Variational Fracture: Twenty Years After*

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Abstract

In this work, we propose to succinctly review, in non technical terms, the current mathematical state of brittle fracture first put forth by A.A. Griffith in his seminal work [21], then re-interpreted under the label "variational fracture" in [19] and subsequent works. We will only address the sharp theory and will limit ourselves to a theoretical exposition, leaving phase-field approximations and other possible implementations to other articles within this volume.

1 Griffith fracture viewed as a variational problem

"The phenomena of rupture and flow in solids" [21] is one hundred years old. Yet that paper's ripples still confound anyone interested in fracture. Our story must start with our understanding of Griffith's work recounted within the semantics of our time.[1]

The environment is two-dimensional. The sample Ω, a bounded open domain of $\mathbb{R}^2$, is filled with a brittle elastic and, say, homogeneous material. The time-dependent loads – a generic term for both hard or soft devices, as well as body forces – are applied slowly so that inertia is thought to be irrelevant (quasi-staticity). Specializing to dead loads, those are body forces denoted by $f_b(t)$ and defined on Ω; surface forces denoted by $f_s(t)$ and defined on $\partial_s\Omega \subset \partial\Omega$; and boundary displacements denoted by $g(t)$ and defined on $\partial_d\Omega := \partial\Omega \setminus \overline{\partial_s\Omega}$.

The type of elastic behavior matters none, as long as it can be represented by a smooth bulk energy $F \mapsto W(F)$ which will be assumed to be a function of the gradient of the deformation field φ; in linearized elasticity W is simply a function of the strain $e(u) := \frac{1}{2}(\nabla u + \nabla u^t)$ with $u(x) := \varphi(x) - x$.

The crack (a collective term) is a discontinuity line for the deformation field. Imagine that we know where and when the crack will nucleate and how and along which path it will propagate.

*The author wishes to acknowledge A. Dumas (fils) who thoroughly investigated a case of rupture in [17], this some seventy years before A.A. Griffith.

[1]Our admittedly personal account of that story will keep the number of references to a minimum. Narrative continuity is our goal, not taxonomy. Nor for that matter exhaustion: many topics worthy of investigation will not even get a mention, be it non-interpenetration, dynamics, energy dissipating evolutions, ...

Then the problem reduces to one of elastic equilibrium with the applied loads at every time. Griffith's premise is that a crack has nucleated and that it follows a connected path $\hat{\Gamma} \subset \overline{\Omega} \setminus \partial_s\Omega$ which has somehow been foretold. Then the crack is fully characterized by its arclength ℓ and will be denoted by $\Gamma(\ell)$, a subset of $\hat{\Gamma}$. Griffith's purport is the prediction of the crack advance along that path, or, in other words, of the length $\ell(t)$ of the crack at time t. For this, he requires an additional ingredient in the form of a surface energy associated to the curves where the deformation is discontinuous.

The surface energy adopted by Griffith is simple. A very rudimentary counting argument for broken atomic bonds in a crystalline structure leads him to the conclusion that such an energy should be proportional to the length of the discontinuity line with a proportionality factor, the fracture toughness, denoted by G_c. Doing so, he restricts his focus to the brittle case, which will also frame the confines of this contribution.

In mechanical parlance, for a crack length ℓ, the domain $\Omega \setminus \Gamma(\ell)$ is at time t in equilibrium with the loads. The elastic solution $\varphi(t,\ell)$ (which may, or may not be unique, depending on the nature of the elastic energy W) yields a potential energy (unique if one believes in elastic energy minimization)

$$\mathcal{P}(t,\ell) := \int_{\Omega\setminus\Gamma(\ell)} W(\nabla\varphi(t,\ell))\,dx - \mathcal{F}(t,\varphi(t,\ell)) \tag{1}$$

with

$$\mathcal{F}(t,\varphi) := \int_\Omega f_b(t).\varphi\,dx + \int_{\partial_s\Omega} f_s(t).\varphi\,ds. \tag{2}$$

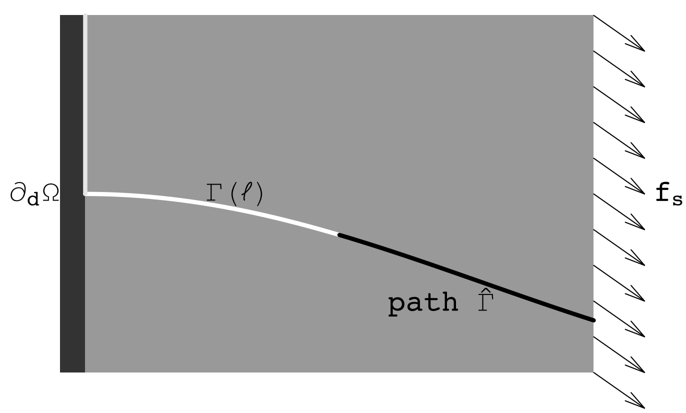

Figure 1: Griffith's cracked solid.

Griffith's proposal for the prediction of $\ell(t)$ is entirely contained in the following aphorism: If $\ell(t)$ is the crack length at time t it is because a slightly longer crack at that time would cost more in surface energy that it would save in potential energy. In signs, for any $1 >> \Delta > 0$,

$$\mathcal{P}(t,\ell(t)+\Delta) + G_c(\ell(t)+\Delta) \geq \mathcal{P}(t,\ell(t)) + G_c\ell(t). \tag{3}$$

So, Griffith's is a local minimality statement about the current state of affairs. And, folklore notwithstanding, it says nothing about the future!

