THE ELASTO-PLASTIC EXQUISITE CORPSE: A SUQUET LEGACY

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Abstract. Pierre Marie Suquet's pioneering work on plasticity paved the way for the mathematical theory of plasticity as we know it today. In this contribution we propose to review the most recent advances on that front and to illustrate how those impact classical problems of quasi-static elasto-plastic evolutions. Most notably, we exhibit new flow rules and derive conditions that prohibit the onset of plastic slips. From these, we obtain new uniqueness results for such evolutions.

Keywords: plasticity, quasi-static evolutions, plastic slips, uniqueness, space of bounded deformations

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1. Introduction

In the late 70's Pierre Marie Suquet laid down the mathematical foundations of the theory of perfect small strain elasto-plasticity, establishing for the first time in [20] the existence of an elasto-plastic evolution for reasonable data. To do this he had to study a functional space that was far from understood at the time, the space $BD(\Omega)$ of functions with bounded deformations [19]. That work was then elaborated upon through various works of R. Temam (see e.g. [21]) and R.V. Kohn and R. Temam (see [17]).

In a nutshell, his approach consisted in passing to the limit in a quadratic (visco-plastic) regularization $H(p) + \varepsilon p^2/2$ of the one-homogeneous dissipation potential $H(p)$ that characterizes elasto-plasticity. Here and below, p stands for the plastic strain, a trace-free (deviatoric) symmetric matrix. In the traditional case of Von Mises elasto-plasticity, $H(p) = \sqrt{2/3}\sigma_c|p|$ where σ_c is the yield stress.

The applied mathematics community proved too obtuse to understand the impact of that result and mathematical interest in plasticity soon subsided with the notable exception of the work of G. Anzellotti and S. Luckhaus on dynamic elasto-plasticity [4] and of that of A. Bensoussan and J. Frehse on stress regularity in Von Mises elasto-plasticity [6]. Pierre himself was apparently discouraged from further venturing into a field he had singlehandedly created.

The elasto-plastic cadaver fell by the wayside and was soon promised to eternal mathematical decay. It was however unexpectedly resuscitated by G. Dal Maso, A. De Simone and M. G. Mora [7]. Their view was different from that of their predecessor. Elasto-plastic evolution was indeed an exquisite example of energy conserving variational evolutions and fit squarely within the general framework advocated in particular by A. Mielke (see e.g. [18]).

Energy conserving variational evolutions are quasi-static and isothermal. They consist in a time-parameterized set of minimization problems for the sum of the elastic energy and of the add-dissipation. The resulting minimizers should also conserve energy throughout the evolution, thereby preventing pernicious losses of energy through any kind of surreptitious dissipation process.

The existence of such an evolution is secured through a time incremental process which is very close in spirit to the traditional implicit Euler scheme of the numerical analyst. Once again, the functional space at play was $BD(\Omega)$, but much more was now known about that space, thanks to the formidable efforts of the italian school of Calculus of Variations created by E. De Giorgi. The fine structure of that space was much better controlled, as detailed in [2], although, to this day, $BD(\Omega)$ remains much more of a challenge than its scalar analogue $BV(\Omega)$.

The more delicate part of [7] is to achieve a good understanding of the stress-strain duality. That duality is a pre-requisite to showing that the obtained evolution satisfies the so-called flow rule: whenever the (deviatoric part of the) stress reaches the boundary of its admissible set, the plastic strain should flow in the direction normal to that set. In a nutshell, the plastic work $\sigma_D \cdot \dot{p}$ – here and below σ_D denotes the deviatoric part of the Cauchy stress – has *a priori* no meaning because $\dot{p}$ is a measure while σ_D is merely a bounded function. The already quoted work of R.V. Kohn and R. Temam adresses precisely that issue, although it only succeeds when the relative boundary of the Dirichlet part of the boundary of the domain Ω is very smooth. In any case, the authors of [7] recover the flow rule, but, at the same time, they also derive additional flow rules that are activated when the plastic strain is not solely characterized by an integrable density. Indeed, the fine structure of $BD(\Omega)$ allows for jumps in the displacement field. Those will be tangential because plasticity is modeled through a deviatoric plastic strain and they will be identified with the expected plastic slips. But the fine structure of $BD(\Omega)$ also allows for a Cantor-like behavior for the displacement field. Such a pathology is not part and parcel of the classical mechanical view of plasticity. In any case, the nature of the additional flow rules is not very explicit in [7].

