FRACTURE WITH HEALING: A FIRST STEP TOWARDS A NEW VIEW OF CAVITATION

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Abstract. Recent experimental evidence on rubber has revealed that the internal cracks that arise out of the process often referred to as cavitation can actually heal.

In this contribution we demonstrate that crack healing can be incorporated into the variational framework for quasi-static brittle fracture evolution that has been developed in the last twenty years. This will be achieved for two-dimensional linearized elasticity in a topological setting, that is when the putative cracks are closed sets with a preset maximum number of connected components.

Other important features of cavitation in rubber such as near incompressibility and the evolution of the fracture toughness as a function of the cumulative history of fracture and healing have yet to be addressed even in the proposed topological setting.

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1. Introduction

1.1. **A simplistic model for cavitation.** Ever since the 1930's, ample experimental evidence points to the specificity of the initiation and propagation of fracture in rubber, or more generally in soft organic solids (see e.g. [7, 16, 17]). While metals, ceramics, and, more generally, crystalline and glassy solids show well defined crack patterns when subject to extreme loading processes, fracture in rubber tends to initiate through the growth of microscopic defects arising in regions under sufficiently high hydrostatic stress. Because of its fluidic elder counterpart, the phenomenon has become known as cavitation.

It was initially thought that cavitation could be explained on pure elastic ground. In the mechanical universe, the most notorious proponents of elastic cavitation were undoubtedly A.N. Gent & P.B. Lindley [16]. In their footstep, J.M. Ball pioneered the first mathematical translation of that idea [4]. There he posited that hyperelasticity can, in and of itself create cavities through solutions of the type $x/|x|$ that are good Sobolev functions, provided that the growth at infinity of the elastic energy be subcritical, that is less than the spatial dimension. In a more classical framework an equivalent viewpoint posits incipient point defects that balloon up to cavities. This insight generated a slew of mathematical studies that did show promise.

However, the spectacle of cavitation as a purely elastic phenomenon is in our opinion unrealistic. On pure theoretical grounds, it strikes us as somewhat peculiar that an innate sense of self would raise material awareness of its energetic elastic health under very large stretches, a prerequisite for any cogent statement of its growth. On more practical grounds, it was recently shown in [22] that, in the classical poker-chip experiments of Gent and Lindley as well as for a different experiment that uses a rubber reinforced by filler particles [26][1], a mere accounting of the elastic properties of the solids, while leading to a superficially adequate qualitative agreement with a number of experimental observations, fails to provide a complete qualitative and, most importantly quantitative rendering of the evolution.

Our guiding principle is therefore that elasticity alone cannot account for the full complexity of the phenomenon of cavitation in rubber. From a macroscopic point of view, one should at the least introduce new internal surfaces within the solid to adequately describe the actual microscopic

[1] We refer to that experiment as the *filler particle experiment*.

mechanisms behind fracture, be it the spatial rearrangement of the underlying macromolecules, or the breakage of chemical bonds. Such a viewpoint would seem to promote a fracture type model in the vein of those adopted for brittle solids, albeit in the context of finite elasticity (see e.g. [10]) and with the additional accounting of near or full incompressibility.[2]

Incompressibility notwithstanding, a refined fracture model was recently advocated in various mathematical works of D. Henao & C. Mora-Corral [19, 25]. There, a surface energy proportional to the perimeter of the cavities in the deformed configuration is considered, in the spirit of surface tension. It is then added to the elastic energy and subsequently viewed, at least in [25], as a conservative contribution. Adopting for a moment a common terminology in the mechanics community,[3] the only source of dissipation is born out of the irreversible creation of a countable number of point discontinuities that will grow into cavities.

The idea of endowing created surfaces with an energy is original and potentially fruitful. This refined viewpoint – or even a classical fracture viewpoint for that matter – may provide a good fit for some of the poker-chip experiments. But both will most likely become exercises in Alt-Reality when it comes to the *filler particle experiments*. Recent such experiments, carried out at high spatio-temporal resolution in [26], showed that some of the created cavities actually vanish during the loading process while others migrate away from the particles. Traditional or revamped theories of fracture do not sustain disappearance, or migration and, while arguably predicting the final location of the cavities, completely fail in their depiction of the path that would lead to the final migrated state.

The full picture of the *filler particle experiment* is actually more intricate. The experiments in [26] have also shown that the regions of the rubber that experience healing appear to acquire different fracture properties from those of the original rubber, thereby hinting at an evolution of the underlying molecular rearrangement and/or chemical bonding due to the healing process.

A full account of such observations is not our purpose at this point. It would certainly involve a healing process, together with a hardening or softening process in the fracture toughness, if such a notion is sensical. Further, near or full incompressibility would certainly be a major partner, although its role has yet to be scripted.

Rather we propose in this contribution to focus solely on healing. The above quoted experiment notwithstanding, there is ample independent evidence that healing does take place in soft organic solids; see e.g. [23, 5, 9]. Now of course, as far as rubber is concerned, healing and near incompressibility should not be viewed as independent agents. We will woefully ignore their relationship in the following study. Mathematical impotence, rather than spite, motivates our choice.

So, as an admittedly childish first step, we propose to incorporate healing in the A. A. Griffith's theory of fracture [18] (suitably re-engineered through a variational lens [15, 6]) for two-dimensional linear elasticity. At first glance such a task would seem simple enough, at least from a modeling standpoint and provided that one is willing to view the healing process as rate independent, which is most likely not so.[4] The naive recipe would be to dissipate some amount of surface energy for crack repair. In other words one would pay, say $c_1 \times$ length of $\Gamma \setminus K$, $c_1 > 0$, for changing the crack K to a different crack Γ and would also pay $c_2 \times$ length of $K \setminus \Gamma$, $c_2 > 0$, for repairing some of K with Γ.

Such petulance must be tempered with the recognition that doing so would result in a model for which healing would never take place because a healed part of the crack would increase the elastic energy while dissipating some surface energy through healing. Thus the healing process,

[2]The addition of an incompressibility constraint is a huge mathematical hurdle from the standpoint of the variational theory of (brittle) fracture and the reader should be alerted to the absence of any mathematically significant result that encompasses both incompressibility and fracture.

[3]While a prevailing one, the postulate that fracture, or cavitation, should be described in terms of entropy production due to some kind of dissipation is just that (see *a contrario* [20]) and our casting of the cavitation model in those terms is mere abidance by the majority view.

[4]Rearranging the molecular structure of the rubber and/or forming new chemical bonds are in all likelihood viscosity driven processes that will shatter rate independence while potentially still variationally tractable; see the recent approach of viscoplasticity using energy-dissipation-balance solutions [24]. As for the problem at hand, the precise nature of viscosity is very unclear as of yet.

if rate independent and proportional to the length of the healed part must actually decrease the dissipated energy. A formal account will be given at the onset of Section 2.

For now, just think of a pre-set connected crack path Γ in a domain Ω and of a connected crack $\Gamma(\ell)$ of length ℓ *starting from a set point* – say the origin – along Γ (which should also contain the origin). Denote by $\mathcal{W}(\ell)$ the potential energy associated to the elastic equilibrium of $\Omega \setminus \Gamma(\ell)$ – the un-cracked part of the domain – under the current loading at time t. Then we impose fealty of the dual fracture/healing process to that of Griffith's fracture [18].

It is thus assumed that the energy dissipated through any putative advancement of the crack is proportional to the add-crack length with c_1 as fracture toughness; similarly that gained through healing is proportional to the subtract-crack length with c_2 as healing toughness. Of course $c_1 > c_2$ so that there indeed be a net dissipation.

To determine $\ell(t)$ a two-pronged formulation is espoused.

- First, a stability criterion à la Griffith is imposed: the energy release rate must satisfy
$$c_2 \le -\frac{\partial \mathcal{W}}{\partial \ell}(\ell(t)) \le c_1.$$
- Then the crack cannot extend unless the second inequality is an equality while it cannot shrink unless the first one is an equality.

Further, because irreversibility is *de facto* abandoned, there is no impediment to surface energy contributing to internal energy as well. In the above cartoon picture of the evolution, this amounts to adding a term like $c\ell, c \ge 0$, to the elastic energy $\mathcal{W}(\ell)$.

Sections 2-4 investigate the setting of anti-plane shear linear elasticity which is undoubtedly the simplest available framework for fracture evolution. The resulting model is presented in Section 2 in its variational reformulation. Section 3 is devoted to the proof of a stability result which is essential in the success of the limit process when passing from a time-incremental to a time-continuous formulation. Section 4 establishes the existence result for an evolution where both cracking and healing are allowed. In Section 5 we generalize the results of Section 4 to the setting of planar elasticity (plane strain or plane stress) in the footstep of similar work on the fracture only case [8].

From a mathematical standpoint, the first existence results for the variational theory of fracture were obtained in [11] in the anti-plane shear case under the topological restriction that the cracks should have no more than m connected components, m being a pre-set connectivity threshold. This restriction was subsequently alleviated in [14]. The present study unfortunately forces us to return to the topological setting of [11], mainly because we do not know how to prove energy conservation in the fully "variational" framework, that is with no restriction on the topology of the cracks (see in particular Remark 1.5 below).

There is by now a vast literature on various aspects of the variational theory of fracture. We trust that the potential readership for this work is well versed in the main tenet of that theory and consequently refrain from any detailed explanation of the expounded formulation. We refer the newcomers to [6] for an exposition of that theory and in particular to [6, Chapter 2] where the link between the variational theory and a formulation of the above two-pronged formulation is unraveled.

At the close of this introduction, we see it fit to put forth the following disclaimer: The model that is advocated below is not meant to be viewed as the final adjudication of cavitation. In view of recent experimental evidence, we merely assert that fracture and healing are essential partners in the cavitation process. We then proceed to incorporate healing into the variational theory of fracture in the mathematically simplest possible manner. Doing so at this time does not preclude subsequent refinements or modifications of the model. Reference [21] presents a much more intricate phase field model that strives to account for both incompressibility and hardening on top of healing.

But it would be presumptuous on our part to pretend that we know how to address the mathematical hurdles that would accompany a rigorous analysis of more complex cavitation models such as that offered in [21]. So, from a mathematical standpoint, the analysis below is the sum total of what lies within our reach for now.

Notation. Given $x \in \mathbb{R}^2$, $r > 0$ and $\nu \in \mathbb{R}^2$, $Q_\nu(x,r) \subset \mathbb{R}^2$ denotes the square of center x with one side orthogonal to ν and length r. When ν is vertical, we will write simply $Q(x,r)$. $B(x,r)$ will denote the disk of center x and radius r.

Given two sets $A, B \subseteq \mathbb{R}^2$, $A\Delta B$ denotes their symmetric difference, while $A \subset\subset B$ will mean $\overline{A} \subseteq B$.

In all that follows $\mathrm{M}^2_{\mathrm{sym}}$ and $\mathrm{M}^2_{\mathrm{skew}}$ denotes the family of symmetric and antisymmetric 2×2-matrices, respectively while $\mathcal{L}_s(\mathrm{M}^2_{\mathrm{sym}})$ stands for the space of symmetric endomorphisms of $\mathrm{M}^2_{\mathrm{sym}}$.

For any mapping $u : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $e(u)$ denotes the symmetrized gradient of u, that is $e(u) := 1/2(\nabla u + \nabla u^T)$.

Also, for any open set A, $\mathscr{LD}(A) := \{u \in L^2_{loc}(A;\mathbb{R}^2) : e(u) \in L^2(A;\mathrm{M}^2_{\mathrm{sym}})\}$.

Finally, we use standard notation for Sobolev spaces and for Hausdorff measures, specifically denoting by $\|\ \|$ the L^2-norm and by $\|\ \|_\infty$ the L^∞-norm. Also, for X Banach space, we denote by $AC([0,T];X)$ the space of X-valued absolutely continuous functions.

1.2. **Mathematical preliminaries - Hausdorff convergence of compact sets.** In the sequel, Hausdorff convergence will play an essential role. For the reader's convenience, we recall a few properties that will be used throughout.

The family $\mathcal{K}(\mathbb{R}^N)$ of closed sets in $\mathbb{R}^N$ can be endowed with the Hausdorff metric d_H defined by

$$d_H(K_1,K_2) := \max\left\{\sup_{x\in K_1} \mathrm{dist}(x,K_2), \sup_{y\in K_2} \mathrm{dist}(y,K_1)\right\}$$

with the conventions $\mathrm{dist}(x,\emptyset) = +\infty$ and $\sup\emptyset = 0$, so that $d_H(\emptyset,K) = 0$ if $K = \emptyset$ and $d_H(\emptyset,K) = +\infty$ if $K \neq \emptyset$.