The interpretation in terms of energy release rate is immediate if letting Δ decrease to 0 and it yields the celebrated

$$G(t) \leq G_c \tag{4}$$

with $G(t) := -\frac{\partial \mathcal{P}}{\partial \ell}(t, \ell(t))$ as a necessary condition for (3) to hold. Please do note that Griffith's local minimality statement does not assume any kind of differentiability of the potential energy with respect to crack length; not so for (4), a source of persistent headaches in the fracture community.

It is fair to say that Griffith is remiss in his prediction of the crack advance. The paternity of the post-Griffith cure is muddled[2] while the cure itself is simple. It consists in asserting that the crack will stay put unless $G(t)$ attains the value G_c in which case the crack is free to extend if that proves benefitial. In other words,

$$\dot{\ell}(t) = 0 \text{ unless } G(t) = G_c. \tag{5}$$

Once again, remark that, while (5) is indeed an additional information, it is one that ascribes some degree of smoothness to both $\ell \mapsto \mathcal{P}(\ell)$ and $t \mapsto \ell(t)$. The headache has increased in virulence.

Summing up, Griffith and post-Griffith reduce to (4), a local minimality statement, together with (5). To this one should append a generally undisputed statement: the irreversibility of the process, that is that $\ell(t) \nearrow$ with t.[3]

In this light, the contribution of the variational theory of fracture can be laid out in few words: Recast Griffith's minimality statement (4) in the language of the calculus of variations and, assuming smoothness, show that the propagation criterion (5) is a mere translation of the conservation of energy. We will not detail the equivalence and refer the interested reader to [8, Chapter 2] instead. The crucial observations are that, first there is no reason to stay two-dimensional (and we will call N the dimension), then that prescribing a putative locus for the (add-)crack (a $N-1$-dimensional set) is unnecessary. Let the crack choose which future path it wishes to borrow provided that minimality and energy conservation are obeyed.

Modulo this, the variational formulation is exactly Griffith's formulation. In particular, arguing against minimality is tantamount to rejecting Griffith altogether and pledging allegiance to the energy release rate viewpoint. Such a pledge cannot be argued against under two normative conditions: the awareness of its origin as a byproduct of minimality and the acceptance of the accompanying regularity hurdles which have been fodder for many illusory investigations in fracture mechanics since its inception.

Freedom of path, while often confusing to the practitioner of fracture, is the trademark feature of the variational approach. It is also true to Griffith's original intent which was arguably forgotten by his next of kin.

Denoting by $\Gamma(t)$ the crack at time t and by $\mathcal{H}^{N-1}$ the $N-1$-dimensional Hausdorff measure (think surface measure), we replace (3), by

`(Um)` $(\Gamma(t), \varphi(t))$ is a local minimizer (in a topology that remains to be specified), or a global minimizer for

$$\mathcal{E}(t; \varphi, \Gamma) := \int_{\Omega \setminus \Gamma} W(\nabla \varphi)\, dx - \mathcal{F}(t, \varphi) + G_c \mathcal{H}^{N-1}(\Gamma), \tag{6}$$

[2] We find telltale signs in the work of J. Rice (see e.g. [27, Section 3]) but have been at pains to isolate a definitive statement of its origin in the literature.

[3] Irreversibility may be debatable when dealing with elastomers or gels.

among all $\Gamma \supset \Gamma(t)$ and all φ with the required regularity on $\Omega \setminus \Gamma$ and $\varphi = g(t)$ on $\partial_d\Omega \setminus \Gamma$. Note that the test φ's depend on the test Γ's that can be pretty much anything as long as they are closed sets with finite "surface" area.

Correspondingly, we also rewrite the irreversibility condition as

`(Ir)` $\Gamma(t) \underset{t}{\nearrow}$.

Finally, we replace (5) by

`(Ec)`
$$\begin{aligned} E(t) &:= \int_{\Omega\setminus\Gamma(t)} W(\nabla\varphi(t))\,dx - \mathcal{F}(t,\varphi(t)) + G_c\mathcal{H}^{N-1}(\Gamma(t)) \\ &= \mathcal{P}(t,\Gamma(t)) + G_c\mathcal{H}^{N-1}(\Gamma(t)) \\ &= E(0) + \int_0^t \Bigg[\int_{\partial_d\Omega\setminus\Gamma(l(s))} \frac{\partial W}{\partial F}(\nabla\varphi(s))\nu.\ \dot{g}(s)\ d\mathcal{H}^{N-1} \\ &\qquad\qquad -\dot{\mathcal{F}}(s,\varphi(s))\Bigg]\ ds, \end{aligned}$$

with an obvious extension of the definition (1) of the potential energy $\mathcal{P}$ and (2) of the loads $\mathcal{F}$. Here ν is the exterior normal to Ω.

We allow the test cracks Γ to be pretty much any closed set in $\overline{\Omega} \setminus \partial_s\Omega$ with finite $N{-}1$-dimensional Hausdorff measure. This allows us to envision very rough cracks, with surface areas that coincide with the usual area when the crack is a rectifiable surface. We do not allow for the crack to lie on $\partial_s\Omega$ because the crack cannot live where soft devices are applied.

We shall refer to `(Um),(Ir),(Eb)`, as *the strong variational evolution.*

From a mathematical standpoint, one should first adjudicate the well-posedness of such an evolution under suitable assumptions on the data, that is on f_b, f_s and g. That has proved to be a difficult task which is not yet fully completed. We will attempt a quick review in Section 2. Then, one should test the formulation against well-known crack evolutions and investigate its range of applicability. This is in turn a formidable task that we will not even attempt to address in these notes. Other contributions in the volume will surely connect the variational formulation to various phase-field approximations, detail the relevant algorithmics and present many beautiful results.

