QUASI-STATIC EVOLUTION FOR THE ARMSTRONG-FREDERICK HARDENING-PLASTICITY MODEL

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ABSTRACT. The Armstrong-Frederick model for nonlinear kinematic hardening is regarded as a benchmark model in contemporary elastoplasticity. This work presents an existence result to an appropriately time-rescaled evolution for that model. To do so, we have to resort to a regularization of the dependence of the convex of plasticity upon the back stress. Such a regularization process seems to be the unfortunate price one has to pay for a successful mathematical analysis.

1. INTRODUCTION

The modeling and prediction of plastic effects has a long history. The first attempts are usually traced back to the observations by Tresca [61] on the occurrence of yield stresses in metal solidification and to the introduction by St. Venant [6] of constitutive equations in plane stress for rigid perfect plasticity. In turn, Lévy [37], von Mises [62] and, later, Prandtl [53] and Reuss [55] investigated the three-dimensional setting. The variational description of perfect rigid elasto-plasticity, as we understand it today, was settled by von Mises [63] early on in 1928. A fundamental tenet of plasticity consists is assuming the the *stress* σ experienced by the body cannot exceed some given yield. Namely, one asks that, throughout the evolution, $\sigma \in K$ for some given *elastic domain* K, a convex subset of $\mathbb{M}^{n\times n}_{\rm sym}$ (symmetric matrices).

A first refinement of the model takes *hardening* effects into account. Hardening modifies the mechanical response of materials by encoding the history of the plastic deformation. It is commonly interpreted as the macroscopic manifestation of dislocation migration. The earliest effort in that direction is attributed to Prandtl [54] although the current formulation of elasto-plasticity with *linear kinematic hardening* was settled at a later stage with the work of Melan [42] and Prager [52]. In a nutshell, the yield criterion is modified and becomes $\sigma - \chi \in K$. The additional stress χ is the so-called *back stress*; in the linear case, it is assumed to be related to the plastic strain p of the material by $\dot\chi = B\dot p$ where B is a given hardening tensor. From that point on, many models have been put forth with a view to a more intricate phenomenology: viscous and thermal effects, solid-solid phase changes . . . The reader is referred to the classical monographs by Hill [36], Lemaitre & Chaboche [38], Lubliner [40], Maugin [41] among others.

Linear kinematic hardening shifts the elastic domain K proportionally to the back stress. In particular, it cannot capture the so-called *Bauschinger effect*, that is the observation that the plastic history of a body determines the resistance of the material to further plasticization. It is often the case that the elastic limit in compression is lowered by a previous tensile loading and vice-versa. The crucial relevance of this effect in applications has triggered the interest for developing suitably *nonlinear kinematic hardening* models. Among these, we focus here on the classical contribution by Armstrong & Frederick [3] where the idea is to add a nonlinear correction term to the rate equation for the plastic strain rate rate. In particular, the flow rule driving the

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back stress χ is augmented as $\dot{\chi} + |\dot{p}|F\chi = B\dot{p}$ where F is a given tensor. Such a modification entails the boundedness of the back stress χ, a desirable feature in many applications. But there is a drawback: the *normality principle* [36] driving the evolution turns out to be state-dependent.

The mathematical analysis of plastic evolution problems originates in the 70s. Well-posedness, regularity, and approximation of the displacement, stress, and plastic strain fields became the main focus. Early existence results in the viscous or hardening cases can be found in the classical monograph by Duvaut & Lions [21] and in Moreau [50]. The more difficult perfectly-plastic case is then tackled in Suquet [58, 59] and Johnson [31, 33] (see also [35] and the monograph [60]). On the numerical side, finite element approximations in plasticity were pioneered by Johnson [32, 34].

After a twenty year lull and inspired by the work of Ortiz & Repetto [51], the interest of the mathematical comunity in plasticity was re-kindled in various works of Mielke together with many collaborators, see e.g. [43] and references therein.

As far as pure small strain elasto-plasticity is concerned, Dal Maso, DeSimone, & Mora [13] revisited the results of Suquet and Johnson in the setting of *energetic formulations* of rate-independent evolutions [27, 44]; see also a first attempt in [22]. That reformulation underlines the relevance of energy conservation and stability and paves the way for a direct use of variational – lower semicontinuity – techniques in this setting. In particular, plastic evolution is obtained as the limit of sequences of time-discretized plastic flows, themselves a result of incremental minimization. In the specific context of plasticity, the viewpoint espoused in [13] proved subsequently successful in the investigation of pressure-sensitive materials [12], brittle materials [19] and, in the setting of hardening, of shape memory alloys [2] and of softening [14, 15]. Moreover, energetic formulations have been considered in the context of strain-gradient plasticity [28, 29], heterogeneous materials [25, 26, 56, 57], homogenization [24, 26, 29, 30, 48], dimensional reduction [20, 39], and also used to derive small-strain plasticity from a model at finite strain in [47].

Absent a standard, state-independent, normality postulate, energetic formulations are also relevant in spite of their variational bias. Dal Maso, DeSimone, & Solombrino [16, 17, 18] investigate the energetic solvability of the so-called *Cam-Clay* model for plasticization in soils. There, the model features an explicit dependence of the elastic domain upon an adequate internal variable. More recently, [4] focuses directly on non-associative models of Mohr-Coulomb or Drucker-Prager type. The first step of the analysis consists in a reformulation of the original plastic model as a quasi-variational evolution inequality [5]. We shall follow the same path below, see Section 2.

The available mathematical contributions to the Armstrong-Frederick nonlinear kinematic hardening model are few. Brokate & Krejčí [8, 9, 10] consider the well-posedness of the constitutive model. The ODE tensorial material relation is proved to admit unique solutions both in the stress-controlled and the strain-controlled case. These papers observe that the Armstrong-Frederick model – and, more generally, the Mróz and the Chaboche models – can be reformulated as a system of an ODE, together with a hysteretic relation. However such a reformulation entails a coordinate change which does not pair well with the equilibrium relation. The only available three-dimensional result for the Armstrong-Frederick model available so far is by Chełmiński [11]: well-posedness of a suitable viscous regularization of the original problem is discussed. However, the obtained a priori estimates are not sufficient to pass to the limit as the viscosity goes to zero in the nonlinear setting of the original problem.

This paper considers the Armstrong-Frederick model in its full three-dimensional setting. The constitutive relation is coupled with the system resulting from quasi-static equilibrium. At first, we recast the model in an equivalent quasi-variational form (Section 2). This results in a dissipation pseudo-potential that explicitly depends on the back stress. Then, we operate a viscous regularization of the model (Section 3) in the spirit of [11] and [59]. The existence of the visco-plastic regularization, an interesting result per se, is obtained through a stable and convergent time-discretization procedure (cf. [11]). We then definitely depart from the approach of [11] in passing to the the non-viscous limit (Section 4). In particular, we establish the quasi-static limit with respect to some properly rescaled time by following an approach first advocated by Efendiev & Mielke [23], see also the recent [45, 46]. That rescaling has already been applied in the plasticity context in [4, 16, 17].

Our result is the first existence result for a the quasi-static rate-independent plastic evolution driven by an Armstrong-Frederick-type model with nonlinear kinematic hardening. Note however that passing to the 0-viscosity limit forces us to focus on a mollification of the constitutive equation by means of a convolution kernel, see Section 2 below. This modification is needed to secure the crucial lower-semicontinuity of the dissipation pseudo-potential. The analysis of the de-mollified Armstrong-Frederick model seems to be out of reach for now.

We stress that our regularization by convolution can be expected to have a moderate impact on the effective material behavior as it acts in space only and may be assumed to be very localized. As such, we claim that it is a worthy compromise toward a better understanding of the Armstrong-Frederick model. We comment on this and other regularizations in the short conclusion (Section 5).

2. Description of the model

This section is devoted to recall the basic features of the model, as well as to the necessary background mathematical material.

2.1. **Notation.**

We denote by $\mathbb{M}^{n\times n}$ the space of 2-tensors in $\mathbb{R}^n$ $(n = 1, 2, 3)$ and by $\mathbb{M}^{n\times n}_{\rm sym}$ and $\mathbb{M}^{n\times n}_{\rm dev}$ the subspaces of symmetric and symmetric-deviatoric tensors. The space $\mathbb{M}^{n\times n}_{\rm sym}$ is naturally endowed with the scalar product $a : b = a_{ij}b_{ij}$ (summation convention) and the corresponding norm $|a|^2 := a : a$ for all $a, b \in \mathbb{M}^{n\times n}_{\rm sym}$. Moreover, $\mathbb{M}^{n\times n}_{\rm sym}$ is orthogonally decomposed as $\mathbb{M}^{n\times n}_{\rm sym} = \mathbb{M}^{n\times n}_{\rm dev} \oplus \mathbb{R}1_2$ where $\mathbb{R}\, 1_2$ is the subspace spanned by the identity 2-tensor. In particular, for all $a \in \mathbb{M}^{n\times n}_{\rm sym}$, we let $a = a_D + tr(a)1_2/n$. The symbols $\otimes$ and $\odot$ stand for the tensor product and the symmetrized tensor product, respectively. Namely, $(u \otimes v)_{ij} = u_iv_j$ and $(u \odot v)_{ij} = (u_iv_j+u_jv_i)/2$ for all $u, v \in \mathbb{R}^n$.

Given a Banach space E and a convex functional $\varphi : E \to (\infty, \infty]$ we let $D(\varphi) = \{x \in E : \varphi(x) < \infty\}$ denote its *effective domain* and $\partial\varphi : E \to E^*$ (dual) be its *subdifferential* (possibly multivalued) defined as

$$y \in \partial\varphi(x) \iff x \in D(\varphi) \ \text{ and } \ \langle y, w-x\rangle \le \varphi(w) - \varphi(x)\ \forall w \in E.$$

Here, the symbol $\langle\cdot,\cdot\rangle$ corresponds to the *duality pairing* between E^* and E. For instance, given the nonempty, convex, and closed set $K \subset E$, its *indicator function* $I_K : E \to [0, \infty]$ defined as

$$I_K(x) = \begin{cases} 0, & x \in K \\ \infty, & \text{else} \end{cases}$$

is convex, proper, lower semicontinuous and its subdifferential is

$$y \in \partial I_K(x) \iff x \in K \ \text{ and } \ \langle y, w-x\rangle \le 0\ \forall w \in K.$$

In other words, $\partial I_K = \{0\}$ in the interior of K, $\partial I_K = \{r\lambda\}$ at ∂K where $r \ge 0$ and λ is an outward normal to ∂K (possibly one of the many, due to non-smoothness), and $\partial I_K = \emptyset$ outside K.

2.2. **The original model revisited.** The context is that of small strains. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set occupied by a homogeneous elasto-plastic material. We denote by $u : \Omega \to \mathbb{R}^n$ the displacement field and by $Eu := (Du + Du^T)/2$ the strain tensor. As is usual in small deformations plasticity, the strain tensor is *additively* decomposed as

$$Eu = e + p,$$

where $e \in \mathbb{M}^{n\times n}_{\text{sym}}$ and $p \in \mathbb{M}^{n\times n}_{\text{dev}}$ respectively stand for the elastic and plastic strains. This is part of what will be referred to as *kinematic compatibility.* The constitutive equation which relates the (Cauchy) stress tensor σ to the elastic part e of the linearized strain is also assumed to be linear, *i.e.*,

$$\sigma = Ae$$

where A is the Hooke elasticity tensor. In the isotropic case, $A = K(1_2 \otimes 1_2) + 2G(1_4 - 1_2 \otimes 1_2/n)$ where $K, G > 0$ are the bulk and the shear modulus and 1_m is the identity m-tensor in $\mathbb{R}^n$. At equilibrium, and if no volume forces are applied to the sample, the stress satisfies

$$\text{div}\sigma = 0 \quad \text{in } \Omega.$$

In the Armstrong-Frederick model, an additional kinematic hardening variable $\alpha \in \mathbb{M}^{n\times n}_{\text{dev}}$ is also introduced. The back stress $\chi \in \mathbb{M}^{n\times n}_{\text{dev}}$ is then related to that variable and to the plastic strain through

$$\chi = B(p - \alpha)$$

where B is a suitable positive-definite symmetric fourth-order tensor. We will call $p - \alpha$ the *back strain.*

We then introduce the internal energy

$$W(e, p - \alpha) := \frac{1}{2}Ae : e + \frac{1}{2}B(p - \alpha) : (p - \alpha).$$

Viewing W as a function of Eu, p, α it also reads as

$$\hat{W}(Eu, p, \alpha) = W(Eu - p, p - \alpha).$$

The thermodynamic force $-\dfrac{\partial \hat{W}}{\partial p} = \sigma_D - \chi$ associated with p is constrained to remain in a compact convex subset K of the set $\mathbb{M}^{n\times n}_{\text{dev}}$:

$$\sigma_D - \chi \in K := \{\tau \in \mathbb{M}^{n\times n}_{\text{dev}} : f(\tau) \le 0\},$$

where $f : \mathbb{M}^{n\times n}_{\text{dev}} \to \mathbb{R}$ is the *yield function.* We assume that

$$\begin{aligned} &f \text{ is convex and Lipschitz} \\ &-f_0 := f(0) < 0 \text{ and } f(0) = \min\{f(\tau); \tau \in \mathbb{M}^{n\times n}_{\text{dev}}\} \\ &\hat{f} := f + f_0 \text{ is positively 1-homogeneous.} \end{aligned} \tag{2.1}$$

In particular, $0 \in \text{int}\, K$.

At each point τ on ∂K we define $N_K(\tau)$ to be the unit exterior normal cone to K at that point, i.e.,

$$N_K(\tau) := \left\{\nu \in \mathbb{M}^{n\times n}_{\text{dev}} : |\nu| = 1 \text{ and } \nu : (\eta - \tau) \le 0, \forall \eta \in K\right\}. \tag{2.2}$$

Now, the thermodynamic force associated with α is

$$-\frac{\partial \hat{W}}{\partial \alpha} = \chi.$$

The Armstrong-Frederick hardening model is characterized by the following flow rule:

$$\big[\dot{p}\big|\dot{\alpha}\big] \in \mathcal{A}(\sigma_D, \chi)$$

where

$$\mathcal{A}(\sigma_D,\chi) := \left\{\lambda\left[\nu\Big|\frac{\nu:(\sigma_D-\chi)}{f_0}B^{-1}F\chi\right] : \nu\in N_K(\sigma_D-\chi) \text{ and } \lambda\ge 0;\ \lambda=0 \text{ if } f(\sigma_D-\chi)<0\right\}.$$

From here onward, the notation $\big[\alpha\big|\beta\big]$ stands for the generic 2-vector of tensors in $\mathbb{M}^{n\times n}_{\text{dev}}$. The tensor F is an additional positive-definite fourth-order tensor which we will take to be equal to B in the remainder of the paper, and this with no loss of generality.

In all fairness, the classical Armstrong-Frederick model is usually restricted to the Von Mises seting in which case $f(\tau)=|\tau|-f_0$. The previous flow rule can then be rephrased in the following form:

$$\begin{aligned}\dot p &= |\dot p|\frac{\partial f}{\partial\tau}(\sigma_D-\chi)\\ \dot\chi + |\dot p|B\chi &= B\dot p,\end{aligned}\tag{2.3}$$

which is that most often encountered in the literature.

Our goal is to obtain a quadruplet $(u(x,t),e(x,t),p(x,t),\alpha(x,t))$ such that

$$\begin{cases} Eu(x,t)=e(x,t)+p(x,t) & \text{kinematic compatibility}\\ \sigma(x,t)=Ae(x,t),\ \chi(x,t)=B(p(x,t)-\alpha(x,t)) & \text{constitutive relations}\\ \operatorname{div}\sigma(x,t)=0 & \text{equilibrium}\\ \sigma_D(x,t)-\chi(x,t)\in K & \text{stress constraint}\\ \big[\dot p\big|\dot\alpha\big](x,t)\in\mathcal{A}(\sigma_D(x,t),\chi(x,t)) & \text{flow rule}\end{cases}$$

together with the Dirichlet boundary condition $u=w$ on $\partial\Omega$. We know from prior works on plasticity that the boundary condition will not always be satisfied because plastic strains may develop at the boundary, so that, as seen later, we will have to replace that condition by

$$p(x,t)=(w-u)(x,t)\odot\nu \text{ on } \partial\Omega$$

where ν stands for the unit normal to $\partial\Omega$.

The resulting model has resisted any incorporation attempt within a standard generalized thermodynamical framework. To our knowledge, there are no existence theorems for such an evolution. We propose to remedy this, albeit on a slightly regularized form of the evolution.

Inspired by prior work on non-associative elasto-plasticity [4], we propose to rewrite the stress constraint and the flow rule in an "equivalent" way. This is done through the introduction, for any $\chi\in\mathbb{M}^{n\times n}_{\text{dev}}$, of the set

$$K(\chi) := \left\{\big[\tau\big|\eta\big]\in\mathbb{M}^{n\times n}_{\text{dev}}\times\mathbb{M}^{n\times n}_{\text{dev}} : f(\tau)+\frac12\eta:\eta\le\frac12\chi:\chi\right\};\tag{2.4}$$

see Figure 1. In particular, recalling that we have set $B=F$ for the sake of notational symplicity, we have the

Lemma 2.1. *The following holds true:*

a) $\sigma_D-\chi\in K \iff \big[\sigma_D-\chi\big|\chi\big]\in K(\chi)$;
b) $\sigma_D-\chi\in \partial K \iff \big[\sigma_D-\chi\big|\chi\big]\in \partial K(\chi)$;
c) $\big[\dot p\big|\dot\alpha\big]\in\mathcal{A}(\sigma_D,\chi) \iff \big[\dot p\big|\dot\alpha\big]\in\partial I_{K(\chi)}\big[\sigma_D-\chi\big|\chi\big]$.

Proof. We have that $\sigma_D-\chi\in K$ iff $f(\sigma_D-\chi)\le 0$ iff $f(\sigma_D-\chi)+\chi^2/2\le 0+\chi^2/2\le\chi^2/2$ iff $\big[\sigma_D-\chi\big|\chi\big]\in K(\chi)$.

Similarly, we can prove that $\sigma_D-\chi\in\partial K$ iff $f(\sigma_D-\chi)=0$ iff $f(\sigma_D-\chi)+\chi^2/2=\chi^2/2$ iff $\big[\sigma_D-\chi\big|\chi\big]\in\partial K(\chi)$.