The Hausdorff metric has good compactness properties (see [3, Theorem 4.4.15]).

Proposition 1.1 (Compactness). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets contained in a fixed compact set of $\mathbb{R}^N$. Then there exists a compact set $K \subseteq \mathbb{R}^N$ such that up to a subsequence*

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

We will repeatedly make use of the following property due to Gołąb; for the proof we refer the reader to [13, Theorem 3.18] or [3, Theorem 4.4.17].

Theorem 1.2 (Gołąb). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact connected sets in $\mathbb{R}^N$ such that*

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

Then K is connected and for every open set $A \subseteq \mathbb{R}^N$

$$\mathcal{H}^1(K\cap A) \le \liminf_{n\to\infty} \mathcal{H}^1(K_n \cap A).$$

Remark 1.3. The lower semicontinuity of Gołąb's Theorem still holds when K_n has a uniformly bounded number of connected components.

Lemma 1.4. *Let $(K_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ be two sequences of compact sets in $\mathbb{R}^N$, each with a uniformly bounded number of connected components. Assume that*

$$K_n \to K \qquad \text{and} \qquad H_n \to H \qquad \text{in the Hausdorff metric.}$$

Then, for any open set $A \subseteq \mathbb{R}^N$,

$$\mathcal{H}^1((K\setminus H)\cap A) \le \liminf_n \mathcal{H}^1((K_n\setminus H_n)\cap A). \tag{1.1}$$

Proof. Let $V \subseteq \mathbb{R}^N$ be an open neighborhood of H. For n large enough we have $H_n \subseteq V$, so that by Gołąb's Theorem

$$\mathcal{H}^1((K\setminus \bar{V})\cap A) \le \liminf_n \mathcal{H}^1((K_n\setminus \bar{V})\cap A) \le \liminf_n \mathcal{H}^1((K_n\setminus H_n)\cap A).$$

Since V is arbitrary, the conclusion follows. □

Remark 1.5. The topological setting for the cracks adopted in the paper, i.e., cracks which are closed and with a preset number of connected components, is motivated precisely by Lemma 1.4. A larger class of admissible cracks, as that adopted in [10] where cracks are just rectifiable, requires suitable convergences of variational type, under which inequality (1.1) is known to fail. But that inequality is in particular an essential ingredient in the proof of the energy inequality (4.19) below to the extent that it establishes that (4.16) holds true.

A simple example for which inequality (1.1) is violated under the variational convergences of [10] is the following: Let K be a segment of unit length, and let H_n be the dotted segment of length $1/2$ obtained from K by dividing it into 2^n equal parts and retaining only every other sub-segment. It is easily proved that $H_n \to \emptyset$ in the variational sense (see [10, Section 4.1]), so that choosing $K_n = K$,

$$\mathcal{H}^1(K \setminus H) = \mathcal{H}^1(K) = 1 \qquad \text{while} \qquad \mathcal{H}^1(K_n \setminus H_n) = \frac{1}{2}.$$

2. Setting of the problem

The reference configuration is an open bounded set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary.

Admissible cracks. Let $m \in \mathbb{N}$ with $m \geq 1$ be given. The class of admissible cracks is given by

$$\mathcal{K}_m^f(\overline{\Omega}) := \{K \subset \overline{\Omega} : K \text{ is compact, with at most } m \text{ connected components and } \mathcal{H}^1(K) < +\infty\}. \tag{2.1}$$

Admissible configurations. Let $\partial_D \Omega \subseteq \partial\Omega$ be open in the relative topology. The class of admissible boundary displacements g is given by the space $H^1(\Omega) \cap L^\infty(\Omega)$. We say that the pair (u, K) is an admissible configuration of our system for g

$$K \in \mathcal{K}_m^f(\overline{\Omega})$$

and

$$u \in H^1(\Omega \setminus K) \qquad \text{with} \qquad u = g \text{ on } \partial_D\Omega \setminus K.$$

We will write $(u, K) \in \mathcal{A}(g)$. Note that the pair $(\nabla u, u)$ can be thought as an element of $L^2(\Omega; \mathbb{R}^3)$ since K has null Lebesgue measure.

The following compactness result will be used several times.

Lemma 2.1. *Let $g_n, g \in H^1(\Omega)$ be such that*

$$g_n \to g \qquad \textit{strongly in } H^1(\Omega).$$

Assume that $(u_n, K_n) \in \mathcal{A}(g_n)$ with

$$(\nabla u_n, u_n) \rightharpoonup (\Phi, u) \qquad \textit{weakly in } L^2(\Omega; \mathbb{R}^3),$$

and

$$K_n \to K \qquad \textit{in the Hausdorff metric.}$$

Then $(u, K) \in \mathcal{A}(g)$, and $\Phi = \nabla u$ on $\Omega \setminus K$.

Proof. Let $\varphi \in C_c^\infty(\Omega \setminus K)$. Then, for n large,

$$\varphi \in C_c^\infty(\Omega \setminus K_n).$$

We can thus write, for $i = 1, 2$,

$$\int_{\Omega \setminus K} \Phi_i \varphi \, dx = \lim_n \int_{\Omega \setminus K_n} \partial_i u_n \varphi \, dx = -\lim_n \int_{\Omega \setminus K_n} u_n \partial_i \varphi \, dx = -\int_{\Omega \setminus K} u \partial_i \varphi \, dx.$$

We deduce that $u \in H^1(\Omega \setminus K)$ with $\nabla u = \Phi$. Let us check that $(u, K) \in \mathcal{A}(g)$. Lest the result be trivial, it is not restrictive to assume that

$$\partial_D \Omega \setminus K \neq \emptyset.$$

Since $\partial_D\Omega$ is open in the relative topology, for every $x_0 \in \partial_D\Omega \setminus K$ we can find an open neighborhood $U \subset \mathbb{R}^2$ of x_0 such that $dist(U, K) > 0$ and $U \cap \Omega$ has a Lipschitz boundary in U given by $\partial_D\Omega \cap U$. Since $K_n \cap U = \emptyset$ for n large, we infer that $u_n \in H^1(\Omega \cap U)$ with

$$u_n \rightharpoonup u \qquad \text{weakly in } H^1(\Omega \cap U),$$

so $u = g$ on $\partial_D\Omega \cap U$. □

Remark 2.2. The choice of $H^1(\Omega)\cap L^\infty(\Omega)$ as the class of admissible displacements allows one to work in $H^1(\Omega \setminus K)$ when dealing with the variational constructions of Section 4. Without an L^∞-bound, the arguments can be adapted provided that we choose the displacements in $L^{1,2}(\Omega \setminus K)$, a Deny-Lions type space [12]. Such will not be the case in Section 5 below (see Remark 5.5).

Energies. We associate to an admissible configuration (u, K) the *elastic energy*

$$\|\nabla u\|^2 = \int_\Omega |\nabla u|^2 \, dx.$$

Here ∇u is viewed as an element of $L^2(\Omega; \mathbb{R}^2)$.

Assume that the system goes from the configuration (u, K) to the configuration (v, Γ). Then

$$\begin{cases} \Gamma \setminus K \text{ is the add-crack,} \\ K \setminus \Gamma \text{ is the healed zone.} \end{cases}$$

We assume the energy dissipated through such a process is

$$c_1 \mathcal{H}^1_D(\Gamma \setminus K) - c_2 \mathcal{H}^1_D(K \setminus \Gamma),$$

with $c_1, c_2 > 0$.

In the expression above and throughout the rest of the paper $\mathcal{H}^1_D$ stands for $\mathcal{H}^1\lfloor_{\Omega\cup\partial_D\Omega}$. This is so because no energy should be dissipated for the part of the crack that lies on the free boundary $\partial\Omega \setminus \partial_D\Omega$.

Summing up, the passage from (u, K) to (v, Γ) involves a change in energy of the form

$$\{\|\nabla v\|^2 - \|\nabla u\|^2\} + c_1 \mathcal{H}^1_D(\Gamma \setminus K) - c_2 \mathcal{H}^1_D(K \setminus \Gamma)\}.$$

Notice that the expression can be rewritten in the form

$$\mathcal{E}(v, \Gamma) - \mathcal{E}(u, K) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K),$$

where

$$\mathcal{E}(v, \Gamma) := \|\nabla v\|^2 + c_2 \mathcal{H}^1_D(\Gamma). \tag{2.2}$$

Indeed,

$$\mathcal{H}^1_D(K \setminus \Gamma) = \mathcal{H}^1_D(K) - \mathcal{H}^1_D(K \cap \Gamma) = \mathcal{H}^1_D(K) - (\mathcal{H}^1_D(\Gamma) - \mathcal{H}^1_D(\Gamma \setminus K))$$

so that

$$c_1 \mathcal{H}^1_D(\Gamma \setminus K) - c_2 \mathcal{H}^1_D(K \setminus \Gamma) = c_2(\mathcal{H}^1_D(\Gamma) - \mathcal{H}^1_D(K)) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K).$$

In view of this new expression, we will assume that

$$c_1 > c_2 > 0. \tag{2.3}$$

See Remark 2.4 below for the case $c_1 = c_2$.

Quasi-static evolutions. Let $T > 0$ and

$$g \in AC([0, T]; H^1(\Omega)), \qquad \|g(t)\|_\infty \le C, t \in [0, T]$$

be a given time dependent boundary displacement.

Given $t \mapsto K(t) \in \mathcal{K}^f_m(\overline{\Omega})$ we set, for $t \le T$,

$$Diss(t) := (c_1 - c_2) \sup \left\{ \sum_{i=0}^n \mathcal{H}^1_D(K(s_{i+1}) \setminus K(s_i)) \, : \, 0 = s_0 < s_1 < \cdots < s_{n+1} = t \right\}.$$

The definition of a quasi-static evolution is the following.

Definition 2.3 (**Quasi-static evolution).** *We say that* $\{t \mapsto (u(t), K(t)) \in \mathcal{A}(g(t)), t \in [0,T]\}$ *is a quasi-static evolution provided that for every* $t \in [0,T]$ *the following items hold true.*

(a) Global stability. *For every* $(v, \Gamma) \in \mathcal{A}(g(t))$

$$\mathcal{E}(u(t), K(t)) \leq \mathcal{E}(v, \Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K(t)), \tag{2.4}$$

where $\mathcal{E}$ *is defined in* (2.2).

(b) Energy balance. *We have*

$$\mathcal{E}(u(t), K(t)) + Diss(t) = \mathcal{E}(u(0), K(0)) + 2\int_0^t \int_\Omega \nabla u(\tau) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau.$$

Remark 2.4. In the spirit of our introductory remarks, we could modify the definition of $\mathcal{E}$ in (2.2) through addition of a term of the form $c\mathcal{H}^1_D(\Gamma)$ with $c \geq 0$, that is a stored surface energy term. The analysis performed in the rest of the paper and Theorems 4.1, 5.4 would remain unchanged in this enlarged setting.

If, in lieu of (2.3), $c_1 = c_2$, the quasi-static evolution is conservative and consists in a time-parametrized set of independent minimization problems: the term $(c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K(t))$ disappears in the Global Stability statement while $Diss(t)$ disappears in the Energy balance statement of Definition 2.3. The existence proofs leading to Theorems 4.1, 5.4 become straightforward.

3. Stability of the global minimality property

A crucial step in the proof of the existence of a quasi-static evolution concerns the stability of the global minimality property (2.4) under Hausdorff convergence for the cracks. The proof is based on a topological version of the jump transfer construction in [14]. Similar ideas have been put forth in [1] in the case of the fracture problem for a flexural linear plate.