For our part, we will next turn our focus on the two pillars of theoretical ignorance, nucleation and kinking, which will be the respective topic of Sections 3 and 4. Because the classical view of kinking revolves around the notion of energy release rate we will begin by revisiting that notion in Section 4, keeping in mind that there is no rationale for the usual constraints on the topology of the cracks. Finally we will venture beyond Griffith in the very short Section 5. There, we will mostly explain why any attempt to forsake minimality altogether is doomed and part with a few concluding words.

2 The mathematics of existence

In the context of image segmentation, D. Mumford and J. Shah [26] had proposed to segment image through the following algorithm: Find a pair K, compact of $\Omega \subset \mathbb{R}^2$ (the picture) representing the contours of the image in the picture, and φ, the true pixel intensity at each point of the picture, an element of $C^1(\Omega \setminus K)$, which minimizes

$$\int_{\Omega\setminus K} |\nabla\varphi|^2 \, dx + k\mathcal{H}^1(K) + \int_\Omega |\varphi - g|^2 \, dx, \tag{7}$$

where g is the measured pixel intensity. The minimization was then shown in [16] to be equivalent to a well-posed one-field minimization problem on a subspace $SBV(\Omega)$ of the space $BV(\Omega)$ of functions with bounded variations on Ω, namely,

$$\int_\Omega |\nabla\varphi|^2 \, dx + k\mathcal{H}^1(S(\varphi)) + \int_\Omega |\varphi - g|^2 \, dx, \tag{8}$$

where $\nabla\varphi$ represents the absolutely continuous part of the weak derivative of φ (a measure), and $S(\varphi)$ the set of jump points for φ.

We recall that a function $\varphi : \Omega \subset \mathbb{R}^N \mapsto \mathbb{R}$ is in $BV(\Omega)$ iff $\varphi \in L^1(\Omega)$ and its distributional derivative $D\varphi$ is a measure with bounded total variation. Then, the theory developed by E. De Giorgi, H. Federer and A.I. Vol'pert (see e.g. [2]) implies that

$$D\varphi = \nabla\varphi(x) \, dx + (\varphi^+(x) - \varphi^-(x))\nu(x)\mathcal{H}^{N-1}_{\lfloor S(\varphi)} + C(\varphi),^{4}$$

with $\nabla\varphi$, the approximate gradient, $\in L^1(\Omega)$ ($\nabla\varphi$ is no longer a gradient), $S(\varphi)$ the complement of the set of Lebesgue points of φ, a $\mathcal{H}^1$ σ-finite and countably 1-rectifiable set (a countable union of compacts included in C^1-hypersurfaces, up to a set of 0 $\mathcal{H}^{N-1}$-measure), $\nu(x)$ the common normal to all those hypersurfaces at a point $x \in S(\varphi)$, $\varphi^\pm(x)$ the values of $\varphi(x)$ "above and below" $S(\varphi)$, and $C(\varphi)$ a measure (the Cantor part) which is mutually singular with dx and with $\mathcal{H}^{N-1}$ (it only sees sets that have 0 Lebesgue-measure and infinite $\mathcal{H}^{N-1}$–measure). The subspace $SBV(\Omega)$ is that of those $\varphi \in BV(\Omega)$ such that $C(\varphi) \equiv 0$.

Thanks to a compactness result due to L. Ambrosio [1], a simple argument of the direct method applied to (8) establishes existence of a minimizer φ^* for that functional. The further result that the pair $\left(\varphi^*_{\lfloor \Omega\setminus\overline{S(\varphi^*)}}, \overline{S(\varphi^*)}\right)$ is a minimizer for (7) is highly non-trivial and makes up the bulk of [16].

In E. De Giorgi's footstep, we thus reformulate the variational evolution in a weak functional framework. To do this, it is more convenient to view the hard device $g(t)$ as living on all of $\mathbb{R}^2$ and to integrate by parts the boundary term involving $\dot{g}(t)$ in `(Eb)`. So, after elementary integrations by parts, we propose *the weak variational evolution:*

For any $t \in [0, T]$,

`(Um)` $(\Gamma(t), \varphi(t))$ is a local (or global) minimizer (in a topology that remains to be specified) for

$$\mathcal{E}(t; \varphi, \Gamma) := \int_\Omega W(\nabla\varphi) \, dx - \mathcal{F}(t, \varphi) + G_c\mathcal{H}^1(\Gamma), \tag{9}$$

among all $\overline{\Omega} \setminus \partial_s\Omega \supset \Gamma \supset \Gamma(t)$ and all $\varphi \equiv g(t)$ on $\mathbb{R}^2 \setminus \overline{\Omega}$ with $S(\varphi) \subset \Gamma$;

`(Ir)` $\Gamma(t) \nearrow_t$;

`(Eb)` $E(t) = E(0) + \int_0^t \left[\int_\Omega \frac{\partial W}{\partial F}(\nabla\varphi(s)).\nabla\dot{g}(s) \, dx - \dot{\mathcal{F}}(s, \varphi(s))\right] ds$ with $E(t) := \mathcal{E}(t; \varphi(t), \Gamma(t))$.

[4]The symbol $\lfloor$ means "restricted to".

The precise functional framework is somewhat delicate and the reader should view $SBV(\Omega)$ as the template. It will be precisely that in the anti-plane shear case but it can be much more involved, especially in the setting of linear elasticity where symmetrized gradients, rather than pure gradients are the relevant measures of the deformation.

Likewise, it is not so that the crack should belong to $\overline{\Omega} \setminus \partial_s \Omega$. Any rigorous analysis will actually require $\partial_s \Omega$, the site of application of the surface forces, to be part of the boundary of a non-brittle piece of the material. In other words, we should single out a thin layer around $\partial_s \Omega$ with infinite fracture toughness. This also will be overlooked in the sequel.