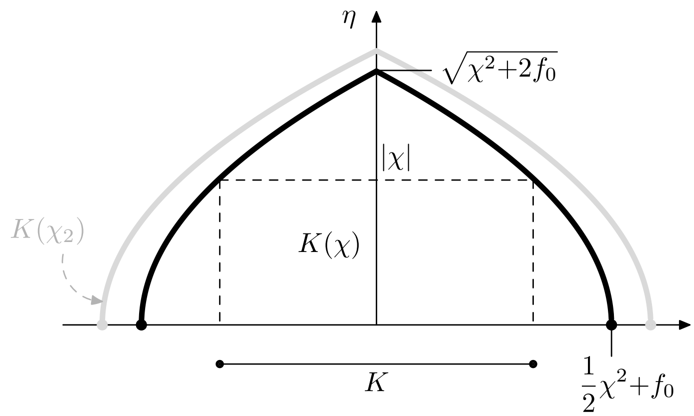

FIGURE 1. The set $K(\chi)$ for $f(\tau)=|\tau|-f_0$. The horizontal axis represents $\mathbb{M}^{n\times n}_{\rm dev}$

The third equivalence is a bit less immediate. Note that $K(\chi)$ is equivalently defined as

$$K(\chi)=\left\{\big[\tau\big|\eta\big]\in\mathbb{M}^{n\times n}_{\rm dev}\times\mathbb{M}^{n\times n}_{\rm dev}:\hat f(\tau)+\frac{1}{2}\eta:\eta\le\frac{1}{2}\chi:\chi+f_0\right\}.$$

Then, $\big[\dot p\big|\dot\alpha\big]\in\partial I_{K(\chi)}\big[\sigma_D-\chi\big|\chi\big]$ iff

$$\dot p:(\tau-(\sigma_D-\chi))+\dot\alpha:(\eta-\chi)\le 0,\ \forall\,\big[\tau\big|\eta\big]\in K(\chi). \tag{2.5}$$

In particular, taking successively $\tau=\sigma_D-\chi$ and $\eta=\chi$ in (2.5), we get that $\dot p=\lambda\nu,\ \nu\in N_K(\sigma_D-\chi)$ and $\dot\alpha=\lambda'\chi$ with $\lambda,\lambda'\ge 0$.

Now, consider $\tau=(1-s)(\sigma_D-\chi)$ $|s|<<1$ and seek η such that $\big[\tau\big|\eta\big]\in\partial K(\chi)$. A simple computation that uses the one-homogeneous character of $\hat f$ leads to $\eta=\sqrt{1+2\dfrac{s}{\chi:\chi}f_0}\;\chi$. Inserting thet $\big[\tau\big|\eta\big]$ into (2.5) yields

$$s\big(-\lambda\nu:(\sigma_D-\chi)+\lambda' f_0\big)+{\rm o}(s)\le 0,$$

hence $\lambda'=\lambda\,\dfrac{\nu:(\sigma_D-\chi)}{f_0}$. $\square$

The transformation devised through Lemma 2.1 highlights the dependence of the flow rule from the state variable χ, thus leading to a so-called quasi-variational inequality.

Unfortunately, as will become clear later, this reformulation does not provide a suitable functional framework for the analysis, most notably because the duality product between the stresses and the plastic strains cannot be successfully defined in the absence of an L^∞-bound on σ_D. But such a bound seems unattainable, unless an L^∞-bound is derived for χ, in which case it becomes trivial since $\sigma-\chi\in K$. We do not know how to obtain such a bound when starting with the definition (2.4) of $K(\chi)$.

To achieve such a bound, we modify Definition 2.4 and incorporate an *a priori* bound on χ in that definition. We set

$$K_M(\chi):=\left\{\big[\tau\big|\eta\big]\in\mathbb{M}^{n\times n}_{\rm dev}\times\mathbb{M}^{n\times n}_{\rm dev}:f(\tau)+\frac{1}{2}|\eta|^2\le\frac{1}{2}T_M(|\chi|^2)\right\} \tag{2.6}$$

where $T_M(r):=\min\{|r|,M\}$. Of course, the previously noted equivalence between the original formulation and the formulation with $K_M(\chi)$ does not hold any longer, at least when $|\chi|>\sqrt{M}$. We will demonstrate at the end of the paper that a proper choice of M actually ensures that the constraint $|\chi| \leq M$ is not saturated, at least for small times, provided that the initial condition on χ is so (see Proposition 4.14).

Note that, in view of the last item in (2.1),

$$\left[\tau\middle|\eta\right] \in K_M(\chi) \Rightarrow |\eta| \leq \sqrt{M - 2f(0)} =: M'. \tag{2.7}$$

We next define the dissipation potential $H_M : (\mathbb{M}^{n\times n}_{\text{dev}})^3 \to \mathbb{R}$ as the support function of $K_M(\chi)$, that is,

$$H_M(\chi, \left[p\middle|\alpha\right]) = \max_{\left[\tau\middle|\eta\right]\in K_M(\chi)} \tau : p + \eta : \alpha,$$

which, for a fixed χ, is convex, sub-additive, and positively 1-homogeneous in (p, α). Further,

$$\left[\dot{p}\middle|\dot{\alpha}\right] \in \partial I_{K_M(\chi)}(\left[\sigma_D - \chi\middle|\chi\right]) \text{ is equivalent to } \left[\sigma_D - \chi\middle|\chi\right] \in \partial H_M(\chi, \left[\dot{p}\middle|\dot{\alpha}\right])$$

where $\partial H_M(\chi, \left[\dot{p}\middle|\dot{\alpha}\right])$ denotes the subdifferential of $H_M(\chi, \left[\cdot\middle|\cdot\right])$ at $\left[\dot{p}\middle|\dot{\alpha}\right]$. We note that, given the displacement $t \mapsto u(t)$, the flow rule for the internal variables $\left[p\middle|\alpha\right]$ can be rewritten in the so-called Biot form as

$$\partial_{[\dot{p},\dot{\alpha}]} D\left(\left[p\middle|\alpha\right], \left[\dot{p}\middle|\dot{\alpha}\right]\right) + \partial_{[p,\alpha]} \hat{W}\left(Eu(t), \left[p\middle|\alpha\right]\right) \ni 0$$

where the *state-dependent* dissipation function D is defined as $D(\left[p\middle|\alpha\right], \left[\dot{p}\middle|\dot{\alpha}\right]) = H_M(\chi, \left[p\middle|\alpha\right]) = H_M(B(p{-}\alpha), \left[\dot{p}\middle|\dot{\alpha}\right])$. Eventually, we are led to investigating the following problem:

$$\begin{cases} Eu(x,t) = e(x,t) + p(x,t), \\ p(x,t) = (w-u)(x,t) \odot \nu(x) \text{ on } \partial\Omega, \\ \sigma(x,t) = Ae(x,t),\ \chi(x,t) = B(p(x,t) - \alpha(x,t)) \\ \text{div}\sigma(x,t) = 0, \\ \left[\sigma_D - \chi\middle|\chi\right](x,t) \in \partial H_M(\chi(x,t), \left[\dot{p}(x,t)\middle|\dot{\alpha}(x,t)\right]). \end{cases} \tag{2.8}$$

Note that, since $\partial H_M(\chi, \left[\dot{p}\middle|\dot{\alpha}\right]) \subset K_M(\chi)$, the last inclusion in (2.8) above entails the stress constraint $\left[\sigma_D - \chi\middle|\chi\right] \in K_M(\chi)$ as well.

2.3. **Properties of the dissipation potential.** We now state and prove a few useful properties of the sets $K_M(\chi)$ and of the dissipation potential H_M.

Lemma 2.2 (**Growth properties of** H_M)**.** *There exist $0 < \kappa < \kappa'_M < \infty$ with κ'_M that may depend on M such that*

$$B_{(\mathbb{M}^{n\times n}_{\text{dev}})^2}(0, \kappa) \subset K_M(\chi) \subset B_{(\mathbb{M}^{n\times n}_{\text{dev}})^2}(0, \kappa'_M) \tag{2.9}$$

or, equivalently,

$$\kappa \left|\left[p\middle|\alpha\right]\right| \leq H_M(\chi, \left[p\middle|\alpha\right]) \leq \kappa'_M \left|\left[p\middle|\alpha\right]\right| \tag{2.10}$$

for every $(\chi, p, \alpha) \in (\mathbb{M}^{n\times n}_{\text{dev}})^3$.

Proof. Since $f(0) < 0$, the continuity of f implies that $f(\tau) + \frac{1}{2}|\eta|^2 < 0 < \frac{1}{2}T_M(|\chi|^2)$, for $\left[\tau\middle|\eta\right] \in B_{(\mathbb{M}^{n\times n}_{\text{dev}})^2}(0, \kappa)$ for some small enough κ. Further, in view of the continuity of f and of (2.7), the other inclusion is obvious. Relations (2.10) follow by convex duality. □

Lemma 2.3 (**Continuity properties of** H_M)**.** *The map H_M is continuous over $(\mathbb{M}^{n\times n}_{\text{dev}})^3$.*

Proof. Let $(\chi_k, p_k, \alpha_k) \to (\chi, p, \alpha)$. We start by proving upper semi-continuity. Since $K_M(\chi_k)$ is compact, for each k there exists $\left[\tau_k\middle|\eta_k\right] \in K_M(\chi_k)$ such that $H_M(\chi_k, \left[p_k\middle|\alpha_k\right]) = \tau_k : p_k + \eta_k : \alpha_k$. By the upper inclusion in Lemma 2.2 above, we can extract a subsequence of $\{\left[\tau_k\middle|\eta_k\right]\}$ – still denoted $\{\big[\tau_k\big|\eta_k\big]\}$ – such that $\big[\tau_k\big|\eta_k\big] \to \big[\tau\big|\eta\big]$. Since $f(\tau_k)+\frac{1}{2}|\eta_k|^2 \le \frac{1}{2}T_M(|\chi_k|^2)$, $\big[\tau\big|\eta\big] \in K_M(\chi)$ and we can pass to the limit in k obtaining

$$\limsup_k H_M(\chi_k, \big[p_k\big|\alpha_k\big]) = \limsup_k \tau_k : p_k + \eta_k : \alpha_k = \tau : p + \eta : \alpha \le H_M(\chi, \big[p\big|\alpha\big]).$$

We now show lower semi-continuity. We first observe that

$$H_M(\chi, \big[p\big|\alpha\big]) := \sup_{\big[\tau\big|\eta\big] \in \operatorname{int} K_M(\chi)} \tau : p + \eta : \alpha,$$

where $\operatorname{int} K_M(\chi)$ denotes the interior of $K_M(\chi)$. Assume that $\big[\tau\big|\eta\big] \in \operatorname{int} K_M(\chi)$, then $f(\tau) + \frac{1}{2}|\eta|^2 < \frac{1}{2}T_M(|\chi|^2)$, and thus $f(\tau) + \frac{1}{2}|\eta|^2 < \frac{1}{2}T_M(|\chi_k|^2)$ for k large enough. Consequently, $\big[\tau\big|\eta\big] \in K_M(\chi_k)$ for k large enough, hence

$$\liminf_k H_M(\chi_k, \big[p_k\big|\alpha_k\big]) \ge \liminf_k \tau : p_k + \eta : \alpha_k = \tau : p + \eta : \alpha.$$

Taking the supremum over all $\big[\tau\big|\eta\big] \in \operatorname{int} K_M(\chi)$ leads to

$$\liminf_k H_M(\chi_k, \big[p_k\big|\alpha_k\big]) \ge H_M(\chi, \big[p\big|\alpha\big]),$$

which completes the proof of the lemma. □

Finally, we show that H is Lipschitz continuous with respect to its first variable.

Lemma 2.4 (**Lipschitz character of** H). *There exists a constant $C_M > 0$ depending on M such that*

$$|H_M(\chi_1, \big[p\big|\alpha\big]) - H_M(\chi_2, \big[p\big|\alpha\big])| \le C_M \left|\big[p\big|\alpha\big]\right| |\chi_1 - \chi_2|$$

for any $\chi_1, \chi_2, p, \alpha \in \mathbb{M}^{n\times n}_{\rm dev}$.

Proof. Assume without loss of generality that $|\chi_1| \le |\chi_2|$, so that $K_M(\chi_1) \subset K_M(\chi_2)$. Since $K_M(\chi_2)$ is compact, there exists $\big[\tau_2\big|\eta_2\big] \in K_M(\chi_2)$ such that $H_M(\chi_2, \big[p\big|\alpha\big]) = \tau_2 : p + \eta_2 : \alpha$.

We have that

$$d_H(\partial K_M(\chi_1), \partial K_M(\chi_2)) \le C_M|\chi_1 - \chi_2|$$

where d_H stands for the Hausdorff distance, for some constant C_M that we choose to be greater than $\max\{2, M\}$ and that depends only on M.

If $\big[\tau_2\big|\eta_2\big]$ is given as before, then there exists $\big[\tau_1\big|\eta_1\big] \in K_M(\chi_1)$ such that $\left|\big[\tau_1\big|\eta_1\big] - \big[\tau_2\big|\eta_2\big]\right| \le C_M|\chi_1-\chi_2|$. Indeed, if $\big[\tau_2\big|\eta_2\big] \in K_M(\chi_1)$, then it suffices to take $\big[\tau_1\big|\eta_1\big] = \big[\tau_2\big|\eta_2\big]$ and the property is trivial. On the other hand, if $\big[\tau_2\big|\eta_2\big] \in K_M(\chi_2) \setminus K_M(\chi_1)$, take $\big[\tau_1\big|\eta_1\big]$ as the minimal-distance projection of $\big[\tau_2\big|\eta_2\big]$ onto the convex set $K_M(\chi_1)$. It follows that

$$\left|\big[\tau_1\big|\eta_1\big] - \big[\tau_2\big|\eta_2\big]\right| \le d_H(\partial K_M(\chi_1), \partial K_M(\chi_2)) \le C_M|\chi_1 - \chi_2|. \tag{2.11}$$

□

Remark 2.5. The sets $K_M(\chi)$ have a very specific form: they can be obtained from one another by rescaling components. In particular, let χ_1 and χ_2 be given. Defining $\alpha = ((|T_M(\chi_2)|^2 + 2f_0)/(|T_M(\chi_1)|^2 + 2f_0))^{1/2}$ and exploiting the 1-homogeneity of $\tau \mapsto \hat{f}(\tau)$ we have that $[\tau, \eta] \in K_M(\chi_1)$ iff $[\alpha^2\tau, \alpha\eta] \in K_M(\chi_2)$. This fact allows us to prove that, for all $\chi_1, \chi_2, \tau, \eta \in \mathbb{M}^{n\times n}_{\rm dev}$,

$$|P_{K_M(\chi_1)}(\big[\tau\big|\eta\big]) - P_{K_M(\chi_2)}(\big[\tau\big|\eta\big])| \le C_M|\chi_1 - \chi_2|$$

where $P_{K_M(\chi)}$ denotes the minimal-distance projection onto the convex set $K_M(\chi)$ and C_M is a positive constant that may depend on M.

Indeed, let $K \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) be some convex and closed set containing 0 and let $\alpha > 1$. Then, Figure 2 demonstrates that, for all $x \in \mathbb{R}^m$,

$$|P_K(x) - P_{\alpha K}(x)| \leq (\alpha-1)\,\mathrm{diam} K. \tag{2.12}$$

Relation (2.12) can be generalized in order to allow different rescalings on different axes. In

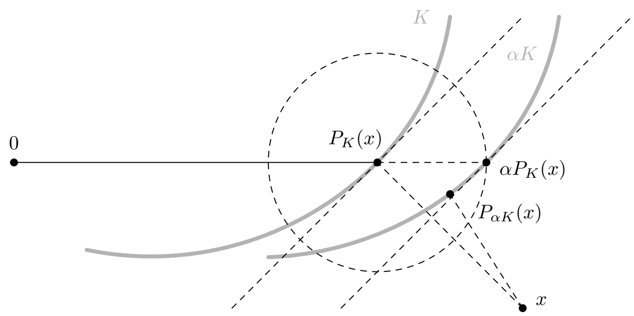

FIGURE 2. The projection $P_{\alpha K}(x)$ belongs to the ball centered in $P_K(x)$ with radius $|\alpha P_K(x) - P_K(x)| = (\alpha-1)|P_K(x)|$.

particular, one can prove that the distance of the two projections $P_{K_M(\chi_i)}([\tau,\eta])$ for $i = 1,2$ can be controlled by the quantity $C(\alpha^2-1)|\tau_1| + C(\alpha-1)|\eta_1|$ where $[\tau_1,\eta_1] = P_{K_M(\chi_1)}([\tau,\eta])$. The assertion follows upon noting that $|\tau_1| \leq C(|T_M(\chi_1)|^2 + 2f_0)$ and $|\eta_1|, |\eta_2| \leq (M^2+2f_0)^{1/2}$ while $\alpha \leq \sqrt{\dfrac{(M^2+2f_0)}{2f_0}}$. Thus

$$|P_{K_M(\chi_1)}(\big[\tau\big|\eta\big]) - P_{K_M(\chi_2)}(\big[\tau\big|\eta\big])| \leq C_M(\alpha^2-1) \leq C'_M|\chi_1 - \chi_2|,$$

where C_M, C'_M are positive constants that only depend on M. ¶

We will be resorting to a *visco-plastic* regularization of the problem. To that effect, we now introduce the perturbed dissipation potential $H_M^\varepsilon : (\mathbb{M}^{n\times n}_{\mathrm{dev}})^3 \to [0,\infty)$ defined, for each $\varepsilon > 0$, as

$$H_M^\varepsilon(\chi, \big[p\big|\alpha\big]) := H_M(\chi, \big[p\big|\alpha\big]) + \frac{\varepsilon}{2}|\,\big[p\big|\alpha\big]\,|^2. \tag{2.13}$$

Its convex conjugate $(H_M^\varepsilon)^* : (\mathbb{M}^{n\times n}_{\mathrm{dev}})^3 \to [0,\infty)$ with respect to the second variable is defined by

$$(H_M^\varepsilon)^*(\chi, \big[\tau\big|\eta\big]) := \sup_{\big[p|\alpha\big] \in (\mathbb{M}^{n\times n}_{\mathrm{dev}})^2} \left\{\tau : p + \eta : \alpha - H_M^\varepsilon(\chi, \big[p\big|\alpha\big])\right\}.$$

Then,

$$(H_M^\varepsilon)^*(\chi, \big[\tau\big|\eta\big]) = \frac{|\,\big[\tau\big|\eta\big] - P_{K_M(\chi)}(\big[\tau\big|\eta\big])|^2}{2\varepsilon}.$$

In particular $(H_M^\varepsilon)^*$ is differentiable in the second variable, and its partial derivative is given by

$$N_M^\varepsilon(\chi, \big[\tau\big|\eta\big]) = \partial (H_M^\varepsilon)^*(\chi, \big[\tau\big|\eta\big]) = \frac{\big[\tau\big|\eta\big] - P_{K_M(\chi)}(\big[\tau\big|\eta\big])}{\varepsilon}. \tag{2.14}$$