Theorem 3.1 (**Stability of the global minimality property).** *Let* c, c' *be fixed positive constants. Let* $g_n, g \in H^1(\Omega)$ *be such that*

$$g_n \to g \qquad \text{strongly in } H^1(\Omega).$$

Assume that $(u_n, K_n) \in \mathcal{A}(g_n)$ *satisfy the following global stability condition: for every* $(v, \Gamma) \in \mathcal{A}(g_n)$,

$$\|\nabla u_n\|^2 + c\mathcal{H}^1_D(K_n) \leq \|\nabla v\|^2 + c\mathcal{H}^1_D(\Gamma) + c'\mathcal{H}^1_D(\Gamma \setminus K_n)$$

and assume further that

$$K_n \to K \qquad \text{in the Hausdorff metric}$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega; \mathbb{R}^2)$$

for some $(u, K) \in \mathcal{A}(g)$. *Then* (u, K) *is a globally stable configuration, that is that, for every* $(v, \Gamma) \in \mathcal{A}(g)$,

$$\|\nabla u\|^2 + c\mathcal{H}^1_D(K) \leq \|\nabla v\|^2 + c\mathcal{H}^1_D(\Gamma) + c'\mathcal{H}^1_D(\Gamma \setminus K).$$

In order to prove Theorem 3.1, we need two geometric results concerning the blow-up behavior of sets in the family $\mathcal{K}^f_1(\mathbb{R}^2)$ of compact connected sets in $\mathbb{R}^2$ with finite length.

Theorem 3.2. *Let* $K \in \mathcal{K}^f_1(\mathbb{R}^2)$. *The following items hold true.*

(a) K *is countably-*$\mathcal{H}^1$ *rectifiable with*

$$K = K_0 \cup \bigcup_{n=1}^{\infty} \gamma_n(I_n),$$

where $I_n \subseteq \mathbb{R}$ *is an open interval,* $\gamma_n : I_n \to \mathbb{R}^2$ *are Lipschitz curves and* $\mathcal{H}^1(K_0) = 0$. *Further, there exists* $N \subseteq K$ *with* $\mathcal{H}^1(N) = 0$ *such that, for every* $x \notin N$, K *admits an approximate tangent line* l_x *at* x *with normal* ν_x.

(b) *Take* $x \in K \setminus N$. *Then for* $r \to 0^+$

$$K_{x,r} := \frac{K - x}{r} \to l_x \qquad \text{locally in the Hausdorff metric.} \tag{3.1}$$

(c) *There exists $N_1 \subseteq K$ with $N \subseteq N_1$ and $\mathcal{H}^1(N_1) = 0$ such that the following property holds. Take $x \in K \setminus N_1$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for every $r < r_0$ the rectangles*

$$R^+_{\varepsilon,r} := Q_{\nu_x}(x,r) \cap \{y \in \mathbb{R}^2 \,:\, (y-x)\cdot\nu_x > \varepsilon r\}$$

and

$$R^-_{\varepsilon,r} := Q_{\nu_x}(x,r) \cap \{y \in \mathbb{R}^2 \,:\, (y-x)\cdot\nu_x < -\varepsilon r\}$$

belong to different connected components of $Q_{\nu_x}(x,r) \setminus K$.

Proof. The rectifiability property of point (a) is proved in [13, Lemma 3.13]. From the general theory of rectifiable sets, we know that K admits an approximate tangent line l_x at $\mathcal{H}^1$-a.e. $x \in K$; see [2, Theorem 2.83].

Now for point (b). Up to an isometry, we may assume $x = 0$ and that the approximate tangent line l is horizontal. Then, by the very definition of an approximate tangent line,

$$\mathcal{H}^1 \lfloor K_r \overset{*}{\rightharpoonup} \mathcal{H}^1 \lfloor l \qquad \text{locally weakly}^* \text{ in } \mathcal{M}_b(\mathbb{R}^2), \tag{3.2}$$

as $r \to 0^+$, where $K_r := \frac{1}{r}K$.

We claim that, for every $R > 0$,

$$K_r \cap \overline{Q}(0,R) \to l \cap \overline{Q}(0,R) \qquad \text{in the Hausdorff metric.} \tag{3.3}$$

Indeed, given any sequence $r_n \to 0$, the compactness of Hausdorff convergence and a diagonal argument imply the existence of a subsequence $(r_{n_h})_{h\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, $m \geq 1$

$$K_{r_{n_h}} \cap \overline{Q}(0,m) \to K_0^m \qquad \text{in the Hausdorff metric.}$$

It is readily checked that, for every $m \geq 1$,

$$K_0^m \subseteq K_0^{m+1} \qquad \text{and} \qquad K_0^m \cap \overline{Q}(0,m) = K_0^{m+1} \cap \overline{Q}(0,m). \tag{3.4}$$

Set $K_0 := \bigcup_{m=1}^\infty K_0^m$. We claim that

$$K_0 = l. \tag{3.5}$$

First, $K_0 \subseteq l$. Indeed, assume by contradiction that $\xi \in K_0 \setminus l$ with $\overline{B}_\eta(\xi) \cap l = \emptyset$ for some $\eta > 0$. Using the measure convergence (3.2), we obtain that

$$\mathcal{H}^1(K_{r_{n_h}} \cap \overline{B}_\eta(\xi)) \to 0. \tag{3.6}$$

But $K_{r_{n_h}}$ is connected by arcs (see [13, Lemma 3.12]), so that, taking $\xi_{n_h} \in K_{r_{n_h}}$ such that $\xi_{n_h} \to \xi$, ξ_{n_h} is connected to 0 through an arc contained in $K_{r_{n_h}}$ so, for h large enough,

$$\mathcal{H}^1(K_{r_{n_h}} \cap \overline{B}_{\eta/2}(\xi_{n_h})) \geq \eta/4.$$

Thus

$$\liminf_{h\to\infty} \mathcal{H}^1(K_{r_{n_h}} \cap \overline{B}_\eta(\xi)) \geq \liminf_{h\to\infty} \mathcal{H}^1(K_{r_{n_h}} \cap \overline{B}_{\eta/2}(\xi_{n_h})) \geq \eta/4,$$

in contradiction with (3.6).

Conversely, $l \subseteq K_0$. Indeed, assume by contradiction that $\xi \in l \setminus K_0$. Then there exists $\eta > 0$ such that $K_{r_{n_h}} \cap B_\eta(\xi) = \emptyset$ for h large, against (3.2).

In view of (3.4) and (3.5) we deduce that for $\varepsilon \to 0$ and for every $R > 0$

$$K_r \cap \overline{Q}(0,R) \to l \cap \overline{Q}(0,R) \qquad \text{in the Hausdorff metric,}$$

that is (3.3). This means that the local convergence of (3.1) holds true, and point (b) is proved.

Let us come to point (c).

Notice that we can reparametrize each Lipschitz curve γ_n by arc length. As a consequence, we may assume that for a.e. $t \in I_n$

$$\gamma_n \text{ is differentiable at } t \text{ with } |\gamma_n'(t)| = 1. \tag{3.7}$$

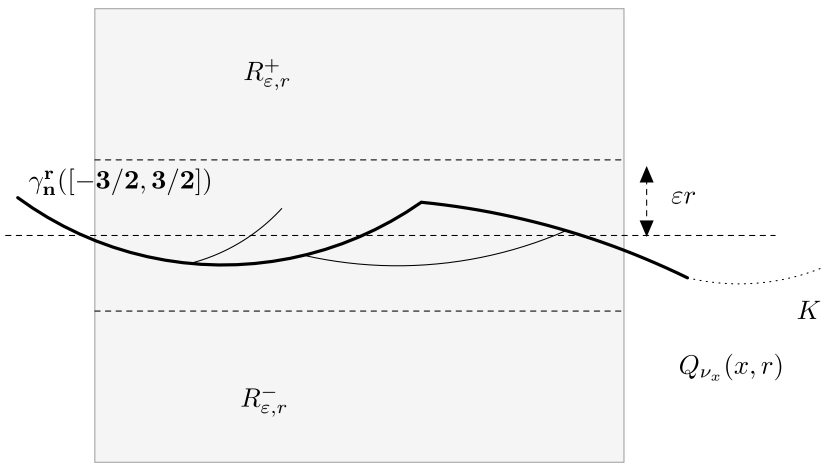

FIGURE 1. Illustration of item (c) in Theorem 3.2; the thick curve is $\gamma_n^r([-3/2,3/2])$.

From point (a), we deduce that there exists $N_1 \subseteq K$ with $\mathcal{H}^1(N_1)=0$, $N\subseteq N_1$ and such that if $x\in K\setminus N_1$, then $x=\gamma_n(t_0)$ for some n, with t_0 satisfying (3.7). It is not restrictive to assume that $x=0$ with a horizontal tangent line l, and that $t_0=0$. By differentiability, for $r\to 0^+$,

$$\gamma_n^r(s) := \frac{1}{r}\gamma_n(rs) \to \gamma_n'(0)s \qquad \text{locally uniformly in } s\in\mathbb{R}. \tag{3.8}$$

In view of (3.1), $\gamma_n'(0)$ is horizontal, and we can assume that $\gamma_n'(0)=(1,0)$.

Let $\varepsilon>0$. Because of (3.8) and since

$$\gamma_n^r(-3/2)\to(-3/2,0) \qquad \text{and} \qquad \gamma_n^r(3/2)\to(3/2,0),$$

we infer that, for r small enough, the (connected) arc $\gamma_n^r([-3/2,3/2])$ satisfies

$$\gamma_n^r([-3/2,3/2])\subseteq\{(x_1,x_2)\in\mathbb{R}^2 \,:\, |x_2|<\varepsilon\}$$

and that $Q(0,1)\setminus\gamma_n^r([-3/2,3/2])$ is disconnected. We deduce that the open rectangles

$$R_\varepsilon^+ := Q(0,1)\cap\{(x_1,x_2)\in\mathbb{R}^2 \,:\, x_2>\varepsilon\}$$

and

$$R_\varepsilon^- := Q(0,1)\cap\{(x_1,x_2)\in\mathbb{R}^2 \,:\, x_2<-\varepsilon\}$$

belong to different connected components of $Q(0,1)\setminus\frac{1}{r}K$. The conclusion of point (c) now follows by rescaling. □

The following result shows that the topological property of point (c) of Theorem 3.2 is essentially stable under Hausdorff convergence. We will need this property for our topological version of the jump transfer.

Proposition 3.3. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_1^f(\mathbb{R}^2)$ and $K\in\mathcal{K}_1^f(\mathbb{R}^2)$ be such that*

$$K_n\to K \qquad \text{in the Hausdorff metric.}$$

Let $N_1\subseteq K$ with $\mathcal{H}^1(N_1)=0$ be as in Theorem 3.2. For every $x\notin N_1$ and $\varepsilon>0$ we can find $r_0>0$ and $\nu_x\in\mathbb{R}^2$ with $|\nu_x|=1$ such that for every $r<r_0$ there exists $n_0\in\mathbb{N}$ and $(\hat K_n)_{n\in\mathbb{N}}$ sequence in $\mathcal{K}_1^f(\mathbb{R}^2)$ with

$$K_n\subseteq\hat K_n, \qquad \hat K_n\setminus K_n\subseteq Q_{\nu_x}(x,r), \qquad \mathcal{H}^1(\hat K_n\setminus K_n)\le 3\varepsilon r$$

such that for $n \geq n_0$ *the rectangles*

$$R^+_{\varepsilon,r} := Q_{\nu_x}(x,r) \cap \{y \in \mathbb{R}^2 \,:\, (y-x)\cdot\nu_x > \varepsilon r\} \tag{3.9}$$

$$R^-_{\varepsilon,r} := Q_{\nu_x}(x,r) \cap \{y \in \mathbb{R}^2 \,:\, (y-x)\cdot\nu_x < -\varepsilon r\} \tag{3.10}$$

belong to different connected components of $Q_{\nu_x}(x,r) \setminus \hat{K}_n$.

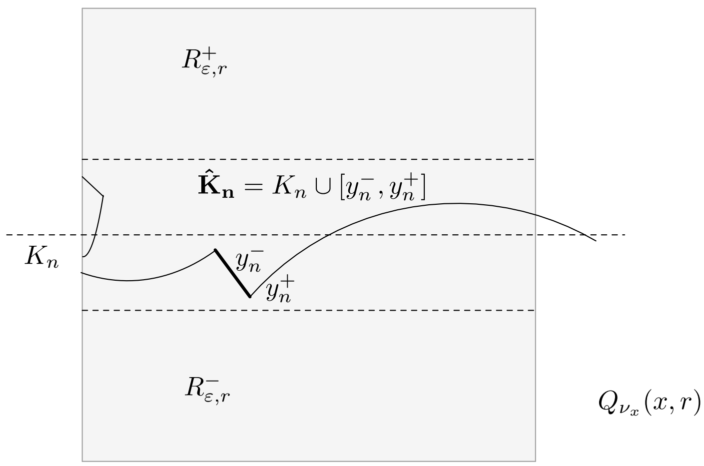

FIGURE 2. Construction of $\hat{K}_n$ in Proposition 3.3.

