|q|$

– total diss.: $\mathcal{D}(0,t;p) := \sup_{part.\ of\ [0,t]} \sum_i \mathcal{H}(p(t_{i+1}) - p(t_i))$

– total energy: $E(t) := 1/2 \int_\Omega Ae(t) \cdot e(t) dx + \mathcal{D}(0,t;p)$

At each time t, $(u(t), e(t), \sigma(t) := Ae(t), p(t))$ satisfies

- Global min.: $1/2 \int_\Omega Ae(t) \cdot e(t) dx \le 1/2 \int_\Omega A\eta \cdot \eta dx + \mathcal{H}(q - p(t))$

(ve)

- Energy cons.: $$\frac{dE}{dt}(t) = \int_\Omega \sigma(t) \cdot E\dot{w}(t) dx$$

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- Note that, if (u,e,p) (resp. (u',e',p')) min. $1/2 \int_\Omega A\eta\cdot\eta dx + \mathcal{H}(q-p)$ (resp p'), then

$$\|e' - e\|_{L^2} \leq C\left\{\|Ew' - Ew\|_{L^2} + |p' - p|^{\frac{1}{2}}_{\Omega\cup\partial_d\Omega}\right\}$$

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- The lower semi-continuity of $\mathcal{H}$ is ensured by Reshetnyak's lower semi-continuity theorem

Variational evolution & classical formulation – 1

- Global minimality $\Leftrightarrow -\mathcal{H}(q) \leq \int_\Omega Ae \cdot \eta \, dx \leq \mathcal{H}(-q)$

  $\forall (v, \eta, q)$ kin. compat. with b.c. $w = 0$

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  equilibrium + Neumann b.c.+ $\sigma \in K$

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- To go further, need to define the duality $\langle \sigma_D, p \rangle$.

Not so clear because σ_D not continuous!

Isues of duality

Here σ and p are arbitrary provided that σ satisfies eqm. + Neumann b.c.+stress adm. & p is assd. to (u, e, p) kin. compatible with w as b.c.

- First define $\langle \sigma_D, p \rangle$ as a distribution:

$$\langle \sigma_D, p \rangle(\varphi) = -\int_\Omega \varphi \sigma \cdot (e - Ew)\, dx - \int_\Omega \sigma \cdot [(u - w) \odot \nabla \varphi]\, dx$$

↑ OK since $\sigma \in L^N$

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- Known result: If $\partial\Omega$ is C^2 and $\partial_{\partial\Omega}[\partial_d\Omega]$ is a C^2 $(N-2)$-hypersurface, then $\langle \sigma_D, p \rangle$ is a finite Radon meas. on $\mathbb{R}^N$ (Kohn-Temam 1983)

- Thm: Ω Lipschitz. Then $\langle \sigma_D, p \rangle$ is a finite Radon meas. on $\mathbb{R}^N \setminus \partial_{\partial\Omega}[\partial_d\Omega]$ and $|\langle \sigma_D, p \rangle| \le \|\sigma_D\|_{L^\infty}|p|$, $\langle \sigma_D, p \rangle_a = \sigma_D \cdot p_a$

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- Technical point: What do we need for $\langle \sigma_D, p \rangle$ to be a finite Radon meas. on all of $\mathbb{R}^N$?
- Open pb.: Can we prove this under the only assumption that e.g. $\mathcal{H}^{N-2}(\partial_{\partial\Omega}[\partial_d\Omega]) < \infty$?

Variational evolution & classical formulation – 2

- Just using the definition of the duality:

$$\langle \sigma_D, p \rangle_{\lfloor \partial_d \Omega} = (\sigma_D \nu)_\tau (w - u) \mathcal{H}^{N-1}_{\lfloor \partial_d \Omega}$$

$$\Downarrow$$

(Ineq) $$H\left(\frac{p}{|p|}\right)|p| \geq \langle \sigma_D, p \rangle \text{ as measures}$$

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- From energy equality + Reshetnyak's lower semi-continuity thm.:

$$\mathcal{H}(\dot p) \leq \dot{\mathcal{D}}(0,t,p) = -\int_\Omega \sigma(t)\cdot(\dot e - E\dot w)(t)\,dx = \langle \sigma_D, \dot p\rangle(\Omega \cup \partial_d\Omega)$$

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Hill's maximal plastic work principle

$$\langle \sigma_D, \dot{p}\rangle(\Omega \cup \partial_d\Omega) = \sup_{\tau_D \text{ adm.}} \langle \tau_D, \dot{p}\rangle(\Omega \cup \partial_d\Omega)$$

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- From $H\left(\frac{\dot p}{|\dot p|}\right)|\dot p| = \langle \sigma_D, \dot p\rangle$, we recover the flow rule, BOTH in Ω and on $\partial_d\Omega$:

$$\dot p_a \in N_K(\sigma_D) \text{ in } \Omega;\ |\dot w - \dot u| \in N_{(K\nu)_\tau}((\sigma_D\nu)_\tau) \text{ on } \partial_d\Omega$$

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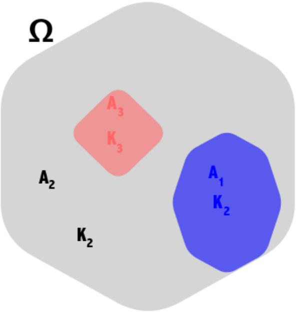
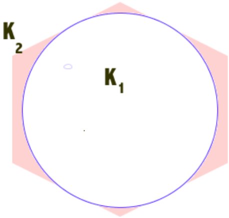
A multiphase domain