Note that, since $\big[0\big|0\big] \in K_M(\chi)$ (see (2.9)),

$$|N_M^\varepsilon(\chi, \big[\tau\big|\eta\big])| \leq \frac{1}{\varepsilon}|\,\big[\tau\big|\eta\big]\,|, \tag{2.15}$$

so that

$$|(H_M^\varepsilon)^*(\chi, \big[\tau_1\big|\eta_1\big]) - (H_M^\varepsilon)^*((\chi, \big[\tau_2\big|\eta_2\big])| \leq \frac{1}{\varepsilon}(\big|\big[\tau_1\big|\eta_1\big]\big| + \big|\big[\tau_2\big|\eta_2\big]\big|)\,\big|\big[\tau_1\big|\eta_1\big] - \big[\tau_2\big|\eta_2\big]\big|\,.$$

Actually, N^ε_M is Lipschitz since we can prove the following

Lemma 2.6 (Lipschitz property of the visco-plastic projection). *Let C_M be the constant in Lemma 2.4, then*

$$|N^\varepsilon_M(\chi_1, \left[\tau_1\middle|\eta_1\right]) - N^\varepsilon_M(\chi_2, \left[\tau_2\middle|\eta_2\right]| \le \frac{C_M}{\varepsilon}\left(|\chi_1-\chi_2| + \left|\left[\tau_1\middle|\eta_1\right] - \left[\tau_2\middle|\eta_2\right]\right|\right).$$

Proof. By definition of N^ε_M and since the projection is 1-Lipschitz,

$$|N^\varepsilon_M(\chi_1, \left[\tau_1\middle|\eta_1\right]) - N^\varepsilon_M(\chi_1, \left[\tau_2\middle|\eta_2\right]| \le \frac{2}{\varepsilon}\left|\left[\tau_1\middle|\eta_1\right] - \left[\tau_2\middle|\eta_2\right]\right|.$$

On the other hand, by Remark 2.5,

$$|N^\varepsilon_M(\chi_1, \left[\tau\middle|\eta\right]) - N^\varepsilon_M(\chi_2, \left[\tau\middle|\eta\right])| \le \frac{C_M}{\varepsilon}|\chi_1-\chi_2|.$$

But $C_M > 2$ by construction, hence the result. ☐

As a final note, given $\chi \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$, define the sets

$$\mathcal{K}(\text{ resp. } \mathcal{K}_M)(\chi) := \{\left[\tau\middle|\eta\right] \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) : \left[\tau\middle|\eta\right](x) \in K(\text{ resp. } K_M)(\chi(x)) \text{ for a.e. } x\in\Omega\}.$$

Then, if $\chi \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$,

$$\|N^\varepsilon_M(\chi, \left[\tau\middle|\eta\right])\|_2 = \frac{\mathrm{dist}_2(\left[\tau\middle|\eta\right], \mathcal{K}_M(\chi))}{\varepsilon}, \tag{2.16}$$

where, for any closed set $\mathcal{C} \subset L^2(\Omega;(\mathbb{M}^{n\times n}_{\rm dev})^2)$, $\mathrm{dist}_2(\left[\tau\middle|\eta\right], \mathcal{C})$ is the L^2-distance from $\left[\tau\middle|\eta\right]$ to $\mathcal{C}$.

2.4. **Mathematical setting.** Throughout the paper, Ω is a bounded connected open set in $\mathbb{R}^n$ with Lipschitz boundary. The Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure are respectively denoted by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$.

We use standard notation for Lebesgue and Sobolev spaces. In particular, for $1 \le p \le \infty$, the L^p-norms of the various quantities are denoted by $\|\cdot\|_p$. The space $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$ is that of all $\mathbb{M}^{n\times n}_{\rm sym}$-valued bounded Radon measures on $\overline{\Omega}$, and the norm in that space is denoted by $\|\cdot\|_1$. By the Riesz representation Theorem, $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$ can be identified with the dual of $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$. Finally, $BD(\Omega)$ stands for the space of functions with bounded deformations on Ω, *i.e.*, $u \in BD(\Omega)$ if $u \in L^1(\Omega;\mathbb{R}^n)$ and $Eu \in \mathcal{M}(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$.We refer to [60] for general properties of that space.

Let $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, and $p \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$ be such that

$$Eu = e + p \text{ in } \Omega,\ p = (w-u)\odot\nu\mathcal{H}^{n-1} \text{ on } \partial\Omega. \tag{2.17}$$

If $\sigma_D \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$ and $\mathrm{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$, it is possible to define the "scalar product" of σ and p as the distribution $[\sigma : p]$ on $\mathbb{R}^n$ by setting

$$[\sigma:p](\varphi) := -\int_\Omega \varphi(u-w)\cdot\mathrm{div}\sigma\,dx - \int_\Omega \sigma:(u-w)\odot\nabla\varphi\,dx - \int_\Omega \sigma:\varphi(e-Ew)\,dx,$$

for every $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. Actually, $[\sigma : p]$ is independent of u, w, and e, provided that kinematic compatibility with p is achieved. It defines a bounded Radon measure on $\overline{\Omega}$. We also define the global duality pairing $\langle\sigma, p\rangle$ by setting

$$\langle\sigma,p\rangle := [\sigma:p](1) = \int_\Omega (w-u)\cdot\mathrm{div}\sigma\,dx - \int_\Omega \sigma:(e-Ew)\,dx. \tag{2.18}$$

It can be proved (see [25, Section 6]) that

$$|\langle\sigma,p\rangle| \le \|\sigma_D\|_\infty\|p\|_1.$$

Moreover, if σ_D further belongs to $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$, then

$$\langle\sigma,p\rangle=\int_{\overline{\Omega}}\sigma_D(x):\frac{dp}{d|p|}(x)\,d|p|(x) \tag{2.19}$$

is the usual duality pairing between $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$ and $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$. In the previous formula we have denoted by $|p|$ the variation measure of p.

If instead p further belongs to $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, then the duality pairing $\langle\sigma,p\rangle$ coincides with the standard product in L^2.

The space $L^1(0,T;\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm sym}))$ is the space of all strongly measurable maps $t\mapsto f(t)\in\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm sym})$ such that

$$\int_0^T\|f(t)\|_\infty\,dt<\infty.$$

Since $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm sym})$ is separable, the dual of the space $L^1(0,T;\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm sym}))$ can be identified to the space $L^\infty_w(0,T;\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev}))$ of all weakly* measurable maps $t\mapsto\lambda(t)\in\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$ such that

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|\lambda(t)\|_1<\infty,$$

through the duality pairing

$$\langle\lambda,f\rangle=\int_0^T\langle\lambda(t),f(t)\rangle_{\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev}),\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm sym})}\,dt.$$

Let A,B be two fourth order tensors satisfying the usual symmetry properties $A_{ijkh}=A_{jikh}=A_{khij}$ (idem for B) for every $i,\,j,\,k,\,h\in\{1,\dots,n\}$, and

$$\gamma|\xi|^2\le A\xi:\xi\le\gamma'|\xi|^2,\quad \gamma|\xi|^2\le B\xi:\xi\le\gamma'|\xi|^2, \tag{2.20}$$

for some $0<\gamma\le\gamma'<\infty$ and every $\xi\in\mathbb{M}^{n\times n}_{\rm sym}$. Then define, for any $(e,\beta)\in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}\times\mathbb{M}^{n\times n}_{\rm dev})$, the internal energy as

$$\mathcal{Q}(e,\beta):=\frac12\int_\Omega\{Ae:e+B\beta:\beta\}\,dx.$$

If $\chi\in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$ and $\big[p\big|\alpha\big]\in L^2(\Omega;(\mathbb{M}^{n\times n}_{\rm dev})^2)$ we define the functionals

$$\mathcal{H}_M(\chi,p):=\int_\Omega H_M(\chi,\big[p\big|\alpha\big])\,dx,\quad \mathcal{H}^\varepsilon_M(\chi,\big[p\big|\alpha\big]):=\int_\Omega H^\varepsilon_M(\chi,\big[p\big|\alpha\big])\,dx,$$

while, if $\chi\in\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$ and $\big[p\big|\alpha\big]\in\mathcal{M}(\overline{\Omega};(\mathbb{M}^{n\times n}_{\rm dev})^2)$ the first functional is defined as

$$\mathcal{H}_M(\chi,\big[p\big|\alpha\big]):=\int_{\overline{\Omega}}H\left(\chi,\frac{d\,\big[p\big|\alpha\big]}{d|\big[p\big|\alpha\big]|}\right)d|\big[p\big|\alpha\big]|.$$

Remark 2.7. The following (lower semi-)continuity results whose proof is identical to [4, Remark 2.8] hold:

1. If $\{\chi_k\}$, $\{\big[p_k\big|\alpha_k\big]\}$ are L^2-sequences , $\chi_k\to\chi$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$, and $\big[p_k\big|\alpha_k\big]\rightharpoonup\big[p\big|\alpha\big]$ weakly in $L^2(\Omega;(\mathbb{M}^{n\times n}_{\rm dev})^2)$, then

$$\mathcal{H}_M(\chi,\big[p\big|\alpha\big])\le\liminf_k\mathcal{H}_M(\chi_k,\big[p_k\big|\alpha_k\big]).$$

   Moreover if $\big[p_k\big|\alpha_k\big]\to\big[p\big|\alpha\big]$ strongly in $L^2(\Omega;(\mathbb{M}^{n\times n}_{\rm sym})^2)$, then

$$\mathcal{H}_M(\chi,\big[p\big|\alpha\big])=\lim_{k\to\infty}\mathcal{H}_M(\chi_k,\big[p_k\big|\alpha_k\big]).$$

2. If $\{\chi_k\} \subset \mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm sym})$, $\{[p_k|\alpha_k]\} \subset \mathcal{M}(\overline{\Omega};(\mathbb{M}^{n\times n}_{\rm dev})^2)$, $\chi_k \to \chi$ uniformly in $\overline{\Omega}$, and $[p_k|\alpha_k] \overset{*}{\rightharpoonup} [p|\alpha]$ weakly* in $\mathcal{M}(\overline{\Omega};(\mathbb{M}^{n\times n}_{\rm dev})^2)$, then
$$\mathcal{H}_M(\chi,[p|\alpha]) \le \liminf_{k\to\infty} \mathcal{H}_M(\chi_k,[p_k|\alpha_k]).$$
¶

When dealing with the visco-plastic approximation of the elasto-plastic problem, we will obtain the first type of convergence on our approximating sequences, while, when letting the viscosity parameter tend to 0, we will only obtain weak convergence in L^2 of the approximating σ-sequence, and convergence in the space of measures of the approximating p-sequence.

Reshetnyak lower semi-continuity Theorem is false when H fails to be (lower semi)-continuous, and this forces us to restrict our analysis to continuous back-stresses; but continuity is not preserved under L^2-weak convergence, which is the best we can prove for the various sequences of stresses that will enter the formulation. Consequently, the analysis will soon grind to a halt for lack of lower semi-continuity of H. This is why we will propose, in the spirit of [16, 18], to introduce a regularization of χ in the definition of $\mathcal{K}_M(\chi)$. This is achieved by introducing a convolution kernel ρ and replacing $\mathcal{K}_M(\chi)$ by $\mathcal{K}_M(\chi*\rho)$ defined below.

We fix $\rho \in \mathcal{C}^1_c(\mathbb{R}^n)$ and set, for $\chi \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$,
$$x \in \overline{\Omega} \mapsto \chi*\rho(x) := \int_\Omega \rho(x-y)\chi(y)\,dy.$$
The convolution $\chi*\rho$ defines an element in $\mathcal{C}^1(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$. By modifying $\mathcal{K}_M(\chi)$ as $\mathcal{K}_M(\chi*\rho)$ we are introducing a length scale in the model, namely, the size of the support of the convolution kernel.

Note that, with our definition of the convolution, if $\chi_\varepsilon \rightharpoonup \chi$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$, then, in particular,
$$\chi_\varepsilon*\rho \to \chi*\rho \text{ uniformly on } \overline{\Omega}. \tag{2.21}$$

Remark 2.8. Before closing this section, let us mention that restoring the lower semicontinuity of H could be achieved by imparting additional compactness on χ. In particular, one could introduce a compactifying term into the energy. As we need the dependence of $K(\cdot)$ upon χ to be continuous, a possibility would be that of augmenting the energy by a gradient term like $\kappa|\nabla\chi|^r$ where r is bigger than the space dimension n. Such a term would introduce a length scale in the model as well. We will not go down that path which in our opinion strays much further away from the original model and would result in one which is more along the lines of gradient-plasticity. Note that, besides the dubious phenomenology that such a model would introduce because of the dependence of the exponent upon the dimension, the introduction of such a gradient term would require extra boundary conditions on the internal variable χ, an option which is often disputed. ¶

For now, we address in the next section the visco-plastic regularization.

3. The visco-plastic model

Here the existence of the solution to the visco-plastic regularization is established. The viscosity parameter $\varepsilon > 0$ is fixed throughout this section.

Consider a boundary displacement $\hat{w} \in H^1(\Omega;\mathbb{R}^n)$. We set
$$A_{\rm reg}(\hat{w}) := \Big\{(v,\eta,q,\beta) \in H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})\times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})\times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}) :$$
$$Ev = \eta + q \text{ a.e. in } \Omega,\ v = \hat{w}\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\Big\}. \tag{3.1}$$

The following existence result for the visco-plastic evolution holds true.

Theorem 3.1 (**Visco-plastic evolution**). *Consider* $w \in H^1(0,T;H^1(\Omega,\mathbb{R}^n))$ *and a quadruplet* $(u_0,e_0,p_0,\alpha_0) \in A_{\rm reg}(w(0))$ *such that* $\mathrm{div}\sigma_0 = 0$ *a.e. in* Ω, *where* $\sigma_0 := Ae_0$. *Then, there exists a unique quadruplet*

$$(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t), \alpha_\varepsilon(t)) \in A_{\rm reg}(w(t)) \quad \forall t \in [0,T]$$

with

$$\begin{aligned} u_\varepsilon &\in H^1(0,T;H^1(\Omega;\mathbb{R}^n)),\\ e_\varepsilon &\in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})),\\ p_\varepsilon &\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})),\\ \alpha_\varepsilon &\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})), \end{aligned}$$

such that, setting $\sigma_\varepsilon(t) := Ae_\varepsilon(t)$ *and* $\chi_\varepsilon := B(p_\varepsilon - \alpha_\varepsilon)(t)$, *the following items are satisfied:*

1. Initial condition: $(u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0), \alpha_\varepsilon(0)) = (u_0, e_0, p_0, \alpha_0)$;
2. Kinematic compatibility: *For every* $t \geq 0$,
$$\begin{aligned} &Eu_\varepsilon(t) = e_\varepsilon(t) + p_\varepsilon(t) \ \text{a.e. in } \Omega,\\ &u_\varepsilon(t) = w(t)\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega; \end{aligned}$$
3. Equilibrium condition: *For every* $t \geq 0$, $\mathrm{div}\,\sigma_\varepsilon(t) = 0$ *a.e. in* Ω;
4. Regularized flow rule: *For a.e.* $t \in [0,T]$,
$$\left[\dot p_\varepsilon(t)\middle|\dot\alpha_\varepsilon(t)\right] = N^\varepsilon_M(\chi_\varepsilon(t), \left[((\sigma_\varepsilon)_D - \chi_\varepsilon)(t)\middle|\chi_\varepsilon(t)\right]) \text{ for a.e. } x \in \Omega,$$
or, equivalently,
$$\left[((\sigma_\varepsilon)_D - \chi_\varepsilon - \varepsilon\dot p_\varepsilon)(t)\middle|\chi_\varepsilon(t) - \varepsilon\dot\alpha_\varepsilon(t)\right] \in \partial H_M(\chi_\varepsilon(t), \left[\dot p_\varepsilon(t)\middle|\dot\alpha_\varepsilon(t)\right]) \text{ for a.e. } x \in \Omega.$$
In particular, $\varepsilon\|\left[\dot p_\varepsilon(t)\middle|\dot\alpha_\varepsilon(t)\right]\|_2 = \mathrm{dist}_2(\left[((\sigma_\varepsilon)_D - \chi_\varepsilon)(t)\middle|\chi_\varepsilon(t)\right], \mathcal{K}_M(\chi_\varepsilon(t)))$;
5. Estimates: *There exists a constant* $C_T > 0$ *depending only on* T *such that, for* $t \in [0,T]$,
$$\begin{cases} \|e_\varepsilon(t)\|_2,\ \|(p_\varepsilon - \alpha_\varepsilon)(t)\|_2 \leq C_T\\ \displaystyle\int_0^T \|\dot p_\varepsilon(s)\|_2^2\,ds \leq \frac{C_T}{\varepsilon},\ \int_0^T \|\dot\alpha_\varepsilon(s)\|_2^2\,ds \leq \frac{C_T}{\varepsilon}. \end{cases}$$

We call such a quadruplet a visco-plastic solution.

Proof. The proof is very similar to that of a related result in non-associative elasto-plasticity; see [4, Theorem 3.1]. We provide a sketch below.

In a first step it is proved by an iteration argument that, if $\hat w \in H^1(\Omega;\mathbb{R}^n)$ and $\left[\hat p\middle|\hat\alpha\right] \in L^2(\Omega;(\mathbb{M}^{n\times n}_{\rm dev})^2)$ then, for $\delta > 0$ small enough, there exists a quadruplet $(u,e,p,\alpha) \in A_{\rm reg}(\hat w)$ satisfying

$$\begin{aligned} &\mathcal{Q}(e, p-\alpha) + \mathcal{H}_M(\chi, \left[p - \hat p\middle|\alpha - \hat\alpha\right]) + \frac{\varepsilon}{2\delta}\|\left[p - \hat p\middle|\alpha - \hat\alpha\right]\|_2^2\\ &\qquad \leq \mathcal{Q}(\eta, q-\beta) + \mathcal{H}_M(\chi, \left[q - \hat p\middle|\beta - \hat\alpha\right]) + \frac{\varepsilon}{2\delta}\|\left[q - \hat p\middle|\beta - \hat\alpha\right]\|_2^2 \end{aligned} \tag{3.2}$$

for any $(v,\eta,q,\beta) \in A_{\rm reg}(\hat w)$, with $\chi := B(p-\alpha)$.

To this effect, we take $(u_0,e_0,p_0,\alpha_0) := (\hat w, E\hat w, 0, 0)$, and for any $k \geq 1$, consider the minimization problem

$$\min_{(v,\eta,q,\beta)\in A_{\rm reg}(\hat w)} \left\{\mathcal{Q}(\eta, q-\beta) + \mathcal{H}_M(\chi_{k-1}, \left[q - \hat p\middle|\beta - \hat\alpha\right]) + \frac{\varepsilon}{2\delta}\|\left[q - \hat p\middle|\beta - \hat\alpha\right]\|_2^2\right\}, \tag{3.3}$$

where $\chi_{k-1} := B(p_{k-1} - \alpha_{k-1})$. Elementary convexity arguments yield the existence of a unique minimizer (u_k,e_k,p_k,α_k) for any $k \geq 1$.