Proof. In view of Theorem 3.2, for every $x \notin N_1$ points (b) and (c) hold true.

Let us fix $x \notin N_1$ and $\varepsilon > 0$, and let $r_0 > 0$ and $\nu_x \in \mathbb{R}^2$ be associated to x according to point (c) of Theorem 3.2. Up to a roto-translation, we may assume

$$x = 0, \qquad \nu_x = (0,1), \qquad l_x = \{x = (x_1,x_2) \,:\, x_2 = 0\}.$$

Notice that, in view of item (b) in Theorem 3.2, we may also assume that

$$K \setminus \overline{Q}(0,r_0) \neq \emptyset.$$

Since $K_n \to K$ in the Hausdorff metric, from the corresponding property of K we deduce that there exists $n_0 > 0$ such that for every $n \geq n_0$

$$K_n \cap \overline{Q}(0,r) \subset \{(x_1,x_2) \in \mathbb{R}^2 \,:\, |x_2| < \varepsilon r\} \tag{3.11}$$

and

$$K_n \setminus \overline{Q}(0,r_0) \neq \emptyset. \tag{3.12}$$

Let $z_n \in K_n \setminus \overline{Q}(0,r_0)$.

Since K_n is connected by arcs, given $x \in K_n \cap Q(0,r)$, we can find an arc contained in K_n with extremes x and z_n. In view of (3.11), (3.12), this arc has to intersect either S_r^- or S_r^+, where $S_r^\pm$ are the vertical segments

$$S_r^\pm := \{\pm r\} \times [-\varepsilon r, \varepsilon r].$$

Modulo reparameterization, we thus infer that there exist (at least) one arc $\gamma^+_{x,r} : [0,1] \to \mathbb{R}^2$ or one $\gamma^-_{x,r} : [0,1] \to \mathbb{R}^2$ with image contained in $K_n \cap \overline{Q}(0,r)$ such that

$$\gamma^\pm_{x,r}(0) = x \qquad \text{and} \qquad \gamma^\pm_{x,r}(1) \in S_r^\pm.$$

Let us consider the intervals contained in $[-r,r]$ given by

$$J^-_{n,r} := \bigcup_{x \in K_n \cap \overline{Q}(0,r)} \pi_1(\gamma^-_{x,r}([0,1])) \qquad \text{and} \qquad J^+_{n,r} := \bigcup_{x \in K_n \cap \overline{Q}(0,r)} \pi_1(\gamma^+_{x,r}([0,1])),$$

obtained by projecting the curves constructed above onto the horizontal axis.

We claim that we can find $\alpha_n^\pm \in J_{n,r}^\pm$ such that

$$|\alpha_n^+ - \alpha_n^-| \to 0. \tag{3.13}$$

If this is the case, since by definition there exists

$$y_n^\pm = (\alpha_n^\pm, \beta_n^\pm) \in K_n \cap \overline{Q(0,r)},$$

we then define $\hat{K}_n$ to be (see Figure 2)

$$\hat{K}_n = K_n \cup [y_n^-, y_n^+],$$

where $[y_n^-, y_n^+]$ is the segment joining y_n^- and y_n^+. In view of (3.11), we have

$$\limsup_{n\to\infty} \mathcal{H}^1([y_n^-, y_n^+]) \leq 2r\varepsilon.$$

Finally, since

$$\gamma^-_{y_n^-,r}([0,1]) \cup [y_n^-, y_n^+] \cup \gamma^+_{y_n^+,r}([0,1]) \subset \hat{K}_n,$$

we deduce that $\hat{K}_n \in \mathcal{K}_1^f(\mathbb{R}^2)$ satisfy the conclusion of the Theorem.

Let us prove claim (3.13). If the relation is not satisfied, we get for n large

$$\inf J_{n,r}^+ - \sup J_{n,r}^- \geq \eta > 0.$$

Since $K_n \to K$ in the Hausdorff metric, we would infer that the projection of $K \cap \overline{Q}(0,r)$ onto the horizontal axis is composed of two distinct intervals contained in $[-r, r]$, against the fact that K disconnects $\overline{Q}(0,r)$. The proof is now concluded. □

Remark 3.4. Let $\Omega \subseteq \mathbb{R}^2$ be open bounded and with a Lipschitz boundary. Assume that the sets K_n, K of Proposition 3.3 are such that $K_n, K \subseteq \overline{\Omega}$. Notice that, for $\mathcal{H}^1$-a.e. $x \in K \cap \partial\Omega$, the tangent lines to K and $\partial\Omega$ at the point x coincide, so that the topological blow-up properties of Theorem 3.2 at the point x hold simultanously for K and $\partial\Omega$. Consequently, the proof of Proposition 3.3 shows that $\hat{K}_n$ can be chosen such that in addition $\hat{K}_n \subseteq \overline{\Omega}$.

We can now prove Theorem 3.1.

Proof of Theorem 3.1. The global stability we need to prove can be rewritten in the form

$$\|\nabla u\|^2 + c\mathcal{H}^1_D(K \setminus \Gamma) \leq \|\nabla v\|^2 + (c + c')\mathcal{H}^1_D(\Gamma \setminus K),$$

for every $(v, \Gamma) \in \mathcal{A}(g)$ (see the computations in Section 2).

We divide the proof into two steps.

Step 1. Let us assume that $K \in \mathcal{K}_1^f(\overline{\Omega})$. Thanks to [11, Lemma 3.6], there exists $H_n \in \mathcal{K}_1^f(\overline{\Omega})$ with $K_n \subseteq H_n$,

$$\mathcal{H}^1(H_n \setminus K_n) \to 0 \quad \text{and} \quad H_n \to K \quad \text{in the Hausdorff metric.} \tag{3.14}$$

We need to introduce the connected sets H_n because it might be so that, although K is connected, the K_n might not be since they are only restricted to be elements of $\mathcal{K}_m^f(\overline{\Omega})$.

Let $V \subseteq \mathbb{R}^2$ be open with $\Gamma \subseteq V$. Let then $U \subset V$ be open with $\bar{U} \subset V$ and $\Gamma \cap K \subset U$. Let also $\varepsilon > 0$ be fixed.

Note that, for $\mathcal{H}^1$-a.e. $x \in \Gamma \cap K$, the tangent lines to Γ and K at the point x coincide. We can thus find $N \subseteq \Gamma \cap K$ with $\mathcal{H}^1(N) = 0$ and such that for $x \in (\Gamma \cap K) \setminus N$ the conclusions of point (c) in Theorem 3.2 hold true with respect to both K and Γ simultaneously.

For $x \in (\Gamma \cap K) \setminus N$, let $r_0(x) > 0$ and $\nu_x \in \mathbb{R}^2$ be given by Proposition 3.3. We may assume in addition that

$$Q_{\nu_x}(x, r_0(x)) \subset U$$

and also, thanks to e.g. [2, Theorem 2.83 (i)], that, for every $r < r_0(x)$,

$$(1-\varepsilon)r \leq \mathcal{H}^1(Q_{\nu_x}(x,r) \cap (K \cap \Gamma)) \leq (1+\varepsilon)r. \tag{3.15}$$

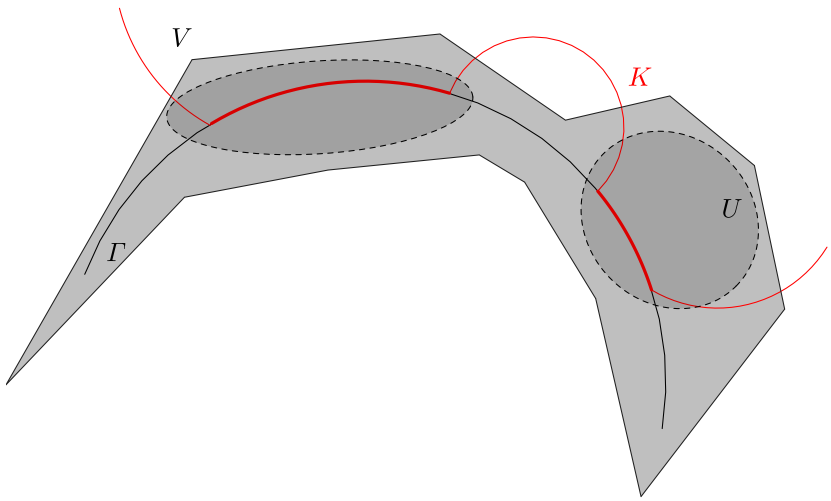

FIGURE 3. Setting the geometry for the proof of Theorem 3.1.

By the Vitali-Besicovitch lemma (see e.g. [2, Theorem 2.19]) we can find a finite number of disjoint such squares $\{Q_{\nu_j(x_j,r_j)}\}_{j=1,\dots,m}$ with $x_j \in K \cap \Gamma$, $\nu_j := \nu_{x_j}$, $r_j < r_0(x_j)$, such that

$$\mathcal{H}^1\left((K \cap \Gamma) \setminus \bigcup_{j=1}^m Q_{\nu_j}(x_j, r_j)\right) < \varepsilon. \tag{3.16}$$

It is no restriction to assume that either $Q_{\nu_j}(x_j, r_j) \subset\subset \Omega$ or $x_j \in \partial\Omega$, with $\partial\Omega \cap Q_{\nu_j}(x_j, r_j)$ given by the graph of a Lipschitz function with respect to a reference frame with ν_j as vertical direction.

We modify H_n in each square according to Proposition 3.3 and Remark 3.4 and find $\hat{H}_n \in \mathcal{K}_1^f(\overline{\Omega})$ with $H_n \subseteq \hat{H}_n$, such that for n large

$$\hat{H}_n = H_n \qquad \text{outside } \bigcup_{j=1}^m Q_{\nu_j}(x_j, r_j),$$

and

$$\mathcal{H}^1(\hat{H}_n \setminus H_n) \le 3\varepsilon \sum_{i=1}^m r_i. \tag{3.17}$$

Moreover, we can assume that the rectangles $R_j^\pm$ associated to $Q_{\nu_j}(x_j, r_j)$ according to (3.9) and (3.10) belong to different connected components $A_{j,n}^\pm$ of $Q_{\nu_j}(x_j, r_j) \setminus \hat{H}_n$. Let us denote by

$$v_j^\pm \in H^1(Q_{\nu_j}(x_j, r_j)) \tag{3.18}$$

the extension of $v\lfloor R_j^\pm$ obtained through a reflection across the line $l_{x_j} \pm \varepsilon r \nu_j$: notice that the Sobolev regularity of $v_j^\pm$ is ensured because, by construction,

$$\Gamma \cap Q_{\nu_j}(x_j, r_j) \subseteq \{x \in \mathbb{R}^2 \,:\, |(x - x_j) \cdot \nu_j| < \varepsilon r\}.$$

Let us set

$$\Gamma_n := \left(\Gamma \setminus \bigcup_{j=1}^m Q_{\nu_j}(x_j, r_j)\right) \cup \bigcup_{j=1}^m \Gamma_n^j$$

(3.19) with

$$\Gamma_n^j := \left(\hat{H}_n \cap \overline{Q}_{\nu_j}(x_j, r_j)\right) \cup \left(\partial Q_{\nu_j}(x_j, r_j) \ \cap \{|(y - x_j)\cdot \nu_j| \le \varepsilon r_j\} \cap \overline{\Omega}\right).$$

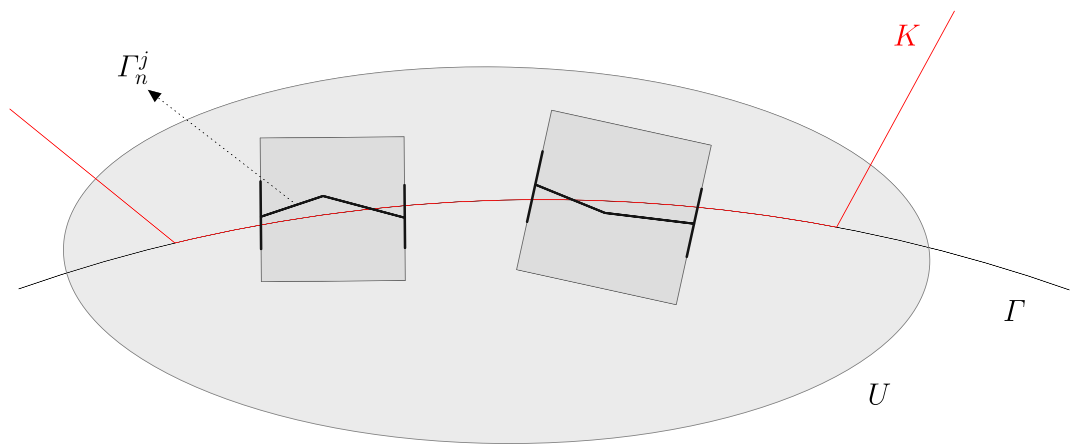

FIGURE 4. The sets Γ_n defined in (3.19).