No ordering property of the K_i's
We will need C^1 interfaces

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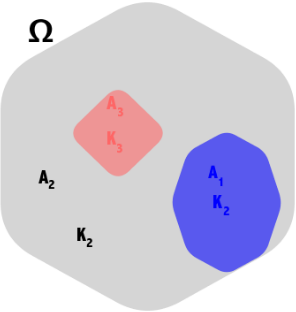
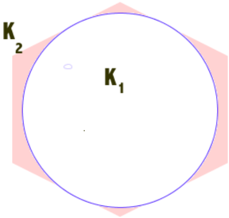


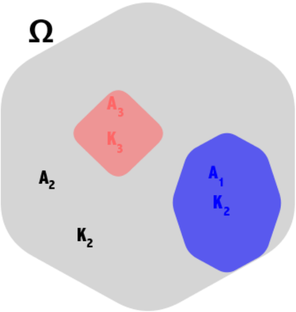


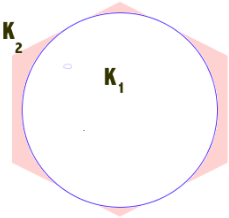
No ordering property of the K_i's
We will need C^1 interfaces

- Define the dissipation :

$H(x, p) := H_i(p) = \sup\{\sigma_D \cdot p : \sigma_D \in K_i\}$ in each phase i. Since we expect p to be a measure, how do we define H on $\bar{\Omega}_i \cap \bar{\Omega}_j$?

A multiphase domain





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- ~~$\inf\{H_i, H_j\}$~~ because destroys convexity $\Rightarrow$ Inf-convolution:

$$H(x,\xi) := \begin{cases} \inf\{H(a \odot \nu(x)) + H(-b \odot \nu(x)); a - b = c\}, & \text{if } \xi = c \odot \nu(x) \\ \infty, & \text{else} \end{cases}$$

$\Downarrow$

destroys l.s.c./ Need to re-establish l.s.c. of $\mathcal{H}$:

Thm: If (u_n, e_n, p_n) kin. compatible and the natural weak conv. hold $(BD \times L^2 \times \mathcal{M}_b)$ then $\mathcal{H}(p) \le \liminf_n \mathcal{H}(p_n)$

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• Existence of a variational evolution

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- Choice of dissipation is the right one for passing to the zero hardening limit in a model with isotropic linear hardening.

Homogenization

- Rescaled heterogeneous variational evolution: x replaced by x/ε for multiphase torus $\mathcal{Y}$ with C^1 interfaces.
- Homogenization: with approp. i.c.'s, $\exists\varepsilon_n$ s.t., for all $t \in [0, T]$,

$$\begin{cases} u_n(t) \xrightharpoonup{*} u(t) & \text{weakly* in } BD(\Omega') \\ e_n(t) \xrightharpoonup{w\text{-}2} E(t) & \text{two-scale weakly in } L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^{N\times N}_{sym}) \\ p_n(t) \xrightharpoonup{w^*\text{-}2} P(t) & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D). \end{cases}$$

Here, $E(x,y)\,\mathcal{L}^N_x \otimes \mathcal{L}^N_y + P - Eu \otimes \mathcal{L}^N_y = E_y\mu$ in $\Omega' \times \mathcal{Y}$ with $\mu \in \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{R}^N)), E_y\mu \in \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^{N\times N}_{sym}), \mu(F \times \mathcal{Y}) = 0$, $\forall F$ Borel $\subseteq \Omega'$.

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- Further $(u(t), E(t), P(t))$ is a two-scale quasistatic evolution: defined as before with explicit y-dependence; for example: dissipation $\mathcal{H}^{hom}(Q) := \int_{\mathcal{Y}\times\Omega\cup\partial_d\Omega} H\left(y, \frac{Q}{|Q|}(x, y)\right) d|Q|$

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• In essence, $P(\cdot,\cdot,y)$ is, for each $y \in \mathcal{Y}$, an internal var. $\Rightarrow \exists$ flow rule in y that expresses normality at the micro level.......

Hardening

Arguing our choice of dissipation – isotropic linear hardening

• Additional variable: $\zeta(t,x)$ measures change of convex of plasticity $K(t,x) := (1-\zeta(t,x))K(x)$, z dual variable to ζ

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$\Updownarrow$ Not so hard to see

Variational evolution for

$(u_h, e_h, p_h, z_h) \in$
$H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) \times L^2(\Omega;\mathbb{M}^{N\times N}_{dev}) \times L^2(\Omega)$:

with i.c. $\equiv 0$

• Global min.: $1/2\int_\Omega A\eta\cdot\eta dx + h^2/2\int_\Omega y^2 dx + \|y-z\|_{L^1}$ among all (v,η,q,y) with $Ev = \eta + q$, $H(x,q) \le z$

• Energy cons. : same as before with
$E_h(t) := 1/2\int_\Omega Ae_h(t)\cdot e_h(t)dx + h^2/2\int_\Omega z_h^2(t)dx + \int_\Omega z_h(t)dx$

Heterogeneous plasticity as limit of model with isotropic hardening $h \searrow 0$

- Usual estimates: $p_h \overset{\mathcal{M}_b}{\rightharpoonup} p$; $u_{h_t} \overset{SBV_p}{\rightharpoonup} u$; $e_{h_t} \overset{L^2}{\rightharpoonup} e$ with (u, e, p) kim. compatible.

- Pass to the limit in EL eqs.:

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- Energy conservation in the limit easy consequence of l.s.c. dissn.

$$+ \ \mathcal{D}(0, t; p) \leq \hat{\mathcal{D}}(0, t; p)$$