That minimizer is easily shown to satisfy

$$\operatorname{div}\sigma_k = 0 \quad \text{a.e. in } \Omega,$$

as well as

$$\big[p_k - \hat p\big|\alpha_k - \hat\alpha\big] = \delta N^\varepsilon_M(\chi_{k-1}, \big[(\sigma_D)_k - \chi_k\big|\chi_k\big]) \text{ a.e. in } \Omega,$$

or, still,

$$\big[e_k\big| - \alpha_k\big] = \big[Eu_k - \hat p\big| - \hat\alpha\big] - \delta N^\varepsilon_M(\chi_{k-1}, \big[(\sigma_D)_k - \chi_k\big|\chi_k\big]) \text{ a.e. in } \Omega.$$

We now prove that $\{e_k, p_k, \alpha_k\}$ is a Cauchy sequence in $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym} \times \mathbb{M}^{n\times n}_{\rm dev} \times \mathbb{M}^{n\times n}_{\rm dev})$. Indeed, from the two relations above, we get

$$\big[p_k - p_{k-1}\big|\alpha_k - \alpha_{k-1}\big] = \delta\left\{N^\varepsilon_M(\chi_{k-1}, \big[(\sigma_D)_k - \chi_k\big|\chi_k\big]) - N^\varepsilon_M(\chi_{k-2}, \big[(\sigma_D)_{k-1} - \chi_{k-1}\big|\chi_{k-1}\big])\right\},$$

while

$$\begin{aligned}\big[e_k - e_{k-1}\big| - (\alpha_k - \alpha_{k-1})\big] = \big[Eu_k - Eu_{k-1}\big|0\big] - \\ \delta\left\{N^\varepsilon_M(\chi_{k-1}, \big[(\sigma_D)_k - \chi_k\big|\chi_k\big]) - N^\varepsilon_M(\chi_{k-2}, \big[(\sigma_D)_{k-1} - \chi_{k-1}\big|\chi_{k-1}\big])\right\}.\end{aligned}$$

Taking the L^2-scalar product of the first relation with $\big[p_k - p_{k-1}\big|\alpha_k - \alpha_{k-1}\big]$ and of the second with $\big[\sigma_k - \sigma_{k-1}\big| - (\alpha_k - \alpha_{k-1})\big]$ and using the fact that, by Lemma 2.6, N^ε_M is Lipschitz continuous (with a Lipschitz constant of order $1/\varepsilon$), we deduce from the first relation that

$$\|\,\big[p_k - p_{k-1}\big|\alpha_k - \alpha_{k-1}\big]\,\|_2^2 \le \frac{C_M\delta}{\varepsilon}(\|(\sigma_D)_k - (\sigma_D)_{k-1}\|_2^2 + \|\chi_k - \chi_{k-1}\|_2^2 + \|\chi_{k-1} - \chi_{k-2}\|_2^2),$$

while (2.20) and the second relation yields

$$\begin{aligned}\gamma\|\,\big[e_k - e_{k-1}\big|\alpha_k - \alpha_{k-1}\big]\,\|_2^2 \le \int_\Omega (\sigma_k - \sigma_{k-1}) : (Eu_k - Eu_{k-1})\,dx + \\ \frac{C_M\delta}{\varepsilon}\|\big[\sigma_k - \sigma_{k-1}\big| - (\alpha_k - \alpha_{k-1})\big]\|_2\,(\|(\sigma_D)_k - (\sigma_D)_{k-1}\|_2 + \|\chi_k - \chi_{k-1}\|_2 + \|\chi_{k-1} - \chi_{k-2}\|_2).\end{aligned}$$

But since $\operatorname{div}\sigma_k = \operatorname{div}\sigma_{k-1} = 0$ a.e. in Ω and $u_k - u_{k-1} \in H^1_0(\Omega; \mathbb{R}^n)$, the integral in the right hand side of the inequality above vanishes, hence, adding the two inequalities above, using the other inequality in (2.20), and setting

$$I_k := \|e_k - e_{k-1}\|_2^2 + \|p_k - p_{k-1}\|_2^2 + \|\alpha_k - \alpha_{k-1}\|_2^2,$$

we obtain

$$I_k \le \frac{C'_M\gamma'\delta}{\varepsilon\gamma}\,(I_k + I_{k-1}),$$

for some constant C'_M depending only on M. Hence, if δ is small enough, say

$$\delta < \frac{\varepsilon\gamma}{3C'_M\gamma'}, \tag{3.4}$$

then

$$I_k \le \frac{1}{2} I_{k-1},$$

which shows that $\{e_k, p_k, \alpha_k\}$ is a Cauchy sequence in $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym} \times \mathbb{M}^{n\times n}_{\rm dev} \times \mathbb{M}^{n\times n}_{\rm dev})$. Since $u_k = \hat w$ on $\partial\Omega$, Poincaré-Korn's inequality then implies that u_k is a Cauchy sequence in $H^1(\Omega; \mathbb{R}^n)$.

The remainder of the proof of (3.2) is straightforward upon application of the first item in Remark 2.7.

We now introduce an incremental problem. Consider a sequence of nested subdivisions $(t_k^i)_{0\le i\le N(k)}$ of the time interval $[0,T]$ with the following properties:

$$\delta_k \searrow 0 \ \text{ as } \ k \nearrow \infty, \quad N(k)\delta_k = T, \quad t_k^i := i\delta_k \ \text{ for } \ i = 1,\dots,N(k),$$
$$\{t_k^i : i = 1,\dots,N(k)\} \subset \{t_l^i : i = 1,\dots,N(l)\}, \ k \le l.$$

We first set $(u_k^0, e_k^0, p_k^0, \alpha_k^0) := (u_0, e_0, p_0, \alpha_0)$ which belongs by assumption to $A_{\rm reg}(w(0))$.

Assume now that k is large enough for δ_k to satisfy (3.4). Then, for $i \in \{1,\dots,N(k)\}$, we define by induction $(u_k^i, e_k^i, p_k^i, \alpha_k^i)$ to be minimizers of (3.2) with $\hat w = w_k^i := w(t_k^i)$, $\hat p = p_k^{i-1}$, $\hat\alpha = \alpha_k^{i-1}$, that is

$$\begin{aligned}\mathcal{Q}(e_k^i, p_k^i - \alpha_k^i) &+ \mathcal{H}_M(\chi_k^i, [p_k^i - p_k^{i-1} | \alpha_k^i - \alpha_k^{i-1}]) + \frac{\varepsilon}{2\delta_k}\| [p_k^i - p_k^{i-1} | \alpha_k^i - \alpha_k^{i-1}] \|_2^2 \\ &\le \mathcal{Q}(\eta, q-\beta) + \mathcal{H}_M(\chi_k^i, [q - p_k^{i-1} | \alpha - \alpha_k^{i-1}]) + \frac{\varepsilon}{2\delta_k}\| [q - p_k^{i-1} | \beta - \alpha_k^{i-1}] \|_2^2, \end{aligned} \tag{3.5}$$

for any $(v,\eta,q,\beta) \in A_{\rm reg}(w_k^i)$, where $\chi_k^i = B(p_k^i - \alpha_k^i)$.

It is easily checked that, if $\sigma_k^i := Ae_k^i$, then

$$\operatorname{div} \sigma_k^i = 0 \text{ a.e. in } \Omega \tag{3.6}$$

$$\left[(\sigma_D)_k^i - \chi_k^i - \frac{\varepsilon}{\delta_k}(p_k^i - p_k^{i-1}) \Big| \chi_k^i - \frac{\varepsilon}{\delta_k}(\alpha_k^i - \alpha_k^{i-1})\right] \in \partial\mathcal{H}(\chi_k^i, [p_k^i - p_k^{i-1} | \alpha_k^i - \alpha_k^{i-1}]). \tag{3.7}$$

Now, by (2.13) and the homogeneity of degree 0 of $\partial\mathcal{H}_M(\cdot,\cdot)$ in its second entry, (3.7) also reads as

$$[(\sigma_D)_k^i - \chi_k^i | \chi_k^i] \in \partial\mathcal{H}_M^\varepsilon\left(\chi_k^i, \left[\frac{p_k^i - p_k^{i-1}}{\delta_k} \Big| \frac{\alpha_k^i - \alpha_k^{i-1}}{\delta_k}\right]\right),$$

which, by convex duality and (2.14), is equivalent to

$$\left[\frac{p_k^i - p_k^{i-1}}{\delta_k} \Big| \frac{\alpha_k^i - \alpha_k^{i-1}}{\delta_k}\right] = N_M^\varepsilon(\chi_k^i, [(\sigma_D)_k^i - \chi_k^i | \chi_k^i]) \text{ a.e. in } \Omega. \tag{3.8}$$

Define, for $t \in [t_k^i, t_k^{i+1})$, the right-continuous piecewise constant interpolations

$$u_k(t) := u_k^i, \quad e_k(t) := e_k^i, \quad \sigma_k(t) := Ae_k^i, \quad p_k(t) := p_k^i, \quad \alpha_k(t) := \alpha_k^i, \quad w_k(t) := w_k^i,$$

and the piecewise affine interpolations

$$\hat e_k(t) := e_k^i + \frac{t - t_k^i}{\delta_k}(e_k^{i+1} - e_k^i), \quad \hat p_k(t) := p_k^i + \frac{t - t_k^i}{\delta_k}(p_k^{i+1} - p_k^i),$$
$$\hat\alpha_k(t) := \alpha_k^i + \frac{t - t_k^i}{\delta_k}(\alpha_k^{i+1} - \alpha_k^i), \quad \hat\sigma_k(t) := A\hat e_k(t), \quad \hat\chi_k(t) := B(\hat p_k(t) - \hat\alpha_k(t)).$$

Take now $(u_k^{i-1} + w_k^i - w_k^{i-1}, e_k^{i-1} + Ew_k^i - Ew_k^{i-1}, p_k^{i-1}, \alpha_k^{i-1}) \in A_{\rm reg}(w_k^i)$ as competitor in (3.5). Then

$$\begin{aligned}\mathcal{Q}(e_k^i, p_k^i - \alpha_k^i) &+ \mathcal{H}_M(\chi_k^i, [p_k^i - p_k^{i-1} | \alpha_k^i - \alpha_k^{i-1}]) + \frac{\varepsilon}{2\delta_k}\| [p_k^i - p_k^{i-1} | \alpha_k^i - \alpha_k^{i-1}] \|_2^2 \\ &\le \mathcal{Q}(e_k^{i-1} + Ew_k^i - Ew_k^{i-1}, p_k^{i-1} - \alpha_k^{i-1}) = \mathcal{Q}(e_k^{i-1}, p_k^{i-1} - \alpha_k^{i-1}) \\ &+ \frac{1}{2}\int_\Omega A(Ew_k^i - Ew_k^{i-1}) : (Ew_k^i - Ew_k^{i-1})\,dx + \int_\Omega \sigma_k^{i-1} : (Ew_k^i - Ew_k^{i-1})\,dx. \end{aligned} \tag{3.9}$$

Since Ew is absolutely continuous in time with values in $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$, then

$$Ew_k^i - Ew_k^{i-1} = \int_{t_k^{i-1}}^{t_k^i} E\dot w(s)\,ds.$$

By (2.20),

$$\frac{1}{2}\int_\Omega A(Ew_k^i - Ew_k^{i-1}) : (Ew_k^i - Ew_k^{i-1})\,dx \le \frac{\gamma'}{2}\left\|\int_{t_k^{i-1}}^{t_k^i} E\dot{w}(s)\,ds\right\|_2^2$$
$$\le \frac{\gamma'}{2}\left(\int_{t_k^{i-1}}^{t_k^i} \|E\dot{w}(s)\|_2\,ds\right)^2 \le \frac{\gamma'}{2}\omega(\delta_k)\int_{t_k^{i-1}}^{t_k^i} \|E\dot{w}(s)\|_2\,ds, \quad (3.10)$$

where $\omega : [0,\infty) \to [0,\infty)$ is an infinitesimal function in 0. In view of (3.9) and (3.10),

$$\mathcal{Q}(e_k^i, p_k^i - \alpha_k^i) + \mathcal{H}_M(\chi_k^i, \left[p_k^i - p_k^{i-1}\middle|\alpha_k^i - \alpha_k^{i-1}\right]) + \frac{\varepsilon}{2\delta_k}\|\left[p_k^i - p_k^{i-1}\middle|\alpha_k^i - \alpha_k^{i-1}\right]\|_2^2$$
$$\le \mathcal{Q}(e_k^{i-1}, p_k^{i-1} - \alpha_k^{i-1}) + \frac{\gamma'}{2}\omega(\delta_k)\int_{t_k^{i-1}}^{t_k^i}\|E\dot{w}(s)\|_2\,ds + \int_{t_k^{i-1}}^{t_k^i}\int_\Omega \sigma_k(s) : E\dot{w}(s)\,dx\,ds.$$

Let $0 \le t_1 \le t_2 \le T$, and consider the unique $j_1,\, j_2 \in \{1,\dots,N(k)\}$ such that $t_1 \in [t_k^{j_1}, t_k^{j_1+1})$ and $t_2 \in [t_k^{j_2}, t_k^{j_2+1})$. Summing up for $i = j_1+1$ to j_2, and using the 1-homogeneity of $\mathcal{H}$ in its second variable, we get

$$\mathcal{Q}(e_k(t_2), p_k(t_2) - \alpha_k(t_2)) + \sum_{i=j_1+1}^{j_2}\delta_k\mathcal{H}_M\left(\chi_k^i, \left[\frac{p_k^i - p_k^{i-1}}{\delta_k}\middle|\frac{\alpha_k^i - \alpha_k^{i-1}}{\delta_k}\right]\right) +$$
$$\frac{\varepsilon}{2}\sum_{i=j_1+1}^{j_2}\delta_k\left\|\left[\frac{p_k^i - p_k^{i-1}}{\delta_k}\middle|\frac{\alpha_k^i - \alpha_k^{i-1}}{\delta_k}\right]\right\|_2^2 \le \mathcal{Q}(e_k(t_1), p_k(t_1) - \alpha_k(t_1)) + \frac{\gamma'}{2}\omega(\delta_k)\int_0^T \|E\dot{w}(s)\|_2\,ds$$
$$+ \int_{t_k^{j_1}}^{t_k^{j_2}}\int_\Omega \sigma_k(s) : E\dot{w}(s)\,dx\,ds.$$

Thus, for every $0 \le t_1 \le t_2 \le T$ with $t_1 \in [t_k^{j_1}, t_k^{j_1+1})$ and $t_2 \in [t_k^{j_2}, t_k^{j_2+1})$,

$$\mathcal{Q}(e_k(t_2), p_k(t_2) - \alpha_k(t_2)) + \int_{t_k^{j_1}}^{t_k^{j_2}}\mathcal{H}_M(\sigma_k(s), \left[\dot{\hat{p}}_k(s)\middle|\dot{\hat{\alpha}}_k(s)\right])\,ds + \frac{\varepsilon}{2}\int_{t_k^{j_1}}^{t_k^{j_2}}\left\|\left[\dot{\hat{p}}_k(s)\middle|\dot{\hat{\alpha}}_k(s)\right]\right\|_2^2\,ds$$
$$\le \mathcal{Q}(e_k(t_1), p_k(t_1) - \alpha_k(t_1)) + \int_{t_k^{j_1}}^{t_k^{j_2}}\int_\Omega \sigma_k(s) : E\dot{w}(s)\,dx\,ds + \omega_k. \quad (3.11)$$

with $\omega_k := \frac{\gamma'}{2}\omega(\delta_k)\int_0^T \|E\dot{w}(s)\|_2\,ds$.

Inequality (3.11) with $t_1 = 0, t_2 = t$ immediately implies the bounds of the fifth item in the statement of the theorem.

From there onward, the proof is exactly that in [4, Sections 3.2, 3.3], using a technique identical to that employed above to establish that $\{e_k(t), p_k(t), \alpha_k(t)\}$ is a Cauchy sequence in $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}} \times \mathbb{M}^{n\times n}_{\mathrm{dev}} \times \mathbb{M}^{n\times n}_{\mathrm{dev}})$, hence, by Poincaré-Korn's inequality, that $\{u_k\}$ is then a Cauchy sequence in $L^\infty(0,T; H^1(\Omega;\mathbb{R}^n))$. □

Remark 3.2. The existence result of Theorem 3.1 holds with $N^\varepsilon_M(\chi_\varepsilon(t),\cdot)$ replaced by $N^\varepsilon_M(\chi_\varepsilon(t) * \rho,\cdot)$ (and, correspondingly, $\partial H_M(\chi_\varepsilon(t),\cdot)$ replaced by $\partial H_M(\chi_\varepsilon(t) * \rho,\cdot)$). We then call a solution quadruplet a *ρ-visco-plastic solution.* In that case,

$$\varepsilon\|\left[\dot{p}_\varepsilon(t)\middle|\alpha_\varepsilon(t)\right]\|_2 = \mathrm{dist}_2((\sigma_\varepsilon)_D(t) - \chi_\varepsilon(t), \mathcal{K}_M(\chi_\varepsilon(t) * \rho)).$$

Remark 3.4 below also applies to that case. ¶

Remark 3.3. Note that, in lieu of the constant C_T, the bounds in item 5. of Theorem 3.1 can be restated in terms of an expression of the form $a\big(\int_0^T \|E\dot w(s)\|_2\,ds\big)$, with $a \geq 0$ continuous and non decreasing. This also applies to the ρ-visco-plastic evolution. ¶

Remark 3.4 (Visco-plastic energy balance). Finally remark that, as in [16, 45, 46], the visco-plastic flow rule in Theorem 3.1 can be equivalently replaced by

1. Modified Stress Constraint: $\big[(\sigma_\varepsilon)_D(t) - \chi_\varepsilon(t) - \varepsilon\dot p_\varepsilon(t)\big|\chi_\varepsilon(t) - \varepsilon\dot\alpha_\varepsilon(t)\big] \in \mathcal{K}_M(\chi_\varepsilon(t))$ for a.e. $t \in [0,T]$, or equivalently, since convex analysis implies that $K_M(\chi) = \partial H_M(\chi, \big[0\big|0\big])$,
$$\big[(\sigma_\varepsilon)_D(t) - \chi_\varepsilon(t) - \varepsilon\dot p_\varepsilon(t)\big|\chi_\varepsilon(t) - \varepsilon\dot\alpha_\varepsilon(t)\big] \in \partial H_M(\chi_\varepsilon(t), \big[0\big|0\big])$$
for a.e. $t \in [0,T]$;
2. Energy equality: $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t), \alpha_\varepsilon(t))$ satisfies the following energy equality, for every $t \in [0,T]$:

$$\mathcal{Q}(e_\varepsilon(t), p_\varepsilon(t) - \alpha_\varepsilon(t)) + \int_0^t \mathcal{H}_M(\chi_\varepsilon(s), \big[\dot p_\varepsilon(s)\big|\dot\alpha_\varepsilon(s)\big])\,ds + \varepsilon\int_0^t \|\,\big[\dot p_\varepsilon(s)\big|\dot\alpha_\varepsilon(s)\big]\,\|_2^2\,ds$$
$$= \mathcal{Q}(e_0, p_0 - \alpha_0) + \int_0^t\int_\Omega \sigma_\varepsilon(s) : E\dot w(s)\,dx\,ds,$$

or still

$$\mathcal{Q}(e_\varepsilon(t), p_\varepsilon(t) - \alpha_\varepsilon(t)) + \int_0^t \mathcal{H}_M(\chi_\varepsilon(s), \big[\dot p_\varepsilon(s)\big|\dot\alpha_\varepsilon(s)\big])\,ds + \int_0^t \|\,\big[\dot p_\varepsilon(s)\big|\dot\alpha_\varepsilon(s)\big]\,\|_2 \times$$
$$\mathrm{dist}_2(\big[(\sigma_\varepsilon)_D(s) - \chi_\varepsilon(s)\big|\chi_\varepsilon(s)\big], \mathcal{K}_M(\chi_\varepsilon(s)))\,ds = \mathcal{Q}(e_0, p_0 - \alpha_0) + \int_0^t\int_\Omega \sigma_\varepsilon(s) : E\dot w(s)\,dx\,ds.$$

The same applies to the non-associative ρ-visco-plastic evolution defined in Remark 3.2 above.