Notice that $\Gamma_n \in \mathcal{K}_1^f(\overline{\Omega})$. Moreover thanks to (3.15), (3.16), (3.17),

$$\begin{aligned}(3.20)\quad \mathcal{H}^1_D(\Gamma_n \setminus K_n) &\le \mathcal{H}^1_D(\Gamma_n \setminus \hat{H}_n) + \mathcal{H}^1(\hat{H}_n \setminus H_n) + \mathcal{H}^1(H_n \setminus K_n) \\ &\le \mathcal{H}^1_D\left(\Gamma \setminus \bigcup_{j=1}^m Q_{\nu_j}(x_j, r_j)\right) + 7\varepsilon \sum_{j=1}^m r_j + \mathcal{H}^1(H_n \setminus K_n) \\ &\le \mathcal{H}^1_D(\Gamma \setminus K) + \varepsilon + 7\varepsilon \frac{1}{1-\varepsilon}\mathcal{H}^1(\Gamma) + \mathcal{H}^1(H_n \setminus K_n),\end{aligned}$$

and, since $\Gamma_n \subseteq V$,

$$\mathcal{H}^1_D(K_n \setminus \Gamma_n) \ge \mathcal{H}^1_D(K_n \setminus \overline{V}).$$

Let us define v_n as follows:

(a) $v_n = v$ outside $\bigcup_{j=1}^m Q_{\nu_j}(x_j, r_j)$;

(b) $v_n := \begin{cases} v_j^+ & \text{in } A^+_{j,n} \\ v_j^- & \text{else} \end{cases}$ in each cube $Q_{\nu_j}(x_j, r_j) \subset\subset \Omega$ where the functions $v_j^\pm$ were defined in (3.18);

(c) $v_n := \begin{cases} v_j^+ & \text{in } A^+_{j,n} \\ g & \text{otherwise.} \end{cases}$ in each boundary cube $Q_{\nu_j}(x_j, r_j)$ (that is those with $x_j \in \partial\Omega$).

Remark that, by construction, $(v_n, \varGamma_n) \in \mathcal{A}(g)$. Moreover,

$$\|\nabla v_n\|^2 \le \|\nabla v\|^2 + 2\sum_{j=1}^m \int_{Q_{\nu_j}(x_j,r_j)\cap\overline{\Omega}} |\nabla v|^2\,dx + \sum_{j=1}^m \int_{Q_{\nu_j}(x_j,r_j)\cap\overline{\Omega}} |\nabla g|^2\,dx \le \|\nabla v\|^2 + 2\int_{U\cap\overline{\Omega}} |\nabla v|^2\,dx + \int_{U\cap\overline{\Omega}} |\nabla g|^2\,dx. \tag{3.21}$$

Let us compare (u_n, K_n) to $(v_n - g + g_n, \varGamma_n) \in \mathcal{A}(g_n)$. Since

$$\|\nabla u_n\|^2 + c\mathcal{H}^1_D(K_n \setminus \varGamma_n) \le \|\nabla v_n - \nabla g + \nabla g_n\|^2 + (c+c')\mathcal{H}^1_D(\varGamma_n \setminus K_n) = \|\nabla v_n\|^2 + (c+c')\mathcal{H}^1_D(\varGamma_n \setminus K_n) + e_n,$$

where

$$|e_n| \le \|\nabla v_n\|\|\nabla g_n - \nabla g\| + \|\nabla g_n - \nabla g\|^2 \to 0,$$

we infer in view of (3.20)-(3.21) that

$$\|\nabla u_n\|^2 + c\mathcal{H}^1_D(K_n \setminus \bar V) \le \|\nabla v\|^2 + 2\int_{U\cap\overline{\Omega}} |\nabla v|^2\,dx + \int_{U\cap\overline{\Omega}} |\nabla g|^2\,dx + e_n + (c+c')\left[\mathcal{H}^1_D(\varGamma \setminus K) + \varepsilon + 7\varepsilon\frac{1}{1-\varepsilon}\mathcal{H}^1(\varGamma) + \mathcal{H}^1(H_n \setminus K_n)\right].$$

Passing to the limit, we obtain, thanks to Gołąb's Theorem and to (3.14),

$$\|\nabla u\|^2 + c\mathcal{H}^1_D(K \setminus \bar V) \le \|\nabla v\|^2 + (c+c')\mathcal{H}^1_D(\varGamma \setminus K) + (c+c')\left[\varepsilon + \frac{7\varepsilon}{1-\varepsilon}\mathcal{H}^1(\varGamma)\right] + 2\int_{U\cap\overline{\Omega}} |\nabla v|^2\,dx + \int_{U\cap\overline{\Omega}} |\nabla g|^2\,dx.$$

Since V, U and ε are arbitrary, we conclude that

$$\|\nabla u\|^2 + c\mathcal{H}^1_D(K \setminus \varGamma) \le \|\nabla v\|^2 + (c+c')\mathcal{H}^1_D(\varGamma \setminus K),$$

so that the minimality condition follows.

Step 2. Let us consider the general case $K \in \mathcal{K}^f_m(\overline{\Omega})$. If $K^1, \dots K^p$ with $p \le m$ are the connected components of K, thanks to [11, Lemma 3.6] we can find $H_n \in \mathcal{K}^f_m(\overline{\Omega})$ with exactly p connected components $H^1_n, \dots, H^p_n$ such that $K_n \subseteq H_n$,

$$H^j_n \to K^j \qquad \text{in the Hausdorff metric} \tag{3.22}$$

and

$$\mathcal{H}^1(H_n \setminus K_n) \to 0.$$

Since the K^j are compact and disjoint, and

$$\varGamma \cap K = \bigcup_{j=1}^p (\varGamma \cap K^j)$$

we can operate on each $\varGamma \cap K^j$ as in Step 1 using the approximation (3.22) and localizing on disjoint neighborhoods U_j of $\varGamma \cap K^j$. The modifications of $\varGamma$ and v which take place on the family of squares contained in U_j are independent from those taking place in the squares contained in U_i with $i \neq j$, so that we can glue them together to get an approximating configuration $(v_n - g + g_n, \varGamma_n) \in \mathcal{A}(g_n)$ and deduce as in Step 1 the global minimality of (u, K). $\square$

4. Existence of a quasi-static evolution

In this section we derive the main result of the paper.

Theorem 4.1 (Existence of a quasi-static evolution). *Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded, with Lipschitz boundary, and let $\partial_D \Omega \subseteq \partial\Omega$ be open in the relative topology. Assume* (2.3) *and let $g \in AC([0,T]; H^1(\Omega))$ be such that*

$$\sup_{t\in[0,T]} \|g(t)\|_\infty < +\infty. \tag{4.1}$$

Let finally $(u_0, K_0) \in \mathcal{A}(g(0))$ be a globally stable configuration (i.e., satisfying property (2.4)*).*

Then there exists a quasi-static evolution $\{t \mapsto (u(t), K(t)) : t \in [0,T]\}$ in the sense of Definition 2.3 such that $(u(0), K(0)) = (u_0, K_0)$.

Remark 4.2. The existence of at least one globally stable initial configuration $(u_0, K_0) \in \mathcal{A}(g(0))$ is straightforward. It is enough to minimize $\mathcal{E}(v,\Gamma)$ over $\mathcal{A}(g(0))$ following *e.g.* an argument identical to that expounded in the proof of Lemma 4.3 below.

As usual, the existence of a quasi-static evolution is obtained by time discretization, establishing the existence of a discrete in time evolution through the direct method of the Calculus of Variations, then studying its limit as the time-step discretization parameter vanishes.

Let $\delta > 0$ be given, and let

$$0 = t_0^\delta < t_1^\delta < \cdots < t_{N_\delta}^\delta = T$$

be a subdivision of the time interval $[0,T]$ with

$$\max_{i=0,\dots,N_\delta-1} \left(t_{i+1}^\delta - t_i^\delta\right) < \delta.$$

We set

$$g_i^\delta := g(t_i^\delta) \qquad \text{and} \qquad (u_0^\delta, K_0^\delta) := (u_0, K_0).$$

The following lemma deals with the existence of incremental configurations.

Lemma 4.3 (Incremental configurations). *Assume* (2.3) *and* (4.1). *Then for $i = 1,\dots,N_\delta$ there exists $(u_i^\delta, K_i^\delta) \in \mathcal{A}(g_i^\delta)$ with $\|u_i^\delta\|_\infty \le \|g_i^\delta\|_\infty$, $(u_0^\delta, K_0^\delta) = (u_0, K_0)$ such that*

$$(u_i^\delta, K_i^\delta) \in Argmin\{\mathcal{E}(v,\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K_{i-1}^\delta) : (v,\Gamma) \in \mathcal{A}(g_i^\delta)\}.$$

Proof. We proceed by induction, assuming that $(u_{i-1}^\delta, K_{i-1}^\delta)$ has been constructed, and showing the existence of (u_i^δ, K_i^δ).

Set

$$\mathcal{F}_i^\delta(u,\Gamma) := \mathcal{E}(v,\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K_{i-1}^\delta).$$

and let $\{(v_n,\Gamma_n)\}_{n\in\mathbb{N}}$ be a minimizing sequence for $\mathcal{F}_i^\delta$ on $\mathcal{A}(g_i^\delta)$, that is

$$I_i^\delta := \inf_{\mathcal{A}(g_i^\delta)} \mathcal{F}_i^\delta \le \mathcal{F}_i^\delta(v_n,\Gamma_n) \le I_i^\delta + 1/n.$$

By truncation, it is not restrictive to assume

$$\|v_n\|_\infty \le \|g_i^\delta\|_\infty.$$

Comparing with the admissible configuration $(g_i^\delta, \emptyset)$ we get

$$\mathcal{E}(v_n,\Gamma_n) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma_n \setminus K_{i-1}^\delta) \le \|\nabla g_i^\delta\|^2.$$

As a consequence, up to a subsequence we may assume

$$(\nabla v_n, v_n) \rightharpoonup (\Phi, v) \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^3)$$

and

$$\Gamma_n \to \Gamma \quad \text{in the Hausdorff metric.}$$

Thanks to Gołąb's Theorem 1.2, we infer $\Gamma \in \mathcal{K}_f^m(\overline{\Omega})$, and, by Lemma 2.1, we deduce that $(v,\Gamma) \in \mathcal{A}(g_i^\delta)$, with $\Phi = \nabla v$ on $\Omega \setminus \Gamma$. In particular

$$\nabla v_n \rightharpoonup \nabla v, \text{ weakly in } L^2(\Omega;\mathbb{R}^2).$$

Moreover, in view of Lemma 1.4

$$\mathcal{H}^1_D(\Gamma) \le \liminf_n \mathcal{H}^1_D(\Gamma_n) \qquad \text{and} \qquad \mathcal{H}^1_D(\Gamma \setminus K^\delta_{i-1}) \le \liminf_n \mathcal{H}^1_D(\Gamma_n \setminus K^\delta_{i-1}),$$

so that

$$\mathcal{F}^\delta_i(v,\Gamma) = I^\delta_i.$$

The thesis follows by setting $(u^\delta_i, K^\delta_i) := (v,\Gamma)$.

□

For $t^\delta_i \le t < t^\delta_{i+1}$, $i = 0,\dots,N_\delta$, we set

$$u^\delta(t) := u^\delta_i,\ g^\delta(t) := g^\delta_i \quad \text{and} \quad K^\delta(t) := K^\delta_i. \tag{4.2}$$

We denote by $i^\delta(t)$ the index such that $t^\delta_{i^\delta(t)} \le t < t^\delta_{i^\delta(t)+1}$.

The following properties follow directly from the construction of the incremental configurations.