Remark that the crack $\Gamma(t)$ can often, but not always, be identified with $\Gamma_0 \cup \left[\bigcup_{s \leq t} S(\varphi(s))\right]$; see [13].

Also note that it may be the case that $\Gamma(t)$ bites into the Dirichlet part $\partial_d \Omega$ of the boundary, thereby preventing $\varphi(t)$ from reaching its value on the boundary. One could argue that this is unreasonable because the boundary of the sample has different material properties from its bulk. Decreasing the fracture toughness along the boundary is an easy fix which does not impact the mathematical results. Increasing it will only push the crack slightly inside the domain, should it be so that it wanted to invade the boundary to start with. In the end, the formulation will remain unchanged and the increase in surface fracture toughness will disappear.

We now briefly review the current mathematical state of the existence theory for such an evolution. Two distinct questions should be raised. First, is there a solution for the weak variational evolution under suitable regularity assumptions for the data. Then, whenever the first question has been answered in the positive, can one infer a solution to the strong variational evolution.

2.1 Existence of a weak variational evolution

The minimality condition (9) is on a collision course with the force loads. This is so because, for any topology that is compatible with small crack surface area, one can slightly increase the crack surface area and cut off a piece of the domain Ω which is subsequently moved rigidly "up or down", thereby driving the term $\mathcal{F}(t, \varphi)$ in (6) to $-\infty$. So the unescapable conclusion is that there are no weak variational evolutions compatible with the application of force loads!

To be clear, this is not a defect of the variational formulation per se. Rather, it is part and parcel of Griffith's formulation from the get-go. The only reason that prevents this problem from arising in more classical formulations is their insistence in specializing add-cracks to essentially be small straight line segments. We view such a truncated view of fracture mechanics as misguided. Very recently, C. J. Larsen has suggested a way around this issue, but the (unpublished) results are still in a preliminary stage.

So we will assume henceforth that $f_b(t) = f_s(t) \equiv 0$. The only loading mechanism is the hard device $g(t)$ applied to $\partial_d \Omega$. The required smoothness for $g(t)$ is $W^{1,1}(0, T; W^{1-1/p,p}(\partial_d \Omega; \mathbb{R}^N))$ with $1 < p < +\infty$ measuring the growth of the energy W at infinity.[5] Under such a condition, existence for the weak variational evolution is guaranteed in the following settings in chronological order:

1w. The generalized anti-plane shear case: $\varphi : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ and $W(F) = 1/2\mu|F|^2$ (see [18]);

[5] Although quite natural, this assumption prohibits jumps on $\partial_d \Omega$ so that a tearing experiment with $u = d$ on one side of a pre-crack and $u = -d$ on the other side is not allowed.

2w. The fully non-linear elastic case: $\varphi : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ and $W(x, F) : \Omega \times \mathbb{R}^{N^2} \to \mathbb{R}$, Caratheodory, C^1 in F and quasiconvex with the usual coercivity assumption $\alpha|F|^p \leq W(x, F)$ and boundedness assumption $W(x, F) \leq \beta(|F|^p + 1)$ with $p > 1$ and $\alpha, \beta > 0$ (see [12]);

3w. The hyperelastic case: $\varphi : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ and $W(x, F) : \Omega \times \mathbb{R}^{N^2} \to \mathbb{R} \cup \{+\infty\}$, Caratheodory, $\equiv +\infty$ on $\{F : \det F \leq 0\}$, C^1 in $\{F : \det F > 0\}$, polyconvex, coercive as in item 2w. Further, W has to satisfy additional assumptions that we will not list here and $g(t)$ must be invertible and C^1 as well as its inverse (see [14]);

4w. The 2d linearly elastic case: $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ and W quadratic (see [20]).

Of note is the current absence of any kind of existence result for the most relevant case as far as applications are concerned, that of 3d linear elasticity! This, which may come as a surprise to mechanicians, is because the technical difficulties inherent to the presence of symmetrized gradients have been an overwhelming roadblock. The removal process has only begun in the last couple of years. Now, this has not prevented many authors from using phase-field approximations for the computation of crack evolutions in linear elasticity. There are no objections to be raised here, provided that those investigations are pursued with full knowledge of their shaky theoretical foundations.[6]

A few remarks about what is meant by a solution to the weak variational problem. First, the solution $\varphi(t)$ (or $u(t)$ in the linearly elastic case) is never unique. Non-uniqueness should not be construed as a specific indictment of the adopted setting because it is a pervading feature of many problems in physics. Uniqueness, on the other hand, is the conspicuous manifestation of convexity, a feature which usually disappears when coupled problems are investigated. And fracture is truly a coupled problem in that it couples elasticity to crack growth.

Then, the solutions $\varphi(t)$ (or $u(t)$ in the linearly elastic case) have no regularity in time. In particular they can experience jumps which would correspond to a sudden increase of the crack surface area. This is a much discussed feature, but one that is unavoidable if remaining true to Griffith. One is at liberty to induce some kind of dissipative mechanism that would activate at such jumps. In that case energy conservation is lost, dragging along the road to perdition both the variational approach advocated in these notes and the energy release rate based approach commonly used in classical LEFM. Let us point the interested reader to e.g. [24] which represents in our opinion the most elaborate effort in that direction because it preserves crack freedom, in contrast to most available literature on the topic.

Finally, the existence results are all about global minimizers at each time and thus do not depend on the kind of topology one would impart on small perturbations of the state of fracture at any given time in a specific experiment.