In turn, in view of Remark 3.3, together with (2.10), this implies the following bound
$$\int_0^T \|\,\big[\dot p_\varepsilon(s)\big|\dot\alpha_\varepsilon(s)\big]\,\|_1 ds \leq b\left(\int_0^T \|E\dot w(s)\|_2 ds\right), \tag{3.12}$$
with $b \geq 0$ continuous and non decreasing. ¶

4. Time rescaling

As in [4, 16, 23, 45, 46] we propose a rescaling of time which will permit to pass to the vanishing viscosity limit in the ρ-visco-plastic evolution. Under that rescaling jumps in the original time correspond to intervals where the mapping from the rescaled time to the original one remains constant.

Given the bounds in the fifth item of Theorem 3.1, we are not able to infer the L^2-regularity of the fields Eu and p when passing to the 0-viscosity limit $\varepsilon \to 0$ and we thus have to redefine the set of admissible evolutions as $A_{\mathrm{reg}}(\hat w)$ from (3.1) as

$$A(\hat w) := \Big\{(v, \eta, q, \beta) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \times \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{\mathrm{dev}}) \times \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{\mathrm{dev}}) :$$
$$Ev = \eta + q \text{ in } \Omega,\ q = (\hat w - v) \odot \nu\mathcal{H}^{n-1} \text{ on } \partial\Omega;\ q - \beta \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{dev}})\Big\}$$

with $\hat w \in H^1(\Omega; \mathbb{R}^n)$. Without loss of generality, we extend $w \in H^1(0,T; H^1(\Omega, \mathbb{R}^n))$ by $w(T)$ for $t \geq T$.

Remark 4.1. In the spirit of [4, Remark 4.13], it can be easily established, through an adequate regularization of any pair $\left[\tau\middle|\eta\right]$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}\times\mathbb{M}^{n\times n}_{\rm dev})$ with τ_D, η in $L^\infty(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$ and $\operatorname{div}\tau_D$ in $L^n(\Omega;\mathbb{R}^n)$ that, for any $(u,e,p,\alpha)\in A(\hat w)$ (see definition above),

$$\mathcal{H}(\chi*\rho,\left[p\middle|\alpha\right])\geq\langle\tau_D,p\rangle-\int_\Omega\eta:(p-\alpha)dx.$$ ¶

The main result of the paper is the following existence result for a rescaled quasistatic evolution model for the Armstrong-Frederick plasticity model of non-linear kinematic hardening.

Theorem 4.2 (**Rescaled Armstrong-Frederick evolution).** *Let $w\in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$; let $(u_0,e_0,p_0,\alpha_0)\in A(w(0))$ be such that*

$$\operatorname{div}\sigma_0=0 \text{ a.e. in } \Omega \text{ and } \left[\sigma_0-\chi_0\middle|\chi_0\right]\in\mathcal{K}(\chi_0*\rho),$$

where $\sigma_0:=Ae_0$ and $\chi_0:=B(p_0-\alpha_0)$. Then, there exist $\overline{T}>0$ and a mapping $[0,\overline{T}]\ni s\mapsto(u^\circ(s),e^\circ(s),p^\circ(s),\alpha^\circ(s),t^\circ(s))$ such that

$$u^\circ:[0,\overline{T}]\to BD(\Omega) \text{ is strongly continuous and a.e. weakly* differentiable};$$

$$\left\{\begin{array}{ll} e^\circ & :[0,\overline{T}]\to L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) \\ p^\circ-\alpha^\circ & :[0,\overline{T}]\to L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})\end{array}\right. \text{ are strongly continuous and a.e. differentiable};$$

$$p^\circ,\alpha^\circ:[0,\overline{T}]\to\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev}) \text{ are 1-Lipschitz};$$

$$t^\circ:[0,\overline{T}]\to[0,\infty) \text{ is nondecreasing and 1-Lipschitz, with } t^\circ(\overline{T})\geq T.$$

Further, setting $\sigma^\circ:=Ae^\circ$, $\chi^\circ:=B(p^\circ-\alpha^\circ)$, the following properties are satisfied:

1. Initial condition: $(u^\circ(0),e^\circ(0),p^\circ(0),\alpha^\circ(0),t^\circ(0))=(u_0,e_0,p_0,\alpha_0,0)$;
2. Kinematic compatibility: *For every $s\in[0,\overline{T}]$, $(u^\circ(s),e^\circ(s),p^\circ(s),\alpha^\circ(s))\in A(w(t^\circ(s))$;*
3. Equilibrium condition: *For every $s\in[0,\overline{T}]$,*
$$\operatorname{div}\sigma^\circ(s)=0 \text{ a.e. in } \Omega;$$
4. Partial stress constraint: *For every $s\in[0,\overline{T}]\setminus U^\circ$,*
$$\left[\sigma^\circ_D-\chi^\circ\middle|\chi^\circ\right](s)\in\mathcal{K}(\chi^\circ(s)*\rho),$$
where $U^\circ:=\{s\in(0,\overline{T}]:t^\circ$ is constant in a neighborhood of $s\}$;
5. L^2-plastic strain for viscous times: *For a.e. $s\in[0,\overline{T}]$ with $\left[\sigma^\circ_D-\chi^\circ\middle|\chi^\circ\right](s)\notin\mathcal{K}(\chi^\circ(s)*\rho)$, $\dot p^\circ(s)\in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$;*
6. Maximum plastic work: *For a.e. $s\in[0,\overline{T}]$,*
$$\begin{aligned}&\left\langle\left[\sigma^\circ_D(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right]-P_{\mathcal{K}(\chi^\circ(s)*\rho)}(\left[\sigma^\circ_D(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right]),\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\right\rangle\\&+\mathcal{H}_M(\chi^\circ(s)*\rho,\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])=\langle\sigma^\circ_D(s),\dot p^\circ(s)\rangle-\int_\Omega\chi^\circ(s):(\dot p^\circ(s)-\dot\alpha^\circ(s))\,dx.\end{aligned}$$
Further, for a.e. $s\in[0,\overline{T}]$ with $\left[\sigma^\circ_D-\chi^\circ\middle|\chi^\circ\right](s)\notin\mathcal{K}(\chi^\circ(s)*\rho)$,
$$\left[\sigma^\circ_D(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right]-P_{\mathcal{K}(\chi^\circ(s)*\rho)}(\left[\sigma^\circ_D(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right]) \text{ is parallel to } \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right] \text{ a.e. in } \Omega.$$

Note that the equality in Item 6 above concides with the actual version of the classical Hill principle [36] whenever the partial stress constraint (Item 4) is fulfilled. This motivates our reference to Item 6 as of *maximum plastic work* principle.

As regards the regularity of solutions, one should mention that there exist rate-independent evolution in which better bounds can be derived. The reader is referred for instance to [49] where higher-order estimates are obtained by considering the time derivative of the flow rule. These techniques are particularly tailored to the case of a translation-invariant dissipation function. The extension of such results in the present setting would require the extra difficulty of allowing for a state-dependent dissipation function instead.

The proof of Theorem 4.2 is given in Subsections 4.1 to 4.4.

4.1. **The rescaled visco-plastic evolution.** First, we note that, by an argument identical to that in [4, Proposition 4.3],

$$\exists(u_0,e_0,p_0,\alpha_0)\in A(w(0)) \text{ with } \mathrm{div}\sigma_0=0 \text{ a.e. in } \Omega \text{ and } \big[\sigma_0-\chi_0\big|\chi_0\big]\in\mathcal{K}_M(\chi_0*\rho), \tag{4.1}$$

where $\sigma_0:=Ae_0$ and $\chi_0:=B(p_0-\alpha_0)$. Using e.g. [16, Lemma 5.1], we can construct a sequence $\{u_0^\varepsilon\}\subset H^1(\Omega;\mathbb{R}^n)$ such that $u_0^\varepsilon=w(0)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $u_0^\varepsilon\to u_0$ strongly in $L^1(\Omega;\mathbb{R}^n)$, and $Eu_0^\varepsilon\rightharpoonup Eu_0$ weakly* in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}})$. Setting $p_0^\varepsilon:=Eu_0^\varepsilon-e_0$ and $\alpha_0^\varepsilon:=\alpha_0-p_0+p_0^\varepsilon$, we get that $(u_0^\varepsilon,e_0,p_0^\varepsilon,\alpha_0^\varepsilon)\in A_{\mathrm{reg}}(w(0))$ satisfies

$$\begin{cases} u_0^\varepsilon\rightharpoonup u_0 \text{ weakly* in } BD(\Omega)\\ p_0^\varepsilon\rightharpoonup p_0 \text{ weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}})\\ \alpha_0^\varepsilon\rightharpoonup \alpha_0 \text{ weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}}).\end{cases} \tag{4.2}$$

Theorem 3.1 and Remark 3.2 then provide, for every $\varepsilon>0$, a unique ρ-visco-plastic (or visco-plastic) solution $(u_\varepsilon(t),e_\varepsilon(t),p_\varepsilon(t),\alpha_\varepsilon(t))$, for $t\in[0,T]$ (any $T<\infty$ will do), with $(u_0^\varepsilon,e_0,p_0^\varepsilon,\alpha_0^\varepsilon)$ as initial condition.

We rescale time as follows:

$$s_\varepsilon^\circ(t):=\int_0^t\left(\|\dot p_\varepsilon(s)\|_1+\|\dot\alpha_\varepsilon(s)\|_1+\|E\dot w(s)\|_2+1\right)ds,$$

so that $s\mapsto t_\varepsilon^\circ(s):=(s_\varepsilon^\circ)^{-1}(s)$ are strictly monotonically increasing and 1-Lipschitz on $[0,\infty)$, *i.e.*,

$$|t_\varepsilon^\circ(s_1)-t_\varepsilon^\circ(s_2)|\le|s_1-s_2|$$

for every s_1 and $s_2\ge0$. Note that $t_\varepsilon^\circ(0)=0$ and that, by virtue of (3.12),

$$\overline{T}:=2b\left(\int_0^T\|E\dot w(s)\|_2\,ds\right)+\int_0^T\|E\dot w(s)\|_2\,ds+T\ge s_\varepsilon^\circ(T),$$

so that $t_\varepsilon^\circ(\overline{T})\ge T$, for each $\varepsilon>0$.

Define, on $[0,\overline{T}]$,

$$\begin{gathered} w_\varepsilon^\circ(s):=w(t_\varepsilon^\circ(s)),\quad u_\varepsilon^\circ(s):=u_\varepsilon(t_\varepsilon^\circ(s)),\quad e_\varepsilon^\circ(s):=e_\varepsilon(t_\varepsilon^\circ(s)),\quad \sigma_\varepsilon^\circ(s):=\sigma_\varepsilon(t_\varepsilon^\circ(s))\\ p_\varepsilon^\circ(s):=p_\varepsilon(t_\varepsilon^\circ(s)),\quad \alpha_\varepsilon^\circ(s):=\alpha_\varepsilon(t_\varepsilon^\circ(s)),\quad \chi_\varepsilon^\circ(s):=\chi_\varepsilon(t_\varepsilon^\circ(s)).\end{gathered}$$

Remark that p_ε° and α_ε° are 1-Lipschitz on $[0,\overline{T}]$, as well as t_ε°. Then, since, by (4.2), both $s\mapsto p_\varepsilon^\circ(s)$ and $s\mapsto\alpha_\varepsilon^\circ(s)$ are uniformly bounded in $L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}}))$, Ascoli's theorem – bounded sets in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}})$ are relatively compact and metrizable for the weak* topology – implies the existence of a Lipschitz and nondecreasing function $t^\circ:[0,\overline{T}]\to[0,\infty)$ and of p°,α° in $\mathrm{Lip}([0,\overline{T}];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}}))$ such that, for some subsequence of ε, still labeled ε,

$$\begin{cases} t_\varepsilon^\circ(s) & \to & t^\circ(s) & \\ p_\varepsilon^\circ(s) & \rightharpoonup & p^\circ(s) & \text{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}})\\ \alpha_\varepsilon^\circ(s) & \rightharpoonup & \alpha^\circ(s) & \text{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}}),\end{cases} \tag{4.3}$$

uniformly on $[0,\overline{T}]$. Further, in view of (3.12),

$$\begin{gathered}\dot p_\varepsilon^\circ\rightharpoonup\dot p^\circ \text{ weakly* in } \mathcal{M}([0,T]\times\overline{\Omega}) \text{ and in } L^\infty_w(0,\overline{T};\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}}))\\ \dot\alpha_\varepsilon^\circ\rightharpoonup\dot\alpha^\circ \text{ weakly* in } \mathcal{M}([0,T]\times\overline{\Omega}) \text{ and in } L^\infty_w(0,\overline{T};\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{dev}})).\end{gathered} \tag{4.4}$$

Clearly, $w^0_\varepsilon(s) \to w^0(s)$ strongly in $H^1(\mathbb{R}^n;\mathbb{R}^n)$, uniformly on $[0,\overline{T}]$. Then, a proof identical to that leading to [4, Lemma 3.4] would establish the following.

Lemma 4.3. *For every $s \in [0,\overline{T}]$, there exists a quadruplet*

$$(u^\circ(s), e^\circ(s), p^\circ(s), \alpha^\circ(s)) \in A(w^\circ(s))$$

such that, with $\sigma^\circ(s) := Ae^\circ(s)$ and $\chi^\circ(s) := B(p^\circ(s) - \alpha^\circ(s))$,

$$\operatorname{div} \sigma^\circ(s) = 0, \ \text{a.e. in } \Omega,$$

and, for any sequence $s_\varepsilon \to s$,

$$\begin{array}{rcl} u^\circ_\varepsilon(s_\varepsilon) & \rightharpoonup & u^\circ(s) \text{ weakly* in } BD(\Omega) \\ e^\circ_\varepsilon(s_\varepsilon) & \rightharpoonup & e^\circ(s) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) \\ p^\circ_\varepsilon(s_\varepsilon) - \alpha^\circ_\varepsilon(s_\varepsilon) & \rightharpoonup & p^\circ(s) - \alpha^\circ(s) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}) \\ \sigma^\circ_\varepsilon(s_\varepsilon) & \rightharpoonup & \sigma^\circ(s) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) \\ \chi^\circ_\varepsilon(s_\varepsilon) & \rightharpoonup & \chi^\circ(s) \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}). \end{array} \tag{4.5}$$

Moreover, $(u^\circ(0), e^\circ(0), p^\circ(0), \alpha^\circ(0)) = (u_0, e_0, p_0, \alpha_0)$, $s \mapsto u^\circ(s)$ is weakly continuous in $BD(\Omega)$, $s \mapsto e^\circ(s)$, $s \mapsto \sigma^\circ(s)$ are weakly continuous in $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, and $s \mapsto p^\circ(s) - \alpha^\circ(s)$, $s \mapsto \chi^\circ(s)$ are weakly continuous in $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$.*

Remark 4.4. Note that the previous result implies in particular that $s \mapsto e^\circ(s)$ and $s \mapsto \sigma^\circ(s)$ are weakly measurable (with values in $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$), hence strongly measurable, so that, in view of the fifth item in Theorem 3.1, e°, σ° both belong to $L^\infty(0,\overline{T}; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$, while $p^\circ - \alpha^\circ, \chi^\circ$ both belong to $L^\infty(0,\overline{T}; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}))$. ¶

4.2. **Stress constraint.** Passing to the 0-viscosity limit in the rescaled modified stress constraint – see Remark 3.4 – is not convenient because the chain rule introduces a term of the form $\varepsilon \dot{p}^\circ_\varepsilon / \dot{t}^\circ_\varepsilon$ which we do not control.

We introduce the left-continuous (resp. right-continuous) inverse of t° defined by $s^\circ_-(t) := \sup\{s : t^\circ(s) < t\}$ (resp. $s^\circ_+(t) := \inf\{s : t^\circ(s) > t\}$) and $S^\circ := \{t \in (0,T) : s^\circ_-(t) < s^\circ_+(t)\}$. Here we use the convention $\sup \emptyset = 0$, so that $s^\circ_-(0) = 0$. Observe that $t^\circ(s^\circ_-(t)) = t^\circ(s^\circ_+(t)) = t$ and that the set S° is at most countable. By [16, Lemma 5.2], we know that for each $t \notin S^\circ$, $s^\circ_\varepsilon(t) \to s^\circ_-(t) = s^\circ_+(t)$. Hence, in view of convergences (4.3), (4.5) and since p°_ε and α°_ε are 1-Lipschitz, we have that, for all $t \notin S^\circ$,

$$\begin{array}{rclll} u_\varepsilon(t) & \rightharpoonup & u^\circ(s^\circ_-(t)) & \text{weakly* in } BD(\Omega), \\ e_\varepsilon(t) & \rightharpoonup & e^\circ(s^\circ_-(t)) & \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}), \\ (p_\varepsilon - \alpha_\varepsilon)(t) & \rightharpoonup & (p^\circ - \alpha^\circ)(s^\circ_-(t)) & \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}), \\ \sigma_\varepsilon(t) & \rightharpoonup & \sigma^\circ(s^\circ_-(t)) & \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}), \\ \chi_\varepsilon(t) & \rightharpoonup & \chi^\circ(s^\circ_-(t)) & \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}), \\ p_\varepsilon(t) & \rightharpoonup & p^\circ(s^\circ_-(t)) & \text{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev}), \\ \alpha_\varepsilon(t) & \rightharpoonup & \alpha^\circ(s^\circ_-(t)) & \text{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev}). \end{array} \tag{4.6}$$

Recall that

$$U^\circ = \{s \in (0,\overline{T}] : t^\circ \text{ is constant in a neighborhood of } s\},$$

and note that $U^\circ = \bigcup_{t\in S^\circ} (s^\circ_-(t), s^\circ_+(t))$, hence that it is open.