Lemma 4.4. *For every $t \in [0,T]$ the following items hold true:*

(a) $(u^\delta(0), K^\delta(0)) = (u_0, K_0)$.
(b) *The pair $(u^\delta(t), K^\delta(t)) \in \mathcal{A}(g^\delta(t))$ satisfies the global stability condition* (2.4).
(c) *Setting*

$$Diss^\delta(t) := (c_1 - c_2) \sum_{j=1}^{i^\delta(t)} \mathcal{H}^1_D(K^\delta_j \setminus K^\delta_{j-1}), \tag{4.3}$$

we have the energy inequality

$$\mathcal{E}(u^\delta(t), K^\delta(t)) + Diss^\delta(t) \le \mathcal{E}(u_0, K_0) + 2\int_0^{t^\delta_i} \int_\Omega \nabla u^\delta(\tau) \cdot \nabla \dot g(\tau)\, dx\, d\tau + e(\delta) \tag{4.4}$$

where $e(\delta) \to 0$ as $\delta \to 0$.

Proof. Point (a) follows since $(u^\delta(0), K^\delta(0)) = (u^\delta_0, K^\delta_0) = (u_0, K_0)$.

On to point (b). By construction, for every $i = 1,\dots,N_\delta$ and $(v,\Gamma) \in \mathcal{A}(g^\delta_i)$,

$$\mathcal{E}(u^\delta_i, K^\delta_i) + (c_1 - c_2)\mathcal{H}^1_D(K^\delta_i \setminus K^\delta_{i-1}) \le \mathcal{E}(v,\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K^\delta_{i-1}).$$

Since

$$\mathcal{H}^1_D(\Gamma \setminus K^\delta_{i-1}) \le \mathcal{H}^1_D(\Gamma \setminus K^\delta_i) + \mathcal{H}^1_D(K^\delta_i \setminus K^\delta_{i-1}),$$

we deduce

$$\mathcal{E}(u^\delta_i, K^\delta_i) \le \mathcal{E}(v,\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K^\delta_i)$$

from which the global stability condition (2.4) follows.

Let us come to point (c). In view of Lemma 4.3 we may write, for every $i = 1,\dots,N_\delta$,

$$\begin{aligned}\mathcal{E}(u^\delta_i, K^\delta_i) + (c_1 - c_2)\mathcal{H}^1_D(K^\delta_i \setminus K^\delta_{i-1}) &\le \mathcal{E}(u^\delta_{i-1} + g^\delta_i - g^\delta_{i-1}, K^\delta_{i-1}) \\ &\le \mathcal{E}(u^\delta_{i-1}, K^\delta_{i-1}) + 2\int_{t^\delta_{i-1}}^{t^\delta_i} \int_\Omega \nabla u^\delta(\tau) \cdot \nabla \dot g(\tau)\, dx\, d\tau + (t^\delta_i - t^\delta_{i-1}) \int_{t^\delta_{i-1}}^{t^\delta_i} \|\nabla \dot g(\tau)\|^2\, d\tau.\end{aligned}$$

Iterating this estimate we obtain for every $t \in [0,T]$

$$\mathcal{E}(u^\delta(t), K^\delta(t)) + (c_1 - c_2) \sum_{j=1}^{i^\delta(t)} \mathcal{H}^1_D(K^\delta_j \setminus K^\delta_{j-1}) \le \mathcal{E}(u_0, K_0) + 2\int_0^{t^\delta_i} \int_\Omega \nabla u^\delta(\tau) \cdot \nabla \dot g(\tau)\, dx\, d\tau + e(\delta)$$

with $e(\delta) \to 0$ as $\delta \to 0$, which is precisely (4.4). □

In order to pass to the continuous in time evolution, we need the following bounds.

Lemma 4.5 (**A priori bounds**). *Let* $\{t \mapsto (u^\delta(t), K^\delta(t)) \,:\, t \in [0,T]\}$ *be the discrete-in-time evolution given by* (4.2). *There exists* $C > 0$ *independent of* δ *such that, for every* $t \in [0,T]$,

$$\|\nabla u^\delta(t)\| + \|u^\delta(t)\|_\infty + \mathcal{H}^1_D(K^\delta(t)) + z^\delta(t) \le C, \tag{4.5}$$

where

$$z^\delta(t) := \sum_{j=1}^{i^\delta(t)} \mathcal{H}^1_D(K^\delta_j \Delta K^\delta_{j-1}). \tag{4.6}$$

Proof. Since by construction and global minimality

$$\|\nabla u^\delta(t)\| \le \|\nabla g^\delta(t)\| \qquad \text{and} \qquad \|u^\delta(t)\|_\infty \le \|g^\delta(t)\|_\infty, \tag{4.7}$$

we deduce from (4.4) that

$$\mathcal{H}^1_D(K^\delta(t)) + \sum_{j=1}^{i^\delta(t)} \mathcal{H}^1_D(K^\delta_j \setminus K^\delta_{j-1}) \le C_1 \tag{4.8}$$

for some $C_1 > 0$. Since $\mathcal{H}^1_D(B \setminus A) - \mathcal{H}^1_D(A \setminus B) = \mathcal{H}^1_D(B) - \mathcal{H}^1_D(A)$,

$$\sum_{j=1}^{i^\delta(t)} \mathcal{H}^1_D(K^\delta_j \setminus K^\delta_{j-1}) - \sum_{j=1}^{i^\delta(t)} \mathcal{H}^1_D(K^\delta_{j-1} \setminus K^\delta_j) = \mathcal{H}^1_D(K^\delta(t)) - \mathcal{H}^1_D(K_0),$$

we also obtain from (4.8) that

$$\sum_{j=1}^{i^\delta(t)} \mathcal{H}^1_D(K^\delta_{j-1} \setminus K^\delta_j) \le C_2 \tag{4.9}$$

for some $C_2 > 0$. The conclusion follows gathering (4.7), (4.8) and (4.9). □

A crucial step in the $\delta \searrow 0$-analysis is the following

Proposition 4.6 (**Compactness of the cracks**). *There exist a sequence* $\delta_n \to 0$ *and a map* $\{t \mapsto K(t) \in \mathcal{K}^f_m(\overline{\Omega}) \,:\, t \in [0,T]\}$ *such that, if*

$$K_n(t) := K^{\delta_n}(t),\ t \in [0,T],$$

then, for every $t \in [0,T]$, *any limit point* H *of* $(K_n(t))_{n\in\mathbb{N}}$ *in the Hausdorff metric is such that*

$$\mathcal{H}^1_D(H \Delta K(t)) = 0.$$

Proof. Let $\delta_n \to 0$ be such that

$$z_n := z^{\delta_n} \to z \qquad \text{pointwise on } [0,T],$$

where z^δ is given in (4.6) and $z : [0,T] \to \mathbb{R}$ is a suitable increasing function. The existence of $(\delta_n)_{n\in\mathbb{N}}$ is a consequence of the bound (4.5) and of Helly's theorem.

Let $D \subseteq [0,T]$ be a countable and dense set containing 0 and the discontinuity points of the function z. Up to a further subsequence (that we will not relabel), we may assume, in view of the compactness of Hausdorff metric and of the bound (4.5), that for every $t \in D$ there exists $K(t) \in \mathcal{K}^f_m(\overline{\Omega})$ such that

$$K_n(t) \to K(t) \qquad \text{in the Hausdorff metric.}$$

Let now $s \notin D$, and let H be a limit point of the sequence $(K_n(s))_{n\in\mathbb{N}}$, that is

$$K_{n_k}(s) \to H \qquad \text{in the Hausdorff metric}$$

for a suitable subsequence $(n_k)_{k\in\mathbb{N}}$. By the definition of z_n, for every $t < s$ and $t \in D$,

$$\mathcal{H}^1_D(K_{n_k}(s) \Delta K_{n_k}(t)) \le z_{n_k}(s) - z_{n_k}(t).$$

Sending $k \to +\infty$ and using Lemma 1.4 we obtain

$$\mathcal{H}^1_D(H \Delta K(t)) \le z(s) - z(t).$$

Let now $t_k \nearrow s$ with $t_k \in D$ and such that

$$K(t_k) \to \tilde{K}(s) \qquad \text{in the Hausdorff metric.}$$

Recalling that s is a continuity point for z, we infer (using again Lemma 1.4) that

$$\mathcal{H}^1_D(H \Delta \tilde{K}(s)) = 0. \tag{4.10}$$

Since $(t_k)_{k\in\mathbb{N}}$ is arbitrary, we deduce that any limit point $\tilde{K}(s)$ of the family $\{K(t) : t \in D\}$ for $t \to s^-$ satisfies (4.10). The proof now follows by choosing $K(s)$ as one on these limit points. □

Remark 4.7. Let $H, K \in \mathcal{K}^f_m(\overline{\Omega})$ be such that

$$\mathcal{H}^1_D(K \Delta H) = 0. \tag{4.11}$$

Then,

(i) K and H differ by at most m points on $\Omega \cup \partial_D \Omega$;
(ii) If $(v, H) \in \mathcal{A}(g)$, then also $(v, K) \in \mathcal{A}(g)$.

Indeed let H_j be a connected component of H which contains a point x not in K. Since H_j is connected by arcs, (4.11) implies that H_j reduces to the point x which proves point (i).

As far as point (ii) is concerned, we know that $(\nabla v, v)$ can be interpreted as an element of $L^2(\Omega; \mathbb{R}^3)$. Let us first check that $v \in W^{1,2}(\Omega \setminus K)$ with gradient on $\Omega \setminus K$ given by ∇v. We can proceed locally near every point $x \in \Omega \setminus K$.

(a) If $x \notin H$, since $u \in W^{1,2}(\Omega \setminus H)$ we deduce $u \in W^{1,2}(B(x,r))$ for some $r > 0$ small enough, with gradient given by ∇v.
(b) If $x \in H$, then, according to point (i), the connected component H_j of H that contains x reduces to the point x. From $u \in W^{1,2}(\Omega \setminus H)$ we then deduce that for some $r > 0$ small enough

$$u \in W^{1,2}(B(x,r) \setminus H_j) = W^{1,2}(B(x,r) \setminus \{x\}) = W^{1,2}(B(x,r)),$$

with gradient given by ∇v.

Concerning the boundary condition, since $u = g$ on $\partial_D \Omega \setminus H$ in the sense of traces, (4.11) then entails that the equality also holds true on $\partial_D \Omega \setminus K$. We thus conclude that $(u, K) \in \mathcal{A}(g)$.

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\delta_n \to 0$ and $\{t \mapsto K(t) : t \in [0,T]\}$ be given by Proposition 4.6. Set

$$(u_n(t), K_n(t)) := (u^{\delta_n}(t), K^{\delta_n}(t)) \qquad \text{and} \qquad Diss_n(t) := Diss^{\delta_n}(t).$$

Up to a further subsequence, the *a priori* bounds of Lemma 4.5, imply that

$$Diss_n \to D \qquad \text{pointwise on } [0,T] \tag{4.12}$$

for some increasing function $D : [0,T] \to [0, +\infty[$.

For every $t \in [0,T]$ take $u(t) \in H^1(\Omega \setminus K(t))$ to be a minimizer of

$$\min_{(v,K(t)) \in \mathcal{A}(g(t))} \|\nabla v\|^2.$$

By strict convexity, $\nabla u(t)$ is uniquely determined by $K(t)$ and $g(t)$, while $u(t)$ is well defined up to a constant on the connected components of $\Omega \setminus K(t)$ which do not touch $\partial_D \Omega$.

We now prove that

$$\{t \mapsto (u(t), K(t)) : t \in [0,T]\}$$

is a quasi-static evolution for the boundary displacement g such that $(u(0), K(0)) = (u_0, K_0)$ according to Definition 2.3.

Step 1: Global stability. Let us check that, for every $t \in [0,T]$, the pair $(u(t), K(t))$ satisfies the global stability condition (2.4), which reads

$$\|\nabla u(t)\|^2 + c_2 \mathcal{H}^1_D(K(t)) \le \|\nabla v\|^2 + c_2 \mathcal{H}^1_D(\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K(t)). \tag{4.13}$$

In view of the bound (4.5), by Lemma 2.1 and by the compactness of the Hausdorff convergence, we may assume that, up to a subsequence,

$$K_n(t) \to H \in \mathcal{K}^f_m(\overline{\Omega}) \qquad \text{in the Hausdorff metric}$$

and

$$(\nabla u_n(t), u_n(t)) \rightharpoonup (\nabla u, u) \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^3)$$

for some $(u, H) \in \mathcal{A}(g(t))$.