Unfortunately, global minimization comes with its own baggage. Let us illustrate this on the example of a tearing experiment for a thin rectangular elastic sheet $\Omega = (0, L) \times (-H, +H)$ with L possibly equal to $+\infty$. Its shear modulus is μ and its toughness G_c. Tearing corresponds to a displacement load $tH\mathbf{e}_3$ on $\{0\} \times (0, +H)$ and $-tH\mathbf{e}_3$ on $\{0\} \times (-H, 0)$. The edges $\{\pm H\} \times (0, L)$ are traction free and no body loads are applied.

[6] It may be so that the physics underlying proper phase-field models is actually a better fit for brittle fracture than the sharp theory. As already stated, ours is only a Griffith universe beyond which we dare not venture for fear of getting sucked into a modeling wormhole.

Seek an anti-plane shear solution, anti-symmetric with respect to $y = 0$, respecting the boundary condition at $x = 0$ (so no jumps there) and a crack along that axis.[7] Within that kinematics, the simplest type of displacement field is of the form

$$\mathbf{v}(x, y, t) = \mathrm{sg}(y)v(t, x)\mathbf{e}_3 \qquad \text{with} \qquad v(t, 0) = tH. \tag{10}$$

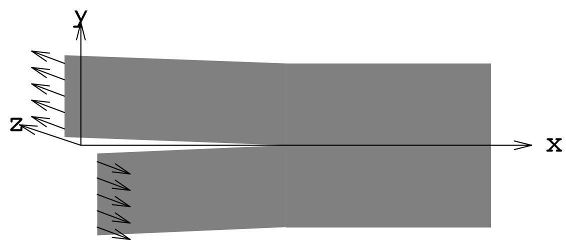

Figure 2: Tearing.

The field $\mathbf{v}(t)$ is discontinuous at the points x on the x-axis where $v(x,t) \neq 0$, that is $S(\mathbf{v}(t)) = \{x \geq 0 \ : \ v(t, x) \neq 0\}$. The energy reads as

$$\mathcal{E}(t, v) = \int_0^L \mu H (v')^2 dx + G_c \mathcal{H}^1(\{x \geq 0 \ : \ v(t, x) \neq 0\}), \ \text{ with } v(t, 0) = tH.$$

If the length L is infinite, it is easily seen that, at fixed t the minimizer $u(t)$ for that energy exists, is unique, continuous on $[0, L]$ and that, if it reaches 0 at some point x, it stays at 0 on $[x, L]$ (see [8, Chapter 3.2]). Consequently, $S(u(t)) = [0, \ell(t))$ for some $0 \leq \ell(t) \leq +\infty$ with $u(t, 0) = tH$ and $u(t, \ell(t)) = 0$. Once this is acquired, then $u(t)$ is shown, through an argument that uses inner variations, to be

$$u(t, x) = tH \left(1 - \frac{x}{l(t)}\right)^+. \tag{11}$$

Remark that, since $\ell(t)$ and $u(t)$ increase with t, irreversibility is guaranteed while energy balance is a byproduct of the smoothness of $u(t)$ on $[0, \ell(t))$.

Then, we find that global minimality yields

$$\ell(t) = tH\sqrt{\mu H / G_c}. \tag{12}$$

If however the length is finite, then one should specify the boundary condition at $x = L$.

If that boundary condition is $u(t, L) = 0$, then $u(t)$ given by (11), (12) is still a minimizer (within the restricted kinematics of (10)) as long as the corresponding $\ell(t)$ stays below L, that is as long as $t \leq t^* := L/H\sqrt{G_c/\mu H}$. If however that boundary condition is a Neumann boundary condition, that is $u'(t, L) = 0$, then global minimality induces in particular a competition between the energy $2tH\sqrt{G_c \mu H}$ which is the energy associated to (11), (12) (a possible minimizer for $t < t^*$) and $G_c L$ which corresponds to a solution where $u = \mathrm{sg}(y)tH$. We get that the length of the crack is $\ell(t) = tH\sqrt{\mu H/G_c}$ for $t \leq t^*/2$, and $\ell(t) = L$ for $t \geq t^*/2$.

[7] This will not produce an exact solution, because that solution will exhibit a discontinuity of the normal stress at the points $(\ell(t), y)$, $y \neq 0$.

Yet for $t < t^*$ the solution (11), (12) cannot possibly know that, at $x = L$, the boundary condition is Neumann, or Dirichlet for that matter. Global minimality has endowed the crack with prescience, hardly a physical concept.

Finding evolutions that are only locally, but not globally minimizing (9) is an arduous task at present. It might be so that the phase-field approximations which are used in the computations of crack evolutions fare better in that respect although the link between the sharp theory and phase-field evolutions is built upon global minimality (see e.g. [8, Chapter 8]).

2.2 Existence of a strong variational evolution

As already stated, passing from the weak to the strong is a challenging task, even in the simplistic setting of image segmentation. As far as fracture evolutions are concerned, the only available results can be found in [15], [4]. They both apply to the 2d anti-plane shear case, that is item 1w in the previous list with $N = 2$. Reference [15] is the first existence theorem ever proved in the variational setting. It does so at the expense of the imposition of a topological restriction on the cracks, that is, roughly, that they be connected. By contrast, the later reference [4] states that, in the weak setting, one can choose $\Gamma(t)$ to be closed and $\varphi(t)$ to be in $H^1(\Omega \setminus \gamma(t))$.