Then, the following partial stress constraint property holds:

Lemma 4.5 (**Partial stress constraint**). *For every $s \notin U^\circ$, one has*

$$\left[(\sigma_D^\circ)(s) - \chi^\circ(s) \middle| \chi^\circ(s)\right] \in \mathcal{K}_M(\chi^\circ(s) * \rho). \tag{4.7}$$

Proof. Thanks to the energy equality in the second item of Remark 3.4 and to the fifth item in Theorem 3.1,

$$\varepsilon \left[\dot{p}_\varepsilon(t) \middle| \dot{\alpha}_\varepsilon(t)\right] \to 0 \text{ strongly in } L^2(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}} \times \mathbb{M}^{n\times n}_{\text{dev}})), \tag{4.8}$$

and also a.e. in $\Omega \times (0,T)$. Recall the modified stress constraint from that same remark, namely

$$\left[(\sigma_\varepsilon)_D(t) - \chi_\varepsilon(t) - \varepsilon\dot{p}_\varepsilon(t) \middle| \chi_\varepsilon(t) - \varepsilon\dot{\alpha}_\varepsilon(t)\right] \in \partial H_M(\chi_\varepsilon(t) * \rho, \left[0 \middle| 0\right])$$

for a.e. $t \in [0,T]$. Then, for any $\left[q\middle|\beta\right] \in (\mathbb{M}^{n\times n}_{\text{dev}})^2$ and for a.e. $(x,t) \in \Omega \times [0,T]$,

$$H_M((\chi_\varepsilon(t) * \rho)(x), \left[q\middle|\beta\right]) \geq ((\sigma_\varepsilon)_D(t) - \chi_\varepsilon(t) - \varepsilon\dot{p}_\varepsilon(t)) : q + (\chi_\varepsilon(t) - \varepsilon\dot{\alpha}_\varepsilon(t)) : \beta.$$

Because of the fifth convergence in (4.6) and of the bound on $p_\varepsilon - \alpha_\varepsilon$ in item 5. of Theorem 3.1, one has that $\chi_\varepsilon(t) * \rho \to \chi^\circ(s^\circ_-(t)) * \rho$ a.e. in Ω and in $L^p(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$, $p < \infty$. Consider a measurable subset $E \subset \Omega$. Integrating the relation above over E, recalling Lemma 2.4 and convergence (4.8), we may pass to the limit in the latter inequality and obtain, for a.e. $t \in [0,T]$,

$$H_M((\chi^\circ(s^\circ_-(t)) * \rho)(x), \left[q\middle|\beta\right]) \geq (\sigma^\circ_D(s^\circ_-(t),x) - \chi^\circ(s^\circ_-(t),x)) : q + \chi^\circ(s^\circ_-(t),x) : \beta \text{ for a.e. } x \in \Omega,$$

or, equivalently,

$$\left[\sigma^\circ_D(s^\circ_-(t)) - \chi^\circ(s^\circ_-(t),x) \middle| \chi^\circ(s^\circ_-(t),x)\right] \in \mathcal{K}_M(\chi^\circ(s^\circ_-(t),x) * \rho).$$

By the left continuity of s°_- and the weak continuity in $L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ of σ° and χ°, we conclude that the previous relation actually holds for every $t \in [0,T]$. A similar argument would lead to

$$\left[\sigma^\circ_D(s^\circ_+(t)) - \chi^\circ(s^\circ_+(t),x) \middle| \chi^\circ(s^\circ_+(t),x)\right] \in \mathcal{K}_M(\chi^\circ(s^\circ_+(t),x) * \rho)).$$

Since $s^\circ_-(t) < s^\circ_+(t)$ if and only if $t^\circ(s)$ is constant over the interval $[s_0^-(t), s_0^+(t)]$, we finally obtain (4.7). $\square$

Let us now introduce the following sets:

$$A^\circ := \{s \in [0,\overline{T}] : \text{dist}_2(\left[\sigma^\circ_D(s) - \chi^\circ(s) \middle| \chi^\circ(s)\right], \mathcal{K}_M(\chi^\circ(s) * \rho)) > 0, \quad B^\circ := [0,\overline{T}] \setminus A^\circ.$$

By Lemma 4.5, the inclusion $A^\circ \subset U^\circ$ holds and, in view of the right inclusion in (2.9), one has that $(\sigma^\circ_D(s), \chi^\circ(s)) \in L^\infty(\Omega; (\mathbb{M}^{n\times n}_{\text{dev}})^2)$ for all $s \in B^\circ$. The function $\sigma^\circ_D(s)$ may fail to belong to $L^\infty(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ for $s \in A^\circ$ but this will be compensated by a higher regularity on the plastic strain.

The following lemma, whose proof is identical to that of [4, Lemmata 3.7, 3.8] holds.

Lemma 4.6 (**Lower semi-continuity of the viscous dissipation**). *The set A° is relatively open in $[0,\overline{T}]$, and, for every $S \in [0,\overline{T}]$,*

$$\int_{A^\circ \cap [0,S]} \| \left[\dot{p}^\circ(s) \middle| \dot{\alpha}^\circ(s)\right] \|_2 \, \text{dist}_2(\left[\sigma^\circ_D(s) - \chi^\circ(s) \middle| \chi^\circ(s)\right], \mathcal{K}_M(\chi^\circ(s) * \rho))\, ds$$
$$\leq \liminf_{\varepsilon \to 0} \int_{A^\circ \cap [0,S]} \| \left[\dot{p}^\circ_\varepsilon(s) \middle| \alpha^\circ_\varepsilon(s)\right] \|_2 \, \text{dist}_2(\left[(\sigma^\circ_\varepsilon)_D(s) - \chi^\circ_\varepsilon(s) \middle| \chi^\circ_\varepsilon(s)\right], \mathcal{K}_M(\chi^\circ_\varepsilon(s) * \rho))\, ds. \tag{4.9}$$

Moreover, $\left[p^\circ \middle| \alpha^\circ\right] \in W^{1,1}_{\text{loc}}(A^\circ; (L^2(\Omega; \mathbb{M}^{n\times n}_{\text{dev}}))^2)$, while $e^\circ \in W^{1,1}_{\text{loc}}(A^\circ; L^2(\Omega; \mathbb{M}^{n\times n}_{\text{dev}}))$ and $u^\circ \in W^{1,1}_{\text{loc}}(A^\circ; H^1(\Omega; \mathbb{R}^n))$.

Remark 4.7. If (a,b) is any connected component of A°, then $u^\circ(s_1) - u^\circ(s_2) \in H^1_0(\Omega; \mathbb{R}^n)$ for any $a < s_1 \leq s_2 < b$. ¶

Remark 4.8. Since $A^\circ \ni s \mapsto (u^\circ(s), e^\circ(s), p^\circ(s), \alpha^\circ(s))$ is absolutely continuous with values in (the reflexive space) $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$ we deduce, from [7, Appendix], that the derivative $(\dot u^\circ(s), \dot e^\circ(s), \dot p^\circ(s), \dot\alpha^\circ(s))$ exists for a.e. $s \in A^\circ$ for the strong topology of $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm dev})$. Moreover, for a.e. $s \in A^\circ$, that derivative belongs to $A(0)$ since $s \mapsto w^\circ(s)$ is constant in each connected components of A°. Finally $\operatorname{div} \dot\sigma^\circ(s) = 0$ a.e. in Ω. ¶

The dissipated energy is lower semicontinuous.

Lemma 4.9 (**Lower semi-continuity of the dissipated energy**). *For every $S \in [0,\overline{T}]$, we have*

$$\liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}_M(\chi^\circ_\varepsilon(s) * \rho, \left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right])\, ds \geq \int_0^S \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\, ds.$$

Proof. According to (4.3), (4.4), for any $\varphi \in C(\bar\Omega;\mathbb{M}^{n\times n}_{\rm dev}),\ \psi \in C([0,\overline{T}])$,

$$\int_{[0,\overline{T}]\times\bar\Omega} \varphi(x)\psi(s) d\left[\dot p^\circ\middle|\dot\alpha^\circ\right] = \int_0^{\overline{T}} \psi(s) \int_{\bar\Omega} \varphi(x) d\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\ ds,$$

so that, by virtue of (4.4), the measure $\left[\dot p^\circ\middle|\dot\alpha^\circ\right]$ disintegrates with respect to the one-dimensional Lebesgue measure $\mathcal{L}^1_s$ as:

$$\left[\dot p^\circ\middle|\dot\alpha^\circ\right] = |\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]|(\Omega)\mathcal{L}^1_s \overset{\rm gen}{\otimes} \frac{\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]}{|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]|(\Omega)}, \tag{4.10}$$

where $\overset{\rm gen}{\otimes}$ stands for the *generalized product* (see [1, Sect. 2.5])

Appealing to Lemma 2.4, and since $p^\circ_\varepsilon, \alpha^\circ_\varepsilon$ are equi-Lipschitz on $[0,\overline{T}]$, for a.e. $s \in [0,\overline{T}]$,

$$\begin{aligned}
&|\mathcal{H}_M(\chi^\circ_\varepsilon(s) * \rho, \left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right]) - \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right])| \\
&\leq C_M \|(\chi^\circ_\varepsilon(s) - \chi^\circ(s)) * \rho\|_\infty \, \|\left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right]\|_1 \\
&\leq C_M \|(\chi^\circ_\varepsilon(s) - \chi^\circ(s)) * \rho\|_\infty,
\end{aligned}$$

while, by (4.5) and (2.21), $\chi^\circ_\varepsilon(s) * \rho \to \chi^\circ(s) * \rho$ uniformly on $\overline{\Omega}$. Dominated convergence yields

$$\liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}_M(\chi^\circ_\varepsilon(s) * \rho, \left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right])\, ds = \liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right])\, ds.$$

But the weak L^2-continuity in time of $s \mapsto \chi^\circ(s)$ established in Lemma 4.3 implies that $\chi^\circ * \rho \in \mathcal{C}([0,\overline{T}] \times \overline{\Omega};\mathbb{M}^{n\times n}_{\rm sym})$, so that $H_M([\sigma^\circ(s) * \rho](x), \xi)$ is continuous in (x, s, ξ) and convex, one-homogeneous in ξ. In view of (4.4), Reshetnyak's Theorem (see, *e.g.*, [1, Theorem 2.38]) yields

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right]) ds \geq \\
&\qquad \int_{[0,S]\times\overline{\Omega}} H_M\left([\chi^\circ(s) * \rho](x), \frac{d\left[\dot p^\circ\middle|\dot\alpha^\circ\right]}{d\left|\left[\dot p^\circ\middle|\dot\alpha^\circ\right]\right|}(s,x)\right) d\left|\left[\dot p^\circ\middle|\dot\alpha^\circ\right]\right|(s,x).
\end{aligned}$$

But [1, Corollary 2.29] applied to (4.10) yields

$$\left|\left[\dot p^\circ\middle|\dot\alpha^\circ\right]\right| = |\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]|(\Omega)\mathcal{L}^1_s \overset{\rm gen}{\otimes} \frac{|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]|}{|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]|(\Omega)}.$$

An application of [1, Theorem 2.28] implies that

$$s \mapsto \int_{\overline{\Omega}} H\left([\chi^\circ(s) * \rho](x), \frac{d\left[\dot p^\circ \middle| \dot\alpha^\circ\right]}{d|\left[\dot p^\circ \middle| \dot\alpha^\circ\right]|}(s,x)\right) \frac{1}{|\left[\dot p^\circ(s) \middle| \dot\alpha^\circ(s)\right]|(\Omega)} d|\left[\dot p^\circ(s) \middle| \dot\alpha^\circ(s)\right]|(x)$$
$$\in L^1(0,S;|\left[\dot p^\circ(s) \middle| \dot\alpha^\circ(s)\right]|(\Omega)\mathcal{L}^1_s),$$

that is that $s \mapsto \mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s) \middle| \dot\alpha^\circ(s)\right])$ is $\mathcal{L}^1_s$-measurable and that

$$\int_{[0,S]\times\overline{\Omega}} H_M\left([\chi^\circ(s) * \rho](x), \frac{d\left[\dot p^\circ \middle| \dot\alpha^\circ\right]}{d|\left[\dot p^\circ \middle| \dot\alpha^\circ\right]|}(s,x)\right) d|\left[\dot p^\circ \middle| \dot\alpha^\circ\right]|(s,x) = \int_0^S \mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s) \middle| \dot\alpha^\circ(s)\right])ds,$$

which completes the proof of the lemma. □

4.3. **Rescaled principle of maximum plastic work.** It is common mechanical knowledge that, in classical elasto-plasticity, the flow rule is equivalent to what is usually referred to as Hill's principle of maximum plastic work. The equivalent statement in our (rescaled) context is that

$$\langle \tau_D, \dot p^\circ(s)\rangle - \int_\Omega \chi : (\dot p^\circ(s) - \dot\alpha^\circ(s))dx \le \langle \sigma_D^\circ(s), \dot p^\circ(s)\rangle - \int_\Omega \chi^\circ(s) : (\dot p^\circ(s) - \dot\alpha^\circ(s))dx$$
$$\text{for a.e. } s \in (0,\overline{T}), \text{ and every } \left[\tau \middle| \chi\right] \in L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}} \times \mathbb{M}^{n\times n}_{\text{dev}})$$
$$\text{with } \operatorname{div}\tau = 0 \text{ and } (\tau_D - \chi)(x) \in K_M([\chi^\circ(s)*\rho](x)) \text{ for a.e. } x \in \Omega.$$

Further, since H is a support function, this is also formally equivalent to

$$\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s) \middle| \dot\alpha^\circ(s)\right]) = \langle \sigma_D^\circ(s), \dot p^\circ(s)\rangle - \int_\Omega \chi^\circ(s) : (\dot p^\circ(s) - \dot\alpha^\circ(s))dx \text{ for a.e. } s \in (0,\overline{T}). \tag{4.11}$$

The maximum plastic work identity (the sixth item of Theorem 4.2) can be viewed as a variant of the previous equality, accounting for the fact that the stress constraint could not be met at all times, but only in B^0.

Of course, the relations above are only meaningful if, on the one hand, the duality $\langle\cdot,\cdot\rangle$ is meaningful for the quantities involved (see (2.18)), that is here if $\dot p^\circ(s) \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\text{dev}})$ can be associated to a pair $(\dot u^\circ(s), \dot e^\circ(s))$ so that the kinematic compatibility (2.17) is satisfied, and, on the other hand, if the integrals are well defined, that is if $\dot p^\circ(s) - \dot\alpha^\circ(s) \in L^2(\Omega;\mathbb{M}^{n\times n}_{\text{dev}})$.

The proof of the sixth item of Theorem 4.2 (the remaining item in that theorem that needs proof) will be the object of Theorem 4.13 below, the proof of which is in turn based on the following derivability result.

Theorem 4.10 (**Derivability of the strain and back strain**). *The maps*

$$s \mapsto e^\circ(s) \quad \text{and} \quad s \mapsto p^\circ(s) - \alpha^\circ(s)$$

are differentiable for a.e. $s \in [0,\overline{T}]$ for the strong $L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})$-topology. Moreover, for every $0 \le s_1 \le s_2 \le \overline{T}$,

$$\mathcal{Q}(e^\circ(s_2), p^\circ(s_2) - \alpha^\circ(s_2)) - \mathcal{Q}(e^\circ(s_1), p^\circ(s_1) - \alpha^\circ(s_1)) =$$
$$\int_{s_1}^{s_2}\int_\Omega \{\sigma^\circ(s,x) : \dot e^\circ(s,x) + \chi^\circ(s) : (\dot p^\circ(s) - \dot\alpha^\circ(s))\}\,dx\,ds. \tag{4.12}$$

Finally, the map $s \mapsto (u^\circ(s), e^\circ(s), p^\circ(s) - \alpha^\circ(s))$ is strongly continuous in $BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\text{dev}})$.

Remark 4.11. In view of the Lipschitz character of $p^\circ(s)$, of Theorem 4.10, and upon appealing to Poincaré-Korn's inequality in $BD(\Omega)$, we conclude that the map $s \mapsto u^\circ(s)$ is weakly* differentiable in $BD(\Omega)$ for a.e. $s \in (0, \overline{T})$, and that the quadruplet $(\dot u^\circ(s), \dot e^\circ(s), \dot p^\circ(s), \dot\alpha^\circ(s))$ belongs to $A(\dot w^\circ(s))$ for those s's. ¶

Proof. The proof of Theorem 4.10 is based on the following energy equality:

Proposition 4.12 (Energy equality in rescaled times). *For every $S \in [0, \overline{T}]$,*

$$\mathcal{Q}(e^\circ(S), p^\circ(S) - \alpha^\circ(S)) + \int_0^S \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\, ds + \int_0^S \| \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right] \|_2 \operatorname{dist}_2(\left[\sigma_D^\circ(s) - \chi^\circ(s)\middle|\chi^\circ(s)\right], \mathcal{K}_M(\chi^\circ(s) * \rho)\, ds$$
$$= \mathcal{Q}(e^\circ(0), p^\circ(0) - \alpha^\circ(0)) + \int_0^S \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\, dx\, ds. \quad (4.13)$$

We will not provide the proof of Proposition 4.12 since it is a verbatim adaptation of that of [4, Propositions 3.15, 3.18]. Note however that the proofs in [4] in turn closely follow that of [16, Equation (8.2)] (the difficult point is to show the $\geq$ in the equality of Proposition 4.12).

In particular, from the above proposition we deduce that, for every $0 \le s_1 \le s_2 \le \overline{T}$,

$$\mathcal{Q}(e^\circ(s_2), p^\circ(s_2) - \alpha^\circ(s_2)) + \int_{s_1}^{s_2} \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\, ds \le$$
$$\mathcal{Q}(e^\circ(s_1), p^\circ(s_1) - \alpha^\circ(s_1)) + \int_{s_1}^{s_2} \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\, dx\, ds. \quad (4.14)$$

Thanks to this relation, we are now in a position to assert the almost everywhere differentiability in time of $s \mapsto e^\circ(s)$ and of $s \mapsto p^\circ(s) - \alpha^\circ(s)$ (see below).