From item (b) in Lemma 4.4 and Theorem 3.1 we infer that (u, H) satisfies the global stability condition

$$\|\nabla u\|^2 + c_2\mathcal{H}^1_D(H) \le \|\nabla v\|^2 + c_2\mathcal{H}^1_D(\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus H) \tag{4.14}$$

for every $(v, \Gamma) \in \mathcal{A}(g(t))$. Note that, by Proposition 4.6,

$$\mathcal{H}^1_D(H \Delta K(t)) = 0.$$

Then Remark 4.7 implies that $(u, K(t)) \in \mathcal{A}(g(t))$, so that the minimality property (4.14) becomes

$$\|\nabla u\|^2 + c_2\mathcal{H}^1_D(K(t)) \le \|\nabla v\|^2 + c_2\mathcal{H}^1_D(\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K(t))$$

for every $(v, \Gamma) \in \mathcal{A}(g(t))$. Comparing with the admissible configuration $(u(t), K(t))$ yields

$$\|\nabla u\|^2 \le \|\nabla u(t)\|^2,$$

so that, by the very definition of $u(t)$, we get $\nabla u(t) = \nabla u$ and conclude that (4.13) is satisfied.

From the arguments above, passing to subsequences is not necessary and we infer that

$$\nabla u_n(t) \rightharpoonup \nabla u(t) \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^2) \tag{4.15}$$

for every $t \in [0, T]$.

Step 2: Energy balance. Let us first prove that, for every $t \in [0, T]$,

$$Diss(t) \le D(t). \tag{4.16}$$

Indeed, for every $0 = s_0 < s_1 < \cdots < s_{k+1} = t$,

$$(c_1 - c_2)\sum_{h=0}^{k} \mathcal{H}^1_D(K_n(s_{h+1}) \setminus K_n(s_h)) \le Diss_n(t). \tag{4.17}$$

According to Proposition 4.6, up to a further subsequence, we have that

$$K_n(s_j) \to H(s_j) \qquad \text{in the Hausdorff metric}$$

with

$$\mathcal{H}^1_D(H(s_j)\Delta K(s_j)) = 0. \tag{4.18}$$

Then, with the help of Lemma 1.4 and of (4.18) we pass to the limit in (4.17) and obtain, in view of (4.12),

$$(c_1 - c_2)\sum_{h=0}^{k} \mathcal{H}^1_D(K(s_{h+1}) \setminus K(s_h)) \le D(t),$$

from which (4.16) easily follows.

Thanks to (4.15),(4.16) and to Gołąb's Theorem, we can pass to the limit in the discrete energy inequality (4.4) and obtain

$$\mathcal{E}(u(t), K(t)) + Diss(t) \le \mathcal{E}(u_0, K_0) + 2\int_0^t \int_\Omega \nabla u(\tau) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{4.19}$$

The opposite inequality in (4.19) holds true, thanks to a by now standard *Riemann sum argument* (see [10, Section 4.4]). In a nutshell, the argument consists in choosing a specific sequence of partitions $\{s^n_i\}_{i=0,\dots,k(n)}$ with $k(n) \overset{n}{\nearrow} \infty$ of the interval $[0, t]$ such that the Riemann sums

$$\sum_{i=0}^{k(n)-1} \int_{s^n_i}^{s^n_{i+1}} \int_\Omega \nabla u(s^n_{i+1}) \cdot \dot{g}(s)\, dxds$$

do converge to

$$\int_0^t \int_\Omega \nabla u(s) \cdot \dot{g}(s)\, dxds.$$

Then one writes the minimality condition for $(u(s_i^n), K(s_i^n)) \in \mathcal{A}(g(s_i^n))$ established in Step 1, tested against $(u(s_{i+1}^n) - g(s_{i+1}^n) + g(s_i^n), K(s_{i+1}^n)) \in \mathcal{A}(g(s_i^n))$ and adds all resulting contributions for $i = 0, ..., k(n) - 1$; see [10, Section 4.4] for the details.

The energy balance

$$\mathcal{E}(u(t), K(t)) + Diss(t) = \mathcal{E}(u_0, K_0) + 2\int_0^t \int_\Omega \nabla u(\tau) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau$$

follows. We conclude that $t \mapsto (u(t), K(t))$ is a quasi-static evolution. The proof is complete. □

Remark 4.8 (Improved convergences). The proof of Theorem 4.1 shows that, for every $t \in [0, T]$,

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^2(\Omega; \mathbb{R}^2), \tag{4.20}$$

$$\mathcal{H}^1_D(K_n(t)) \to \mathcal{H}^1_D(K(t)) \tag{4.21}$$

and

$$Diss_n(t) \to Diss(t).$$

Indeed from the arguments of Step 2 and (4.4) we have

$$\begin{aligned}\mathcal{E}(u_0, K_0) + 2\int_0^t \int_\Omega \nabla u(\tau) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau &= \mathcal{E}(u(t), K(t)) + Diss(t)\\ &\le \liminf_n\, [\mathcal{E}(u_n(t), K_n(t)) + Diss_n(t)] \le \limsup_n\, [\mathcal{E}(u_n(t), K_n(t)) + Diss_n(t)]\\ &\le \limsup_n \left[\mathcal{E}(u_0, K_0) + \int_0^t \int_\Omega \nabla u_n(\tau) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau + e(\delta_n)\right] = \mathcal{E}(u_0, K_0) +\\ &\qquad 2\int_0^t \int_\Omega \nabla u(\tau) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau\end{aligned}$$

from which

$$\lim_n\, [\mathcal{E}(u_n(t), K_n(t)) + Diss_n(t)] = \mathcal{E}(u(t), K(t)) + Diss(t).$$

We thus deduce that

$$\lim_n \mathcal{E}(u_n(t), K_n(t)) = \mathcal{E}(u(t), K(t)) \qquad \text{and} \qquad \lim_n Diss_n(t) = Diss(t),$$

and the first convergence entails immediately (4.20) and (4.21).

Remark 4.9 (The connected case). In the connected case, loss of Hausdorff convergence only takes place at *healing times*, i.e., when $K(t)$ reduces to a point (or is the empty set) on $\Omega \cup \partial_D \Omega$. Indeed, assume the existence of two different subsequences $K_{n_k}(t), K_{\tilde{n}_k}(t)$, with

$$K_{n_k}(t) \to H_1 \qquad \text{in the Hausdorff metric}$$

and

$$K_{\tilde{n}_k}(t) \to H_2 \qquad \text{in the Hausdorff metric}$$

with $H_1 \neq H_2$. Since, in view of Proposition 4.6,

$$\mathcal{H}^1_D(H_1 \Delta K(t)) = \mathcal{H}^1_D(H_1 \Delta K(t)) = 0,$$

it must be that $\mathcal{H}^1_D(H_1 \Delta H_2) = 0$.

According to point (i) in Remark 4.7, those two sets, which are connected, must then reduce to at most a single point on $\Omega \cup \partial_D \Omega$. Since $K(t)$ is also connected, it in turn reduces to at most a single point on $\Omega \cup \partial_D \Omega$.

Finally, taking into account Remark 4.8, at such a time,

$$\mathcal{H}^1_D(K_n(t)) \to 0$$

because $\mathcal{H}^1_D(K(t)) = 0$. So, if Hausdorff convergence does not take place at time t, the approximating cracks are actually vanishing in length.

The argument fails if $m > 1$. In that case, using similar arguments, we can merely assert the existence of a subsequence of $K_n(t)$ such that one of its connected component heals in the limit, which is not much....

5. The case of two-dimensional elasticity.

In this section, we show how to modify the previous arguments in the case of linearized 2d-elasticity.

Admissible configurations. Let the reference configuration be an open bounded set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, while we consider $H^1(\Omega;\mathbb{R}^2)$ as the the class of admissible boundary displacements.

Given $\partial_D\Omega \subseteq \partial\Omega$ open in the relative topology, we say that the pair (u, K) is an admissible configuration for the boundary displacement $g \in H^1(\Omega;\mathbb{R}^2)$ if

$$K \in \mathcal{K}^f_m(\overline{\Omega}) \text{ and } u \in \mathscr{LD}(\Omega \setminus K) \text{ with } u = g \text{ on } \partial_D\Omega \setminus K,$$

where $m \geq 1$ is a fixed number, and $\mathcal{K}^f_m(\overline{\Omega})$ is given in (2.1). We will write $(u, K) \in \mathcal{A}(g)$. The pair $(u, e(u))$ can be thought of as an element of $L^2_{loc}(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathrm{M}^2_{\mathrm{sym}})$ since K has null Lebesgue measure.

Remark 5.1. Let $(u, K) \in \mathcal{A}(g)$, and let $H \in \mathcal{K}^f_m(\overline{\Omega})$ be such that $\mathcal{H}^1_D(K\Delta H) = 0$. Then $(u, H) \in \mathcal{A}(g)$. The proof follows precisely that in Remark 4.7: indeed the local arguments can be reproduced because, in view of Korn's inequality, elements of $\mathscr{LD}(\Omega \setminus K)$ are locally in H^1.

The following compactness result plays the role of Lemma 2.1 in our context.

Lemma 5.2. *Let $g_n, g \in H^1(\Omega;\mathbb{R}^2)$ be such that*

$$g_n \to g \qquad \text{strongly in } H^1(\Omega;\mathbb{R}^2).$$

Assume that $(u_n, K_n) \in \mathcal{A}(g_n)$ with

$$e(u_n) \rightharpoonup \Phi \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^2_{\mathrm{sym}}),$$

and

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

Then there exists $(u, K) \in \mathcal{A}(g)$ such that $\Phi = e(u)$ on $\Omega \setminus K$.

Proof. Let A be a connected component of $\Omega \setminus K$, and let $B \subset\subset A$ be a disk. Consider

$$\mathscr{R} := \left\{ v \in \mathscr{LD}(\Omega \setminus K) : \int_B v \cdot r \, dx = 0 \ \ \forall r \in \mathcal{R} \right\}$$

where $\mathcal{R}$ is the set of infinitesimally rigid motions, i.e.,

$$\mathcal{R} := \{Ax + b \, : \, A \in \mathrm{M}^2_{\mathrm{skew}}, b \in \mathbb{R}^2\}.$$

Define $\hat{u}_n$ to be the $L^2(B)$-orthogonal projection of u_n onto $\mathscr{R}$; clearly $e(\hat{u}_n) = e(u_n)$.

Since K_n Hausdorff-converges to K, any open Lipschitz connected subdomain G compactly embedded in A and containing B is also included, for n large enough, in $\Omega \setminus K_n$. Thus, according to Korn's inequality, $\hat{u}_n \in H^1(G;\mathbb{R}^2)$ and there exists $C_{G,B} > 0$ such that

$$\|\hat{u}_n\|_{L^2(G;\mathbb{R}^2)} \leq C_{G,B}\|e(u_n)\|_{L^2(G;\mathrm{M}^2_{\mathrm{sym}})} \leq C,$$

for some C depending on G, B, hence, up to a subsequence,

$$\hat{u}_n \rightharpoonup u_G, \text{weakly in } H^1(G;\mathbb{R}^2)$$

with

$$e(u_G) = \Phi. \tag{5.1}$$

But u_G also belongs to $\mathscr{R}$. In view of (5.1), it is thus uniquely defined so that the whole sequence $\hat{u}_n$ converges to u_G weakly in $H^1(G;\mathbb{R}^2)$ hence strongly in $L^2(G;\mathbb{R}^2)$. Then taking G to be an increasing sequence of Lipschitz connected open sets with union A, we immediately conclude that $u_G \equiv u$ independent of G with $u \in L^2_{\rm loc}(A;\mathbb{R}^2)$ and $e(u) = \Phi$. Since A is an arbitrary connected component of $\Omega \setminus K$, we infer that $u \in \mathscr{LD}(\Omega \setminus K)$.

The proof that $u = g$ on $\partial_D\Omega \setminus K$ is identical to that in Lemma 2.1 upon renewed use of Korn's inequality. □

Quasi-static evolutions. Let the Hooke's law be given by an element $\mathbb{C} \in L^\infty(\Omega;\mathcal{L}_s(\mathrm{M}^2_{\rm sym}))$ such that

$$a_1|M|^2 \le \mathbb{C}(x)M\cdot M \le a_2|M|^2 \text{ for every } M \in \mathrm{M}^2_{\rm sym}, \tag{5.2}$$

with $a_1, a_2 > 0$. Here $\cdot$ denotes the standard Frobenius matrix inner product.