In a static setting, that is for example at time $t = 0$ in the evolution, one only has to minimize

$$\int_\Omega W(\nabla\varphi)\, dx + G_C \mathcal{H}^1(S(\varphi))$$

among all φ's that are equal to $g := g(0)$ on $\partial_d\Omega \setminus S(\varphi)$. In that case, one more very recent result is available for the linearly elastic framework [9]. It states that, in that setting as well with A as Hooke tensor, there is existence (in some functional space that we will not explicit here) of a minimizer u^* for

$$\int_\Omega 1/2 Ae(u) \cdot e(u)\, dx + G_C \mathcal{H}^1(S(u))$$

among all u's that are equal to $g := g(0)$ on $\partial_d\Omega \setminus S(u)$, a highly non trivial result to start with. Further, it proves that one can choose u^* so that $\mathcal{H}^{N-1}\left(\overline{S(u^*)} \setminus S(u^*)\right) = 0$ and u^* itself lies in $H^1\left(\Omega \setminus \overline{S(u^*)}; \mathbb{R}^N\right)$ (and even better spatial regularity properties for u).

For now, this is it! It is unfortunate, but maybe unavoidable that mathematics and physics live on two drastically different time-scales. This should not thwart the mechanician, or even the computational mathematician in his effort to test the model against more complex situations. Correspondingly, this should not turn the mathematical effort into ridicule for its excrutiatingly slow pace.

As a final note, all above considerations only hold true when Ω is a Lipschitz domain. In particular, if a pre-crack Γ_- is already present in Ω, we are not at liberty to choose Ω to be $\Omega \setminus \Gamma_-$. Rather, we should modify the evolution at time $t = 0$ to ensure that all test cracks contain Γ_-, or, equivalently, replace $\mathcal{H}^{N-1}(\Gamma)$ by $\mathcal{H}^{N-1}(\Gamma \setminus \Gamma_-)$ in the formulation of the evolution.

3 Nucleation, or where Griffith goes awry

As told thus far, our story should equally apply to nucleation. In our language, this amounts to deciding at which time t the crack $\Gamma(t)$ will be such that $\mathcal{H}^{N-1}(\Gamma(t')) \neq 0, t' > t$, or if a pre-crack

Γ_- is present such that $\mathcal{H}^{N-1}(\Gamma(t') \setminus \Gamma_-) \neq 0, t' > t$.

With this in mind, let us consider a monotonically increasing load $g(t) = tg$ where g is a set, time-independent hard device with the required spatial regularity (say for example $H^{1/2}(\partial_d\Omega; \mathbb{R}^N)$ in the linearly elastic case). Then it is clear that, if global minimality is the name of the game, initiation will always occur before time $\bar{t}$ defined as the time for which

$$\bar{t}^2 \int_\Omega 1/2 Ae(u_{el}) \cdot e(u_{el})\; dx = G_C \mathcal{H}^1(\partial_d\Omega)$$

with u_{el} the unique elastic minimizer to

$$\int_\Omega 1/2 Ae(u) \cdot e(u)\; dx$$

among u's in $H^1(\Omega; \mathbb{R}^N)$ with $u = g$ on $\partial_d\Omega$. Indeed, for larger t's, it is cheaper to cut the whole boundary away with a crack, thus driving the elastic energy to 0 while only paying the surface area of $\partial_d\Omega$.

In other words, global minimization will always promote nucleation and thus, as remarked before in the tearing example, endow the crack, or absence thereof, with divine foresight. A natural question would then be whether such a preposterous result can be cured with local minimality. In other words, will nucleation occur when the crack is limited in its ability to explore the entirety of the energetic landscape. The answer can be found in [11] in a specific setting which we now describe.

It is first assumed that the overall setting is that of anti-plane shear with $N = 2$, or else planar linear elasticity. Further, putative cracks are constrained to be closed sets with finite length and a pre-set maximal number p of connected components. So the setting is by its very definition a strong setting. Finally, it is assumed that every point in Ω is a weak singularity for the elastic solution u_{el}, and this uniformly, that is that there exists $C > 0$ such that

$$\int_{B(x,r)\cap\Omega} |\nabla u_{el}|^2\; dx \leq Cr^\alpha$$

for some constant $\alpha > 1$ and for every ball $B(x, r)$ with radius r and every $x \in \bar{\Omega}$. This is the case for example if the elastic solution u_{el} for g is C^1 on $\bar{\Omega}$. In such a case it is proved in [11] that there exists a $\bar{\ell} > 0$ such that

$$\int_\Omega W(e(u_{el}))\; dx < \int_{\Omega\setminus K} W(e(v))\; dx + G_c\mathcal{H}^1(K),$$

for any closed set $K \subset \bar{\Omega}$ with at most p connected components and with $\mathcal{H}^1(K) \leq \bar{\ell}$.

This is a devastating result for nucleation because it states that local minimality in any topology that will let cracks with small enough crack lengths be admissible competitors will prohibit nucleation in a crack-free domain. Of course this case does not cover that of a pre-crack that would re-nucleate because the singularity at the crack tip is not weak (in fact $\alpha = 1$ in such a setting). But the emerging picture, when combining this with the previously evoked result, is bleak: Either resort to global minimality which is an otherwordly statement of crack self-awareness, or give up on nucleation altogether.

Once again, let us stress that the culprit is not the variational attitude, but rather the Griffith attitude. Brittle fracture is not a story that includes nucleation. The observant reader will object that many phase-field models do seem to capture nucleation. Maybe so, but for reasons that have little to do with the sharp theory. For more on this we direct the reader to [23] where nucleation in a phase-field context is debated at length. As argued there, nucleation should most likely be viewed as a threshold on the stress field while propagation should be within the purview of Griffith's theory. The ontological nature of those conflicting imperatives is a steep challenge that we are at a loss to fully confront at present.