Finally, the energy equality immediately implies that

$$Q(e^\circ(s), p^\circ(s) - \alpha^\circ(s)) \in W^{1,1}(0, \overline{T}). \quad (4.15)$$

According to Remark 4.8, we already know that $s \mapsto e^\circ(s)$ and of $s \mapsto p^\circ(s) - \alpha^\circ(s)$ are absolutely continuous in A° with values in the reflexive space $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ (or $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm dev})$). Hence, from [7, Appendix] we conclude that those are differentiable almost everywhere in A° for the strong $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$, respectively $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm dev})$, topology. It suffices to prove the almost everywhere differentiability of e° and of $p^\circ - \alpha^\circ$ in B°.

Let $0 \le s_1 \le s_2 \le \overline{T}$, and assume that $s_1 \in B^\circ$. Thanks to the Lipschitz continuity of H in its first variable (see Lemma 2.4) and to the Lipschitz character of p°, α°, for a.e. $s \in (s_1, s_2)$,

$$\begin{aligned} &\mathcal{H}_M(\chi^\circ(s_1) * \rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]) \\ &\quad \le \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]) + C_M \| \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right] \|_1 \|(\chi^\circ(s) - \chi^\circ(s_1)) * \rho\|_\infty \\ &\quad \le \mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]) + C\|(p^\circ - \alpha^\circ)(s) - (p^\circ - \alpha^\circ)(s_1)\|_2, \end{aligned}$$

for some constant $C > 0$ independent of s and s_1. Next, using (4.14) between s_1 and s_2, we infer that

$$\mathcal{Q}(e^\circ(s_2), p^\circ(s_2) - \alpha^\circ(s_2)) + \int_{s_1}^{s_2} \mathcal{H}_M(\chi^\circ(s_1) * \rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\, ds \le \mathcal{Q}(e^\circ(s_1), p^\circ(s_1) - \alpha^\circ(s_1))$$
$$+ \int_{s_1}^{s_2} \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\, dx\, ds + C \int_{s_1}^{s_2} \|(p^\circ - \alpha^\circ)(s) - (p^\circ - \alpha^\circ)(s_1)\|_2\, ds. \quad (4.16)$$

Since $\chi^\circ(s_1) * \rho$ is continuous, it is uniformly continuous on $\overline{\Omega}$. Thus for each $\varepsilon > 0$, there exists $\delta > 0$ such that if x and $y \in \overline{\Omega}$ are such that $|x-y| < \delta$, then $|[\chi^\circ(s_1) * \rho](x) - [\chi^\circ(s_1) * \rho](y)| < \varepsilon$. Let us split $\overline{\Omega}$ into a finite family of pairwise disjoint sets $\{Q_i\}_{1 \le i \le m_\varepsilon}$ such that

$$\begin{cases} \overline{\Omega} = \bigcup_{i=1}^{m_\varepsilon} \overline{Q_i}, \quad \operatorname{diam}(Q_i) < \delta, \\ \displaystyle\int_{s_1}^{s_2} | \left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)|(\Omega \cap \partial Q_i)\, ds = | \left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)|(\Omega \cap \partial Q_i) = 0 \end{cases}$$

for all $i \in \{1, \dots, m_\varepsilon\}$. Fix a point $x_i \in Q_i$. Then, appealing to Lemma 2.4, for a.e. $s \in (s_1, s_2)$,

$$\int_{\overline{Q_i}} \Bigg| H_M\left([\chi^\circ(s_1) * \rho](x), \frac{d\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)}{d\left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)\right|}(x)\right) \\ - H_M\left([\chi^\circ(s_1) * \rho](x_i), \frac{d\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)}{d\left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)\right|}(x)\right)\Bigg| \, d\left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)\right|(x) \le C_M \varepsilon \left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right]\right|(Q_i).$$

Hence,

$$\int_{s_1}^{s_2} \mathcal{H}_M(\chi^\circ(s_1) * \rho, \left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s))\, ds \\ \ge \sum_{i=1}^{m} \int_{s_1}^{s_2} \int_{\overline{Q_i}} H\left([\chi^\circ(s_1) * \rho](x_i), \frac{d\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)}{d\left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)\right|}(x)\right) d\left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)\right|(x)\, ds \\ - C_M \varepsilon (s_2 - s_1). \quad (4.17)$$

By virtue of [13, Theorem 7.1] applied to $\mathcal{H}_M([\chi^\circ(s_1) * \rho](x_i), \cdot)$, we get, for each $i \in \{1, \dots, m_\varepsilon\}$,

$$\int_{s_1}^{s_2} H\left([\chi^\circ(s_1) * \rho](x_i), \frac{d\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)}{d\left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)\right|}(x)\right) d\left|\left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s)\right|(x)\, ds \\ \ge \int_{\overline{Q_i}} H\left([\chi^\circ(s_1) * \rho](x_i), \frac{d\left[\left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right]}{d\left|\left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right|}(x)\right) d\left|\left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right|(x) \\ \ge \int_{\overline{Q_i}} H\left([\chi^\circ(s_1) * \rho](x), \frac{d\left[\left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right]}{d\left|\left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right|}(x)\right) d\left|\left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right|(x) \\ - C_M \varepsilon \left|\left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right|(Q_i),$$

where we used Lemma 2.4 once again. Using the Lipschitz character of p°, α° once again, we conclude from (4.17) that

$$\int_{s_1}^{s_2} \mathcal{H}_M(\chi^\circ(s_1) * \rho, \left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s))\, ds \ge \mathcal{H}\left(\chi^\circ(s_1) * \rho, \left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right) - 2C_M \varepsilon (s_2 - s_1),$$

and letting, $\varepsilon \searrow 0$, that

$$\int_{s_1}^{s_2} \mathcal{H}_M(\chi^\circ(s_1) * \rho, \left[\dot{p}^\circ \middle| \dot{\alpha}^\circ\right](s))\, ds \ge \mathcal{H}\left(\chi^\circ(s_1) * \rho, \left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right).$$

Thus, (4.16) yields

$$\mathcal{Q}(e^\circ(s_2), p^\circ(s_2) - \alpha^\circ(s_2)) + \mathcal{H}\left(\chi^\circ(s_1) * \rho, \left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right) \le \\ \mathcal{Q}(e^\circ(s_1), p^\circ(s_1) - \alpha^\circ(s_1)) + \int_{s_1}^{s_2} \int_\Omega \sigma^\circ(s) : E\dot{w}^\circ(s)\, dx\, ds + C \int_{s_1}^{s_2} \|(p^\circ - \alpha^\circ)(s) - (p^\circ - \alpha^\circ)(s_1)\|_2\, ds.$$

Since $s_1 \in B^\circ$, $\left[\sigma^\circ - \chi^\circ \middle| \chi^\circ\right](s_1) \in \mathcal{K}_M(\chi^\circ(s_1) * \rho)$ and thus, by virtue of Remark 4.1,

$$\langle \sigma^\circ_D(s_1), p^\circ(s_2) - p^\circ(s_1)\rangle - \int_\Omega \chi^\circ(s_1) : [(p^\circ - \alpha^\circ)(s_2) - (p^\circ - \alpha^\circ)(s_1)]\, dx \le \mathcal{H}\left(\chi^\circ(s_1) * \rho, \left[p^\circ \middle| \alpha^\circ\right](s_2) - \left[p^\circ \middle| \alpha^\circ\right](s_1)\right).$$

Hence,

$$\begin{aligned}\mathcal{Q}(e^\circ(s_2), p^\circ(s_2) - \alpha^\circ(s_2)) &+ \langle \sigma^\circ_D(s_1), p^\circ(s_2) - p^\circ(s_1)\rangle \\ &- \int_\Omega \chi^\circ(s_1) : [(p^\circ - \alpha^\circ)(s_2) - (p^\circ - \alpha^\circ)(s_1)]\, dx \\ &\le \mathcal{Q}(e^\circ(s_1), p^\circ(s_1) - \alpha^\circ(s_1)) + \int_{s_1}^{s_2}\int_\Omega \sigma^\circ(s) : E\dot{w}^\circ(s)\, dx\, ds \\ &\quad + C\int_{s_1}^{s_2} \|(p^\circ - \alpha^\circ)(s) - (p^\circ - \alpha^\circ)(s_1)\|_2\, ds.\end{aligned}$$

Since

$$\begin{aligned}&\mathcal{Q}(e^\circ(s_2), p^\circ(s_2) - \alpha^\circ(s_2)) - \mathcal{Q}(e^\circ(s_1), p^\circ(s_1) - \alpha^\circ(s_1)) \\ &= \mathcal{Q}(e^\circ(s_2) - e^\circ(s_1), (p^\circ(s_2) - \alpha^\circ(s_2)) - (p^\circ(s_1) - \alpha^\circ(s_1))) + \int_\Omega \{\sigma^\circ(s_1) : (e^\circ(s_2) - e^\circ(s_1)) \\ &\qquad + \chi^\circ(s_1) : [(p^\circ - \alpha^\circ)(s_2) - (p^\circ - \alpha^\circ)(s_1)]\}\, dx,\end{aligned}$$

we obtain

$$\begin{aligned}&\mathcal{Q}(e^\circ(s_2) - e^\circ(s_1), (p^\circ(s_2) - \alpha^\circ(s_2)) - (p^\circ(s_1) - \alpha^\circ(s_1))) \le \\ &\quad \int_{s_1}^{s_2}\int_\Omega (\sigma^\circ(s) - \sigma^\circ(s_1)) : E\dot{w}^\circ(s)\, dx\, ds + C\int_{s_1}^{s_2} \|(p^\circ - \alpha^\circ)(s) - (p^\circ - \alpha^\circ)(s_1)\|_2\, ds.\end{aligned}$$

In deriving the inequality above, we have also made use of kinematic compatibility, and of the duality (2.18), together with the fact that $\sigma^\circ(s_1)$ is divergence free.

In view of the coercivity (2.20) of $\mathcal{Q}$, Cauchy-Schwartz inequality and the fact that $\|E\dot{w}^\circ(s)\|_2 \le 1$ for a.e. s, we obtain that

$$\begin{aligned}&\|e^\circ(s_2) - e^\circ(s_1)\|_2^2 + \|(p^\circ(s_2) - \alpha^\circ(s_2)) - (p^\circ(s_1) - \alpha^\circ(s_1))\|_2^2 \le \\ &\qquad C\int_{s_1}^{s_2} \|(p^\circ(s) - \alpha^\circ(s)) - (p^\circ(s_1) - \alpha^\circ(s_1))\|_2\, ds,\end{aligned}$$

for some constant $C > 0$ independent of s_1 and s_2. Hence a form of Gronwall Lemma implies that

$$\|e^\circ(s_2) - e^\circ(s_1)\|_2 + \|(p^\circ(s_2) - \alpha^\circ(s_2)) - (p^\circ(s_1) - \alpha^\circ(s_1))\|_2 \le L(s_2 - s_1), \tag{4.18}$$

for some constant $L > 0$ (independent of s_1 and s_2) for every $0 \le s_1 \le s_2 \le \overline{T}$ with $s_1 \in B^\circ$. Thus, by [17, Theorem 3.1], $s \mapsto e^\circ(s)$ and $s \mapsto p^\circ(s) - \alpha^\circ(s)$ are differentiable almost everywhere in B° for the strong $L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ topology.

Then, we deduce that

$$\frac{d}{ds}\mathcal{Q}(e^\circ(s), p^\circ(s) - \alpha^\circ(s)) = \int_\Omega \{\sigma^\circ(s) :\ \dot{e}^\circ(s)\, dx + \chi^\circ(s) : (\dot{p}^\circ(s) - \dot{\alpha}^\circ(s))\}\, dx$$

for a.e. $s \in [0, \overline{T}]$, and thus relation (4.12) follows, since, thanks to (4.15), the right hand-side of the previous equality is in $L^1(0, \overline{T})$.

By Remark 4.8 together with (4.18), we conclude that $s \mapsto (e^\circ(s), p^\circ(s) - \alpha^\circ(s))$ is actually strongly continuous into $L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\text{dev}})$, and recalling that $p^\circ(s)$ is Lipschitz into

$\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{\rm dev})$, together with Poincaré-Korn's inequality and the H^1-regularity of w°, that $s\mapsto u^\circ(s)$ is strongly continuous into $BD(\Omega)$. □

We now establish the maximum plastic work identity for the vanishing viscosity limit.

Theorem 4.13 (**Maximum plastic work**). *For a.e. $s\in A^\circ$, one has*

$$\int_\Omega \big(\left[(\sigma^\circ)_D-\chi^\circ)\middle|\chi^\circ\right](s)-P_{\mathcal{K}_M(\chi^\circ(s)*\rho)}(\left[\sigma^\circ_D-\chi^\circ)\middle|\chi^\circ\right](s))\big)\cdot\left[\dot p^\circ\middle|\dot\alpha^\circ\right](s)\,dx \\ +\mathcal{H}_M(\chi^\circ(s)*\rho,\left[\dot p^\circ\middle|\dot\alpha^\circ\right](s))=\int_\Omega\{\sigma^\circ(s):\dot p^\circ(s)-\chi^\circ(s):(\dot p^\circ-\dot\alpha^\circ)(s)\}\,dx, \quad (4.19)$$

and for a.e. $s\in B^\circ$,

$$\mathcal{H}_M(\chi^\circ(s)*\rho,\left[\dot p^\circ\middle|\dot\alpha^\circ\right](s))=\langle\sigma^\circ_D(s),\dot p^\circ(s)\rangle-\int_\Omega\chi^\circ(s):(\dot p^\circ(s)-\dot\alpha^\circ(s))dx. \quad (4.20)$$

Further,

$$\left[\sigma^\circ_D(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right]-P_{\mathcal{K}_M(\chi^\circ(s)*\rho)}(\left[\sigma^\circ_D(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right]) \\ \textit{is parallel to } \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\ \textit{a.e. in } \Omega. \quad (4.21)$$

Proof. For a null test function, the flow rule in the fourth item of Theorem 3.1, once integrated over $\Omega\times(0,t^\circ_\varepsilon(S)),\ S\in[0,\overline{T}]$ yields

$$\int_0^{t^\circ_\varepsilon(S)}\int_\Omega\left[(\sigma_D)_\varepsilon(s)-\chi_\varepsilon(s)-\varepsilon\dot p_\varepsilon(s)\middle|\chi_\varepsilon(s)-\varepsilon\dot\alpha_\varepsilon(s)\right]\cdot\left[\dot p_\varepsilon(s)\middle|\dot\alpha_\varepsilon(s)\right]\,dx\,ds\geq \\ \int_0^{t^\circ_\varepsilon(S)}\mathcal{H}_M(\chi_\varepsilon(s)*\rho,\left[\dot p_\varepsilon(s)\middle|\dot\alpha_\varepsilon(s)\right])\,ds.$$

Rescaling time with the map t°_ε, the above implies, thanks to the chain rule, to the fourth item in Theorem 3.1, and to the 1-homogeneous character of $\left[p\middle|\alpha\right]\mapsto H_M(\chi,\left[p\middle|\alpha\right])$,

$$\liminf_{\varepsilon\to0}\left\{\int_0^S\mathcal{H}_M(\chi_\varepsilon(s)*\rho,\left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right])\,ds\right. \\ \left.+\int_0^S\|\left[\dot p^\circ_\varepsilon(s)\middle|\dot\alpha^\circ_\varepsilon(s)\right]\|_2\,\mathrm{dist}_2(\left[(\sigma^\circ_\varepsilon)_D(s)-\chi^\circ_\varepsilon(s)\middle|\chi^\circ_\varepsilon(s)\right],\mathcal{K}_M(\chi^\circ_\varepsilon(s)*\rho))\,ds\right\} \\ \leq\limsup_{\varepsilon\to0}\int_0^S\int_\Omega\{\sigma^\circ_\varepsilon(s):\dot p^\circ_\varepsilon(s)-\chi^\circ_\varepsilon(s):(\dot p^\circ_\varepsilon(s)-\dot\alpha^\circ_\varepsilon(s))\}\,dx\,ds.$$

Then, according to Lemmas 4.6 and 4.9, we get in particular that

$$\int_0^S\mathcal{H}_M(\chi^\circ(s)*\rho,\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\,ds \\ +\int_0^S\|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\|_2\,\mathrm{dist}_2(\left[\sigma^\circ(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right],\mathcal{K}_M(\chi^\circ(s)*\rho))\,ds \\ \leq\limsup_{\varepsilon\to0}\int_0^S\int_\Omega\{\sigma^\circ_\varepsilon(s):\dot p^\circ_\varepsilon(s)-\chi^\circ_\varepsilon(s):(\dot p^\circ_\varepsilon(s)-\dot\alpha^\circ_\varepsilon(s))\}\,dx\,ds. \quad (4.22)$$

Kinematic compatibility implies that

$$\int_0^S\int_\Omega\sigma^\circ_\varepsilon(s):\dot p^\circ_\varepsilon(s)\,dx\,ds=\int_0^S\int_\Omega\sigma^\circ_\varepsilon(s):E\dot u^\circ_\varepsilon(s)\,dx\,ds-\int_0^S\int_\Omega Ae^\circ_\varepsilon(s):\dot e^\circ_\varepsilon(s)\,dx\,ds$$

and, since $\operatorname{div} \sigma_\varepsilon^\circ(s) = 0$ a.e. in Ω and $u_\varepsilon^\circ(s) = w_\varepsilon^\circ(s)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, then this yields in turn

$$\int_0^S \int_\Omega \sigma_\varepsilon^\circ(s) : \dot p_\varepsilon^\circ(s)\, dx\, ds = \int_0^S \int_\Omega \sigma_\varepsilon^\circ(s) : E\dot w_\varepsilon^\circ(s)\, dx\, ds - \int_0^S \int_\Omega A e_\varepsilon^\circ(s) : \dot e_\varepsilon^\circ(s)\, dx\, ds$$
$$= \int_0^S \int_\Omega \sigma_\varepsilon^\circ(s) : E\dot w_\varepsilon^\circ(s)\, dx\, ds - \frac{1}{2}\int_\Omega A e_\varepsilon^\circ(S) : e_\varepsilon^\circ(S)\, dx + \frac{1}{2}\int_\Omega A e_0 : e_0\, dx.$$