We associate to an admissible configuration (u, K) the *elastic energy*

$$\mathcal{Q}(e(u)) := \frac{1}{2}\int_\Omega \mathbb{C}(x)e(u)(x)\cdot e(u)(x)\,dx.$$

As in Section 2, let $T > 0$ and $g \in AC([0,T];H^1(\Omega;\mathbb{R}^2))$ be a given time dependent boundary displacement, and let

$$c_1 > c_2 > 0 \tag{5.3}$$

be two given constants. In analogy with the scalar case (see Definition 2.3), we define a quasi-static evolution in the case of linearized elasticity as follows.

Definition 5.3 (Quasi-static evolution). *We say that $\{t \mapsto (u(t),K(t)) \in \mathcal{A}(g(t)), t \in [0,T]\}$ is a quasi-static evolution provided that for every $t \in [0,T]$ the following items hold true.*

(a) Global stability. *For every $(v,\Gamma) \in \mathcal{A}(g(t))$*

$$\mathcal{E}(u(t),K(t)) \le \mathcal{E}(v,\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K(t)),$$

where, for $(u,K) \in \mathcal{A}(g)$,

$$\mathcal{E}(u,K) := \mathcal{Q}(e(u)) + c_2\mathcal{H}^1_D(K).$$

(b) Energy balance. *We have*

$$\mathcal{E}(u(t),K(t)) + Diss(t) = \mathcal{E}(u(0),K(0)) + \int_0^t\int_\Omega \mathbb{C}e(u(\tau))\cdot e(\dot g(\tau))\,dx\,d\tau,$$

where

$$Diss(t) := (c_1 - c_2)\sup\left\{\sum_{i=0}^n \mathcal{H}^1_D(K(s_{i+1}) \setminus K(s_i)) \,:\, 0 = s_0 < s_1 < \cdots < s_{n+1} = t\right\}.$$

Existence of quasi-static evolutions. The main result of the Section is the following

Theorem 5.4 (Existence of a quasi-static evolution for 2d-elasticity). *Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded Lipschitz domain and $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology. Let $g \in AC([0,T];H^1(\Omega;\mathbb{R}^2))$ and assume (5.2) and (5.3) hold true. Let finally $(u_0,K_0) \in \mathcal{A}(g(0))$ be a globally stable configuration (i.e., satisfying property (2.4)).*

Then, there exists a quasi-static evolution $\{t \mapsto (u(t),K(t)) : t \in [0,T]\}$ in the sense of Definition 5.3 such that $(u(0),K(0)) = (u_0,K_0)$.

Proof. We proceed as in Section 4 by constructing incremental configurations $(u_i^\delta, K_i^\delta) \in \mathcal{A}(g_i^\delta)$. We consider

$$(u_i^\delta, K_i^\delta) \in Argmin\{\mathcal{E}(v,\Gamma) + (c_1 - c_2)\mathcal{H}^1_D(\Gamma \setminus K_{i-1}^\delta) \,:\, (v,\Gamma) \in \mathcal{A}(g_i^\delta)\}. \tag{5.4}$$

The variational problems are well posed thanks to Lemma 5.2 and to Gołąb Theorem.

Interpolating in time, we obtain the discrete in time evolution

$$\{t \mapsto (u^\delta(t), K^\delta(t) \,:\, t \in [0,T]\}$$

such that, defining $Diss^\delta$ as in (4.3),

$$\mathcal{E}(u^\delta(t), K^\delta(t)) + Diss^\delta(t) \le \mathcal{E}(u_0, K_0) + \int_0^{t_i^\delta} \int_\Omega \mathbb{C}e(u^\delta(\tau)) \cdot e(\dot g(\tau))\, dx\, d\tau + e(\delta)$$

with $e(\delta) \to 0$ as $\delta \to 0$. In view of (5.2), this inequality yields the uniform bound

$$\|e(u^\delta(t))\| + \mathcal{H}^1_D(K^\delta(t)) + z^\delta(t) \le C,$$

where z^δ is defined as in (4.6).

Thanks to Lemma 5.2, the proof is now completely analogous to that of Theorem 4.1, provided that we adapt Theorem 3.1 to our context.

Specifically, it suffices to prove the following. Let $c, c' \ge 0$, and let $g_n, g \in H^1(\Omega;\mathbb{R}^2)$ be such that

$$g_n \to g \qquad \text{strongly in } H^1(\Omega;\mathbb{R}^2).$$

Assume that $(u_n, K_n) \in \mathcal{A}(g_n)$ satisfy the following global stability condition: for every $(v,\Gamma) \in \mathcal{A}(g_n)$,

$$\mathcal{Q}(e(u_n)) + c\mathcal{H}^1_D(K_n) \le \mathcal{Q}(e(v)) + c\mathcal{H}^1_D(\Gamma) + c'\mathcal{H}^1_D(\Gamma \setminus K_n)$$

and assume further that

$$K_n \to K \qquad \text{in the Hausdorff metric}$$

$$e(u_n) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega;\mathrm{M}^2_{\mathrm{sym}})$$

for some $(u,K) \in \mathcal{A}(g)$. Then (u,K) is a globally stable configuration, that is that, for every $(v,\Gamma) \in \mathcal{A}(g)$,

$$\mathcal{Q}(e(u)) + c\mathcal{H}^1_D(K) \le \mathcal{Q}(e(v)) + c\mathcal{H}^1_D(\Gamma) + c'\mathcal{H}^1_D(\Gamma \setminus K). \tag{5.5}$$

Notice that, in view of [8, Theorem 1], there exists $v_m \in H^1(\Omega \setminus \Gamma;\mathbb{R}^2)$ with $v_m = g$ on $\partial_D\Omega$ and such that

$$e(v_m) \to e(v) \qquad \text{strongly in } L^2(\Omega;\mathrm{M}^2_{\mathrm{sym}}).$$

As a consequence, it is sufficient to establish (5.5) in the case $(v,\Gamma) \in \mathcal{A}(g)$ with

$$v \in H^1(\Omega \setminus \Gamma;\mathbb{R}^2). \tag{5.6}$$

This is a great simplification, since we can employ the same construction as that in the proof of Theorem 3.1 working on each component.

Specifically, if $v := (v^1, v^2)$, we fix the neighborhood U, V, ε as in Step 1 of the proof of Theorem 3.1, and construct the associated Γ_n, v^1_n, v^2_n (approximations of the scalar functions v^1, v^2). The crucial estimate (3.21) now reads as follows (we can estimate in the squares the symmetrized gradient by the full gradient thanks to (5.6))

$$\begin{aligned}\mathcal{Q}(e(v_n)) \le \mathcal{Q}(e(v)) + 2a_2 \sum_{j=1}^m \int_{Q_{\nu_j}(x_j,r_j)\cap\overline{\Omega}} |\nabla v|^2\, dx + a_2 \sum_{j=1}^m \int_{Q_{\nu_j}(x_j,r_j)\cap\overline{\Omega}} |\nabla g|^2\, dx \\ \le \mathcal{Q}(e(v)) + 2a_2 \int_{U\cap\overline{\Omega}} |\nabla v|^2\, dx + \int_{U\cap\overline{\Omega}} |\nabla g|^2\, dx,\end{aligned}$$

where a_2 is the coercivity constant in (5.2). Comparing (u_n, K_n) with $(v_n - g + g_n, \Gamma_n) \in \mathcal{A}(g_n)$ and using the previous inequality we deduce that

$$\begin{aligned}\mathcal{Q}(e(u)) + c\mathcal{H}^1_D(K \setminus \bar V) \le \mathcal{Q}(e(v)) + (c+c')\mathcal{H}^1_D(\Gamma \setminus K) \\ + (c+c')\left[\varepsilon + \frac{7\varepsilon}{1-\varepsilon}\mathcal{H}^1(\Gamma)\right] + 2a_2 \int_{U\cap\overline{\Omega}} |\nabla v|^2\, dx + a_2 \int_{U\cap\overline{\Omega}} |\nabla g|^2\, dx,\end{aligned}$$

so that the global stability follows since V, U and ε are arbitrary. □

Remark 5.5. Notice that even if an L^∞-bound for the boundary displacement g is assumed, the functional framework for the displacement u^δ_i in the incremental problems (5.4) cannot reduce to $H^1(\Omega \setminus K^\delta_i)$ since truncation fails in the case of energies that depend on the symmetrized gradient.

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References

[1] F. Acanfora and M. Ponsiglione. Quasi static growth of brittle cracks in a linearly elastic flexural plate. *Ann. Mat. Pura Appl. (4)*, 185(2):293–317, 2006.
[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.
[3] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.
[4] John M. Ball. Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Philos. Trans. Roy. Soc. London Ser. A*, 306(1496):557–611, 1982.
[5] B.J. Blaiszik, S.L.B. Kramer, S.C. Olugebefola, J.S. Moore, N.R. Sottos, and S.R. White. Self-healing polymers and composites. *Annual Review of Materials Research*, 40:179–211, 2010.
[6] B. Bourdin, G. A. Francfort, and J.-J. Marigo. *The variational approach to fracture*. Springer, New York, 2008. Reprinted from J. Elasticity **9**1 (2008), no. 1-3 [MR2390547], With a foreword by Roger Fosdick.
[7] Warren F. Busse. Physics of rubber as related to the automobile. *J. Appl. Phys.*, 9:438–451, 1938.
[8] Antonin Chambolle. A density result in two-dimensional linearized elasticity, and applications. *Arch. Ration. Mech. Anal.*, 167(3):211–233, 2003.
[9] P. Cordier, F. Tournilhac, C. Soulie-Ziakovic, and L. Leibler. Self-healing and thermoreversible rubber from supramolecular assembly. *Nature*, 451:977–980, 2008.
[10] G. Dal Maso, G.A. Francfort, and R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Ration. Mech. Anal.*, 176(2):165–225, 2005.
[11] G. Dal Maso and R. Toader. A model for the quasi-static growth of brittle fractures: existence and approximation results. *Arch. Ration. Mech. Anal.*, 162(2):101–135, 2002.
[12] J. Deny and J. L. Lions. Les espaces du type de Beppo Levi. *Ann. Inst. Fourier, Grenoble*, 5:305–370 (1955), 1953–54.
[13] Kenneth J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.
[14] G.A. Francfort and C.J. Larsen. Existence and convergence for quasi-static evolution in brittle fracture. *Comm. Pure Appl. Math.*, 56(10):1465–1500, 2003.
[15] G.A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46(8):1319–1342, 1998.
[16] A.N. Gent and P.B. Lindley. Internal rupture of bonded rubber cylinders in tension. *Proc. Royal Soc. London, A*, 2:195–205, 1959.
[17] A.N. Gent and B. Park. Failure processes in elastomers at or near a rigid inclusion. *J. Mater. Sci.*, 19:1947–1956, 1984.
[18] Alan A. Griffith. The phenomena of rupture and flow in solids. *Phil. Trans. Roy. Soc. London*, CCXXI-A:163–198, 1920.
[19] D. Henao and C. Mora-Corral. Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity. *Arch. Ration. Mech. Anal.*, 197(2):619–655, 2010.
[20] Alexei A. Ilyushin. On the postulate of stability. *J. Appl. Math. Mech.*, 25:746–752, 1961.
[21] A. Kumar, G.A. Francfort, and O. Lopez-Pamies. Fracture and healing of elastomers: A phase-transition theory and numerical implementation. *J. Mech. Phys. Solids*, 112:523–551, 2018.
[22] V. Lefèvre, K. Ravi-Chandar, and O. Lopez-Pamies. Cavitation in rubber: An elastic instability or a fracture phenomenon? *Int. J. Fracture*, 192:1–23, 2015.
[23] F.B. Madsen, L. Yu, and A.L. Skov. Self-healing, high-permittivity silicon dielectric elastomer. *ACS Macro Lett.*, 5:1196–1200, 2016.
[24] A. Mielke, R. Rossi, and G. Savaré. Global existence results for viscoplasticity at finite strain. *Arch. Ration. Mech. Anal.*, 227(1):423–475, 2018.
[25] Carlos Mora-Corral. Quasistatic evolution of cavities in nonlinear elasticity. *SIAM J. Math. Anal.*, 46(1):532–571, 2014.
[26] X. Poulain, V. Lefèvre, O. Lopez-Pamies, and K. Ravi-Chandar. Damage in elastomers: nucleation and growth of cavities, micro-cracks, and macro-cracks. *Int. J. Fracture*, 205(1):1–21, 2017.

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