An astute overseer of fracture might call our attention to cohesive fracture because it does generate a threshold for nucleation as explained in e.g. [8, Chapter 4]. Unfortunately, cohesive fracture is no panacea. It perniciously spreads fracture throughout the bulk of the material inducing a kind of plastic behavior which militates against well-defined cracks. This is because any minimization principle in the spirit of that implemented in (9) above fails to produce a minimizer, even at the initial time. Relaxation occurs resulting in a bulk energy with linear growth at infinity, together with the possibility of diffuse cracking in the form of a Cantor part for the gradient of the deformation; see e.g. [7] where a static minimization problem involving both a bulk energy and a surface energy à la Barenblatt is investigated. Further, the evolution also raises delicate questions of irreversibility and of the dissipative cost of crack creation.

The relevant mathematics are still in their infancy, even for cohesive models that are woefully oversimplistic. As of today, there are no existence results for cohesive fracture evolution, except for a very specific example in the simplest 1d setting [6].

Of course, mathematical impotence does not impugn physical soundness and it may turn out that a well-rounded model of cohesive fracture will resolve the nucleation conundrum while leaving Griffith fracture otherwise undisturbed.

4 Energy release rates and kinking, or not...

Although a familiar concept to the practitioner of fracture, energy release rates are shrouded in the veils of imprecision in the mechanics literature. We will only address the 2d setting since the 3d setting is messier by several orders of magnitude. The idea is very simple. Assume a pre-crack γ, extend that crack in some direction by a small straight line segment originating at the tip and compute the derivative of the potential energy of the system as the add-crack length goes to 0, this for a fixed set of boundary conditions on the boundary $\partial\Omega$ of the domain .

This simple recipe soon leads to fracture pandemonium. First, if the pre-crack is not (essentially) a straight line segment near the crack tip, then the $\sqrt{r}$-singularity at that tip is folklore, rather than fact.[8] Then, the existence of an energy release rate associated with the process described above is usually not addressed directly. Rather, Irwin's formula which relates an hypothetical energy release rate to the stress intensity factors is appealed to and the limit of those intensity factors as the add-crack length tends to 0 is computed. This is usually a very painful task if tackled with care (see [25]). It is also powerless to address more complex topologies for the add-cracks.

A good definition of the energy release rate starts with the idea that the family of connected add-cracks $\Gamma(\ell)$ should be such that $\Gamma(\ell)$ decreases with decreasing ℓ's and that their length behaves

[8] To our knowledge one has to appeal to results in [22] which deal exclusively with polygonal domains.

like ℓ as $\ell \searrow 0$, *i.e.*,

$$\begin{cases} \Gamma(\ell') \subset \Gamma(\ell),\ \ell' \leq \ell \\ \lim_{\ell \to 0} \dfrac{\mathcal{H}^1(\Gamma(\ell))}{\ell} = 1. \end{cases} \tag{13}$$

Then, we will call energy release rate the limit, if it exists, of

$$-1/\ell \left\{ 1/2 \int_\Omega (Ae(u_{\Gamma(\ell)}) \cdot e(u_{\Gamma(\ell)}) - Ae(u_\emptyset) \cdot (u_\emptyset)\ dx \right\}$$

where u_Γ is the elastic solution to the elasticity problem with $\Omega \setminus (\gamma \cup \Gamma)$ as domain. Such a limit exists as demonstrated in [10, Section 3], at least for a subsequence of ℓ's. Actually, this is so under the only condition that the Hausdorff limit Γ_0 (the limit in the sense of the distance between two curves) of $\Gamma(\ell)/\ell$ exists, that is without (13). Further, it is shown there that the limit only depends on Γ_0; this will be a connected closed set of length at most 1. We will thus denote this energy release rate by $\mathcal{G}(\Gamma_0)$.

That limit is given through a minimization problem in $\mathbb{R}^2$ with a semi-infinite straight crack in the direction of γ at the tip of which Γ_0 is added. As such, $\mathcal{G}(\Gamma)$ can be defined for any closed set Γ with $\mathcal{H}^1(\Gamma) \leq 1$.

With the knowledge that, for a large collection of $\Gamma(\ell)$ that includes the classical view of a straight add-crack with length $\ell \searrow 0$, Γ_0 is a line segment of length 1, one is naturally led to investigating

$$\mathcal{G}_{max} := \sup\{\mathcal{G}(\Gamma) : \Gamma \text{ closed }, \mathcal{H}^1(\Gamma) \leq 1\}.$$

That supremum can be shown to be a maximum. However, provided that the pre-crack is not in mode I, that is that the stress intensity factor K_{II} at the crack tip is not 0, the maximal value is not attained by a straight line segment of length 1 (see [10, Section 4]), but by some set Σ which looks like a hook (two line segments joined at one end with a non-zero angle).

That result does not produce a sequence $\Gamma(\ell)$ that satisfies (13) for which the associated energy release rate would be maximal, but only a sequence $\varepsilon\Sigma$ such that

$$\lim_{\varepsilon \to 0} -1/2\varepsilon \int_\Omega (Ae(u_{\varepsilon\Sigma}) \cdot e(u_{\varepsilon\Sigma}) - Ae(u_\emptyset) \cdot (u_\emptyset))\ dx = \mathcal{G}_{max}. \tag{14}$$

This is because $\varepsilon'\Sigma \not\subset \varepsilon\Sigma$ for $\varepsilon' < \varepsilon$.