But the first integral in the last term in the string of equalities above also reads as

$$\int_0^{t_\varepsilon^\circ(S)} \int_\Omega \sigma_\varepsilon(t) : E\dot w(t)\, dx\, dt,$$

and, thanks to the fourth convergence in (4.6), to the uniform convergence of t_ε° to t°, to the fifth item in Theorem 3.1 and to the dominated convergence theorem, it converges to

$$\int_0^{t^\circ(S)} \int_\Omega \sigma^\circ(s_0^-(t)) : E\dot w(t)\, dx\, dt = \int_0^S \int_\Omega \sigma^\circ(s_0^-(t^\circ(s))) : E\dot w^\circ(s)\, dx\, ds,$$

where we used the change of variable $t = t^\circ(s)$. But since $E\dot w^\circ(s) = 0$ for all $s \in U^\circ$ and $s_0^-(t^\circ(s)) = s$ for a.e. $s \notin U^\circ$, we get that

$$\int_0^S \int_\Omega \sigma_\varepsilon^\circ(s) : E\dot w_\varepsilon^\circ(s)\, dx\, ds \to \int_0^S \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\, dx\, ds.$$

Now,

$$\int_0^S \int_\Omega \chi_\varepsilon^\circ(s) : (\dot p_\varepsilon^\circ(s) - \dot\alpha_\varepsilon^\circ(s))\, dx\, ds = \int_0^S \int_\Omega B(p_\varepsilon^\circ(s) - \alpha_\varepsilon^\circ(s)) : (\dot p_\varepsilon^\circ(s) - \dot\alpha_\varepsilon^\circ(s))\, dx\, ds$$
$$= \frac{1}{2}\int_\Omega B(p_\varepsilon^\circ(S) - \alpha_\varepsilon^\circ(S)) : (p_\varepsilon^\circ(S) - \alpha_\varepsilon^\circ(S))\, dx - \frac{1}{2}\int_\Omega B(p_0 - \alpha_0) : (p_0 - \alpha_0)\, dx,$$

so inequality (4.22) reads as

$$\begin{aligned}
&\int_0^S \mathcal{H}_M(\chi^\circ(s) * \rho, [\dot p^\circ(s) | \dot\alpha^\circ(s)])\, ds \\
&\qquad + \int_0^S \| [\dot p^\circ(s) | \dot\alpha^\circ(s)] \|_2 \operatorname{dist}_2([\sigma^\circ(s) - \chi^\circ(s) | \chi^\circ(s)], \mathcal{K}_M(\chi^\circ(s) * \rho))\, ds \\
&\quad \le \int_0^S \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\, dx\, ds - \liminf_{\varepsilon\to 0} \mathcal{Q}(e_\varepsilon^\circ(S), p_\varepsilon^\circ(S) - \alpha_\varepsilon^\circ(S)) + \mathcal{Q}(e_0, p_0 - \alpha_0). \qquad (4.23)
\end{aligned}$$

But, in view of convergences (4.5), weak lower semi-continuity immediately implies that

$$\liminf_{\varepsilon\to 0}\ \mathcal{Q}(e_\varepsilon^\circ(S), p_\varepsilon^\circ(S) - \alpha_\varepsilon^\circ(S)) \ge \mathcal{Q}(e^\circ(s), p^\circ(s) - \alpha^\circ(s)),$$

thus, by Theorem 4.10 and since, in view of Remark 4.11, the quadruplet $(\dot u^\circ(s), \dot e^\circ(s), \dot p^\circ(s), \dot\alpha^\circ(s))$ belongs to $A(\dot w^\circ(s))$ and $\operatorname{div} \dot\sigma^\circ(s) = 0$ a.e. in Ω for a.e. $s \in [0, \overline{T}]$, we can apply the duality formula

(2.18). We deduce that (4.23) reads as

$$
\begin{aligned}
&\int_0^S \mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\,ds\\
&+\int_0^S \|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\|_2\,\mathrm{dist}_2(\left[\sigma_D^\circ(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right],\mathcal{K}_M(\chi^\circ(s)*\rho))\,ds\\
&\le \int_0^S\int_\Omega \sigma^\circ(s):E\dot w^\circ(s)\,dx\,ds-\int_0^S\int_\Omega\{\sigma^\circ(s):\dot e^\circ(s)-\chi^\circ(s):(\dot p^\circ(s)-\dot\alpha^\circ(s))\}\,dx\,ds\\
&=\int_0^S\left\{\langle\sigma_D^\circ(s),\dot p^\circ(s)\rangle-\int_\Omega\chi^\circ(s):(\dot p^\circ(s)-\dot\alpha^\circ(s))\,dx\right\}ds.
\end{aligned}\tag{4.24}
$$

Recalling Remark 4.1, we obtain, for a.e. $s\in B^\circ$,

$$
\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\ge\langle\sigma_D^\circ(s),\dot p^\circ(s)\rangle-\int_\Omega\chi^\circ(s):(\dot p^\circ(s)-\dot\alpha^\circ(s))\,dx.
$$

On the other hand, since $\dot p^\circ(s)\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{dev}})$ for a.e. $s\in A^\circ$, the duality pairing $\langle\sigma^\circ(s),\dot p^\circ(s)\rangle$ coincides with the L^2 product for a.e. $s\in A^\circ$, so that

$$
\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\ge\int_\Omega P_{\mathcal{K}_M(\chi^\circ(s)*\rho)}\left(\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]\right)\cdot\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\,dx.
$$

Consequently, since, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\|_2\,\mathrm{dist}_2(\left[\sigma^\circ(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right],\mathcal{K}_M(\chi^\circ(s)*\rho))\ge\\
&\qquad\int_\Omega\left(\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]-P_{\mathcal{K}_M(\chi^\circ(s)*\rho)}\left(\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]\right)\right)\cdot\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\,dx,
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])+\|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\|_2\,\mathrm{dist}_2(\left[\sigma^\circ(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right],\mathcal{K}_M(\chi^\circ(s)*\rho))\\
&\ge\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\\
&+\int_\Omega\left(\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]-P_{\mathcal{K}_M(\chi^\circ(s)*\rho)}\left(\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]\right)\right)\cdot\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\,dx\\
&\ge\int_\Omega\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]\cdot\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\,dx
\end{aligned}
$$

for a.e. $s\in A^\circ$.

In conclusion, for a.e. $s\in[0,\overline{T}]$,

$$
\begin{aligned}
&\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])+\|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\|_2\,\mathrm{dist}_2(\left[\sigma^\circ(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right],\mathcal{K}_M(\chi^\circ(s)*\rho))\\
&\qquad\qquad\ge\langle\sigma_D^\circ(s),\dot p^\circ(s)\rangle-\int_\Omega\chi^\circ(s):(\dot p^\circ(s)-\dot\alpha^\circ(s))\,dx,
\end{aligned}
$$

and, with (4.24), we obtain that, for a.e. $s\in[0,\overline{T}]$,

$$
\begin{aligned}
&\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])+\|\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\|_2\,\mathrm{dist}_2(\left[\sigma_D^\circ(s)-\chi^\circ(s)\middle|\chi^\circ(s)\right],\mathcal{K}_M(\chi^\circ(s)*\rho))\\
&=\mathcal{H}_M(\chi^\circ(s)*\rho, \left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right])\\
&+\int_\Omega\left(\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]-P_{\mathcal{K}_M(\chi^\circ(s)*\rho)}\left(\left[(\sigma_D^\circ-\chi^\circ)(s)\middle|\chi^\circ(s)\right]\right)\right)\cdot\left[\dot p^\circ(s)\middle|\dot\alpha^\circ(s)\right]\,dx\\
&=\langle\sigma_D^\circ(s),\dot p^\circ(s)\rangle-\int_\Omega\chi^\circ(s):(\dot p^\circ(s)-\dot\alpha^\circ(s)\,dx.
\end{aligned}
$$

The proof of (4.19), (4.20), and (4.21) is complete. ∎

4.4. **Removal of the cap.** We propose in this short section to derive a *classical* partial flow rule, at least for a.e. $s \in B^0$. We will then show that we can actually get rid of the artificial bound M on the back stress in the definition of K and still keep the stress constraint and the flow rule. Of course the set of points B^0 may *a priori* depend on M, so that we have not completely removed the impact of the presence of that cut-off on the evolution. Also, in all fairness, we do not even know how to establish the existence of a strictly positive s_1 such that $[0, s_1] \subset B^0$. In other words, the set of points A^0 where we do not know whether the (rescaled) stress constraint is met might contain $s = 0$ in its closure, in which case our result is truly useless....

The following proposition holds.

Proposition 4.14 (Partial flow rule). *For a.e. $x \in \Omega$ and a.e. $s \in B^0$,*

$$\dot{p}^\circ(s,x) - \dot{\alpha}^\circ(s,x) \in \partial I_{\hat{K}_{M,\chi}(s,x,\sigma^\circ_D(s,x))}(\chi^\circ(s,x))$$

where, for every $\pi \in \mathbb{M}^{n\times n}_{\rm dev}$ and for all $(x,s) \in \Omega \times [0, \bar{S}]$,

$$\hat{K}_{M,\chi^\circ}(s,x,\pi) := \left\{\eta \in \mathbb{M}^{n\times n}_{\rm dev} : \left[\pi - \eta\middle|\eta\right] \in K((\chi^\circ(s) * \rho)(x))\right\}.$$

Further, define $s_1 := \max\{s : [0,s] \in B^0\}$. We can choose α_0 in (4.1) *and M large enough, so that, for every $s \in [0, s_1]$,*

$$\left[\sigma^\circ_D(s) - \chi^\circ(s)\middle|\chi^\circ(s)\right] \in \mathcal{K}(\chi^\circ(s) * \rho),$$

and the flow rule in the sixth item of Theorem 4.2 is satisfied with H in lieu of H_M.

Proof. Recall Remark 4.1. For every $\tau, \eta \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm dev})$ such that $\operatorname{div} \tau = 0$, $\left[\tau_D - \eta\middle|\eta\right] \in \mathcal{K}(\chi^\circ(s) * \rho)$,

$$\mathcal{H}_M(\chi^\circ(s) * \rho, \left[\dot{p}^\circ(s)\middle|\dot{\alpha}^\circ(s)\right]) \geq \langle \tau_D, \dot{p}^\circ(s)\rangle - \int_\Omega \eta : (\dot{p}^\circ(s) - \dot{\alpha}^\circ(s))\, dx.$$

Now, with the result of Theorem 4.13, this implies that

$$\begin{aligned}\|\left[\dot{p}^\circ(s)\middle|\dot{\alpha}^\circ(s)\right]\|_2 \operatorname{dist}_2(&\left[\sigma^\circ_D(s) - \chi^\circ(s)\middle|\chi^\circ(s)\right], \mathcal{K}(\chi^\circ(s) * \rho)) \\ &- \langle(\tau_D - \sigma^\circ_D(s)), \dot{p}^\circ(s)\rangle - \int_\Omega (\eta - \chi^\circ(s)) : (\dot{p}^\circ - \dot{\alpha}^\circ)(s)\, dx \leq 0,\end{aligned}$$

hence, a fortiori,

$$-\langle(\tau_D - \sigma^\circ_D(s)), \dot{p}^\circ(s)\rangle - \int_\Omega (\eta - \chi^\circ(s)) : (\dot{p}^\circ - \dot{\alpha}^\circ)(s)\, dx \leq 0. \tag{4.25}$$

Note that, for all $s \in B^0$, $\chi^\circ(s,x) \in \hat{K}_{M,\chi^\circ}(s,x,\sigma^\circ_D(s,x))$ a.e. in Ω. Thus, for such s's, (4.25) (with $\tau = \sigma^\circ(s)$) implies that

$$-\int_\Omega (\eta - \chi^\circ(s)) : (\dot{p}^\circ - \dot{\alpha}^\circ)(s)\, dx \leq 0,$$

which, by convex analysis arguments, is equivalent to

$$-(\dot{p}^\circ(s,x) - \dot{\alpha}^\circ(s,x)) \in \partial I_{\hat{K}_{M,\chi^\circ}(s,x,\sigma^\circ_D(s,x))}(\chi^\circ(x)), \text{ for a.e. } x \in \Omega,$$

or, still, in view of the Lipschitz character of f, to the existence of $\lambda_{s,x} \geq 0$ such that

$$\dot{p}^\circ(s,x) - \dot{\alpha}^\circ(s,x) = \lambda_{s,x}\left(\frac{\partial f}{\partial \sigma}(\sigma^\circ_D(s,x) - \chi^\circ(s,x)) - \chi^\circ(s,x)\right), \tag{4.26}$$

since, by (2.6), $\partial\hat{K}_{M,\chi^\circ}(s,x,\pi) := \left\{\eta \in \mathbb{M}^{n\times n}_{\rm dev} : f(\pi - \eta) + |\eta|^2/2 = \frac{1}{2}T_M(|(\chi^\circ(s) * \rho)(x)|^2)\right\}$.

Set

$$F := \max\{(\partial f/\partial\sigma)(\pi) : \pi \in \partial B_{\mathbb{M}^{n\times n}_{\rm dev}}(0,1)\},$$

and take M to be $> \gamma' F/\gamma$ with γ, γ' the coercivity and boundedness constants in (2.20). Then, define $\beta := p^\circ - \alpha^\circ$, multiply (4.26) by $1_{\{x\in\Omega : |\beta(s,x)|>F/\gamma\}}(x)\beta(s,x)$ and integrate the resulting expression over Ω. We get

$$\begin{aligned}
&\int_\Omega 1_{\{x\in\Omega:|\beta(s,x)|>F/\gamma\}}(x)\beta(s,x) : \dot\beta(s,x)\, dx \\
&= \int_\Omega \lambda_{s,x}\, 1_{\{x\in\Omega:|\beta(s,x)|>F/\gamma\}}(x) \left(\beta(s,x) : \frac{\partial f}{\partial\sigma}(\sigma_D^\circ(s,x) - \chi^\circ(s,x)) - B\beta(s,x) : \beta(s,x)\right)\, dx \\
&\le \int_\Omega \lambda_{s,x}\, 1_{\{x\in\Omega:|\beta(s,x)|>F/\gamma\}}(x)|\beta(s,x)|(F - \gamma|\beta(s,x)|)\, dx \le 0.
\end{aligned}$$

But the left hand-side of the previous inequality is the time derivative of

$$\frac{1}{2}\int_\Omega \left(|\beta(s,x)|^2 - \frac{F^2}{\gamma^2}\right)^+ dx$$

which is thus a decreasing function over any closed interval $[a,b]$ included in B^0. Thus if $|\chi^\circ(a,x)| \le \gamma' F/\gamma < M$, then

$$|\chi^\circ(s,x)| \le M,\ s \in [a,b],$$

so that both the stress constraint and the flow rule in Theorem 4.2 do not activate the bound M. In other words $\mathcal{K}_M$ can be replaced by $\mathcal{K}$ (resp. H_M by H) for those times.

Remark that the last part of this proof follows closely that of [11, Lemma 1]. □

Remark 4.15. If $s_1 > 0$, then the previous proposition actually demonstrates that, for reasonable initial data, Theorem 4.2 holds for K in lieu of K_M on $[0, \overline{T}]$ in which case we can immediately recover the corresponding evolution in un-rescaled time over $[0,T]$. Unfortunately, we do not know how to prove that the L^∞-bound on χ is preserved when crossing intervals in A^0. ¶

5. Concluding remarks

In guise of conclusion, we discuss below the impact of the regularization of the stress constraint via the kernel ρ.

The *regularized stress constraint* a priori reads as

$$\left[\sigma_D - \chi\middle|\chi\right](x) \in K((\chi*\rho)(x)) \iff f(\sigma_D(x) - \chi(x)) + \frac{1}{2}|\chi^2(x)| \le \frac{1}{2}T_M(|(\chi*\rho)(x)|^2), \text{ a.e. in } \Omega.$$

By letting $\|\rho\|_{L^1} \le 1$, we clearly have that

$$\frac{1}{2}\int_\Omega T_M(|(\chi*\rho)(x)|^2)\, dx \le \frac{1}{2}\int_\Omega |\chi*\rho|^2\, dx \le \frac{1}{2}\int_\Omega |\chi|^2\, dx.$$

Hence, we conclude that

$$\left[\sigma_D - \chi\middle|\chi\right](x) \in K((\chi*\rho)(x)) \text{ a.e. in } \Omega \implies \int_\Omega f(\sigma_D - \chi)\, dx \le 0.$$

In other words, the regularized stress constraint entails the fulfillment of the original stress constraint $f(\sigma_D - \chi) \le 0$ in some integrated (weaker) form. This shows that the regularized flow is not activated at the boundary $\sigma_D(x) - \chi(x) \in \partial K$, that is $\left[\sigma_D - \chi\middle|\chi\right](x) \in \partial K(\chi(x))$, but rather at $\left[\sigma_D - \chi\middle|\chi\right](x) \in \partial K((\chi*\rho)(x))$. Correspondingly, the flow rule of the regularized model $\left[\dot p\middle|\dot\alpha\right] \in \partial I_{K(\chi*\rho)}(\sigma_D - \chi, \chi)$ (see again Theorem 4.2) differs from that of the original Armstrong-Frederick model, namely $\left[\dot p\middle|\dot\alpha\right] \in \partial I_{K(\chi)}(\sigma_D - \chi, \chi)$. Note however that, by choosing the support of ρ to be contained in a suitably small interval centered at 0, the dynamics of the regularized model can be made arbitrarily close to that of Armstrong-Frederick, at least formally.

Before closing this section we mention an *alternative* regularization approach to the Armstrong-Frederick model. This corresponds to a mollification of the original Armstrong-Frederick relation (2.3), the new relation being

$$\dot{\chi} + |\dot{p}|F(\chi * \rho) = B\dot{p},$$

or, equivalently,

$$\left[\dot{p}\middle|\dot{\alpha}\right](x,t) \in \mathcal{A}(\sigma_D(x,t), (\chi * \rho)(x,t)). \tag{5.1}$$

It is thus assumed here that the nonlinear hardening term arises in a nonlocal fashion via space-averaging. This modification of the original Armstrong-Frederick model seems new. It has the effect of taming the quadratic (nonlinear) term in the original Armstrong-Frederick flow rule.

Note that, in the notation of Section 2, for $x \in \Omega$ and t given,

- the stress constraint is satisfied iff $\left[(\sigma_D - \chi)\middle|(\chi * \rho)\right](x,t) \in K((\chi * \rho)(x,t))$;
- $(\sigma - \chi)(x,t) \in \partial K$ iff $\left[(\sigma_D - \chi)\middle|(\chi * \rho)\right](x,t) \in \partial K((\chi * \rho)(x,t))$;
- the flow rule (5.1) is satisfied iff $\left[\dot{p}\middle|\dot{\alpha}\right](x,t) \in \partial I_{K((\chi*\rho)(x,t))}\left[\sigma_D - \chi\middle|(\chi * \rho)\right](x,t)$.

The alternative regularized model has the advantage that it preserves the original non-regularized stress constraint, although it does modify the flow rule.

An existence result for quasi-static evolution driven by (5.1) can then be obtained by faithfully reproducing the analysis presented in this paper.

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