Yet this result has tremendous consequences. If a pre-crack which is not in mode I is assumed to extend from time $t = 0$ continuously in time and length from its tip along a smooth path, then there is universal agreement that it should do so while keeping its energy release rate $\mathcal{G}(t)$ (a meaningful notion in that setting) equal to G_c. So, in particular $\mathcal{G}(0) = G_c$. But, if at time $t = 0$, Griffith's postulate of local minimality applies, then, for any test crack $\varepsilon\Delta$ with $\mathcal{H}^1(\Delta) \leq 1$, we must have

$$-1/2\varepsilon \int_\Omega (Ae(u_{\varepsilon\Delta}) \cdot e(u_{\varepsilon\Delta}) - Ae(u_\emptyset) \cdot (u_\emptyset))\ dx \leq G_c \tag{15}$$

for ε small enough. But $\mathcal{G}(0) < \mathcal{G}_{max}$ according to the previous considerations. Since, for $\Delta = \Sigma$ (15) and (14) imply that $\mathcal{G}_{max} \leq G_c$, we get that $\mathcal{G}(0) < G_c$, a contradiction.

In other words, assuming Griffith's local minimality, pre-cracks cannot grow smoothly in time and length from their crack tips along a smooth path unless in mode I.[9]

Consequently, the mechanician's notion of a kink (a sudden change of direction in the crack path) is incompatible with Griffith's minimality principle. Throughout the decades since Griffith's work, various criteria have been put forward and bitterly fought for in the name of crack path freedom. Most notable among those are the principle of local symmetry and the G_{max}-principle. They were meant to expand Griffith's vision of crack growth. Yet, in view of the above, they cannot coexist with Griffith. But, in his absence, they become meaningless..... Should we venture to posit that there is no such thing as a kink?

Adjudication in the matter at hand can only come from the conflation of finely tuned experiments with a more complex view of crack extension. Or else one should completely abandon Griffith's theory and thus upend fracture mechanics in unprecedented ways.

5 Beyond Griffith

From the very moment that Griffith's variational position was re-asserted in [19], minimality, be it global or local, was assailed in the mechanics community. The detractors often appealed to the artificial nature of the minimality criterion which did not derive from any first principle. Those were sometimes the very people who saw no ill in viewing elastic equilibrium in a finite deformation context as the result of a minimization principle for the elastic energy, an equally unmotivated principle.

As we saw in the previous sections, minimality is fundamental in the mathematical rejection of the classical view of two dimensional kinking while it also plays a pivotal role in the possibility of crack nucleation within the advocated framework.

So, removing minimality altogether is indeed a natural question. In e.g. finite elasticity, doing so amounts to viewing the Euler-Lagrange equations associated with energy minimization as the prime engine for equilibrium. There, the main issue is that, while minimizers of the energy do exist within the framework developed by J.M. Ball [5], one does not know how to obtain solutions for the associated Euler-Lagrange equations because small outer variations of the minimizers, the usual staple for deriving those from a minimization principle, do not respect the requirement that orientation be preserved (the positive determinant constraint).

In the setting of brittle fracture, the situation is more dire. What are the Euler-Lagrange equations associated with the variational evolution? In the Mumford-Shah image segmentation problem (8), one can venture to generate Euler-Lagrange equations through smooth outer variations of the field φ and inner variations of the field φ consisting in replacing x by $x+\varepsilon\eta(x)$ with $\eta \in C^1$ in φ. Provided that $\overline{S(\varphi)}$ is a graph of the form $\{z \in \mathbb{R}^{N-1}, \psi(z)\}$ with $\psi \in C^{1,\gamma}$ for some $0 < \gamma < 1$,

[9] Actually the result is more general and does not allow cracks whose length is asymptotically like that of a line segment as that length tends to 0. Also the assumption that the pre-crack is straight can be greatly relaxed (see [3]).

the resulting equations turn out to be (see [2, Chapter 7.4])

$$\begin{cases} \triangle\varphi = \varphi - g \text{ in } \Omega \\ \dfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \overline{S(\varphi)}^{+} \cup \overline{S(\varphi)}^{-} \\ -k \text{ div}\left(\dfrac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = [|\,|\nabla\varphi|^2 + |\varphi - g|^2|] \end{cases} \tag{16}$$

where $[|f|]$ stands for the jump of f and the superscripts $\pm$ stand for the upper and lower "sides" of the jump set $S(\varphi)$.

Unfortunately, similar equations do not hold in fracture because irreversibility prohibits the type of inner variations used in deriving (16). Furthermore, even in the Mumford-Shah setting, it is not clear that (16) are the sum total of all possible independent equations that a minimizer satisfies. The illusion that some kind of system of equations should govern crack evolution is born out of the derivative view of fracture evolution as a statement about energy release rates. But, as explained in the previous section, the very notion of energy release rate is a fraught one because it hinges on the kind of topological feature that one is willing to impart on the possible add-cracks.

We are now at a mathematical and modeling crossroad. The mathematical backbone of Griffith's fracture universe is nearly fully grown. The resulting picture is that of a truncated universe in which nucleation will forever be clothed in esotericism and kinking will not be tolerated. Yet fracture can easily strive in such an environment and produce, through well articulated approximations, results with a qualitative and quantitative fit. With a modicum of effort, it may even be so that those approximations will capture nucleation and reconcile with kinking.

The choice is ours. Should we adopt the somewhat dubious physics underlying those approximations and forfeit the sharp theory imagined by Griffith and its next of kin. In such a case, fracture is passé and damage gradient models reign supreme. Of course one will have to tinker with those models as suggested, perhaps clumsily, in [23].

Or should we seek a sharp substitute for Griffith's theory which would prompt us to explore an alter-verse. The only one that seems even remotely accessible for now is cohesive fracture. But, at this juncture, it is a perilous journey because we know next to nothing when it comes to cohesiveness, in spite of a slew of recent efforts in that direction.

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