Mechanical wisdom is, as it should be, skeptical of the possibility of diffuse, Cantor-like plasticity and would thus prompt the mathematical observer of plastic behavior to seek regularity as a way out: test plastic fields may exhibit Cantor-like behavior, but the actual evolution won't. Such a statement would be facilitated, should uniqueness of the displacement field hold true. Indeed, it is known in the mathematical world since the work of P. M. Suquet, and in the world at large since that of R. Hill [15] and even before, that the stress field, hence the elastic strain field, are uniquely determined, so that uniqueness of the displacement field is equivalent to uniqueness of the plastic strain field.

A remarkable result of A. Demyanov delivers a fatal blow to this program. Indeed, he proved in [8, Section 10.2] that, in a one-dimensional setting, uniqueness is achieved for a very specific and very smooth loading process, yet that process can be chosen so that any measure which does not charge atoms can be attained as a plastic strain! Could it be that this intriguing result is a byproduct of dimensionality?

In this contribution, we propose to paint in broad brushstrokes the more recent advances in our understanding of the intimate structure of the solutions to a quasi-static elasto-plastic evolution. This is done in the most accessible context, that of Von Mises plasticity which is at present the only setting for which local stress regularity is known to hold in the light of [6]. To do so we first recall in Section 2 the concept of energy conserving minimizing movements within the framework of elasto-plasticity as detailed in [7] and formally demonstrate how one recovers the classical equations of elasto-plasticity from the variational evolution. We also derive a flow rule on the Dirichlet part of the boundary of the domain Ω. That flow rule is not implied by the bulk flow rule. It is an additional equation which was uncovered in [9, Equation (3.12)] and which seems strangely absent from the vast mechanics literature on elasto-plasticity.

We then discuss in Section 3 the flow rule on plastic slips and for the potential Cantor parts of the plastic strain. We derive in that context a condition for the existence of a plastic slip first postulated by B. Halphen and J. Salençon [13, 12]. Specifically, we give a condition on the Cauchy stress field which, when it is satisfied, prohibits the onset of plastic slips in a given region. This section is a summary of results obtained in [11].

Finally, we debate uniqueness in Section 4 through the thorough investigation of two specific examples: the bi-axial test and the spherical cavity under internal pressure. The first one has already been detailed in [10], while the other one is new. These are, to our knowledge, the only *bona fide* three-dimensional examples where uniqueness is generically established. Further, when non-uniqueness does occur in the first example, we exhibit a generalization of A. Demyanov's result [8] to the extent that we conclude to the existence of possible solutions that do exhibit Cantor like plastic strains. Finally, although the evolution exhibited in the second example is "well-known", we believe that the uniqueness of that evolution has never been established up to now.

In all that follows, we propose to keep the mathematical intricacies to a minimum while emphasizing the mechanical aspects of the obtained results and encouraging the mathematically inquisitive reader to consult the references provided herein. We also refer to e.g. [3] for background material, especially concerning finer measure theoretical points.

Notationwise, we adopt the following:

For $B \subseteq \mathbb{R}^3$, the symbol $A \subset\subset B$ means that the closure of A is compact and contained in B. The symbol $\lfloor_A$ stands for "restricted to A".

We denote by $\mathrm{M}^3_{\mathrm{sym}}$ the set of 3×3-symmetric matrices and by M^3_D the set of trace-free (deviatoric) elements of $\mathrm{M}^3_{\mathrm{sym}}$. The identity matrix in $\mathrm{M}^3_{\mathrm{sym}}$ is denoted by $\mathtt{i}$. If π is an element of $\mathrm{M}^3_{\mathrm{sym}}$, then π_D denotes its deviatoric part. The symbol $\cdot$ denotes the classical (Frobenius) inner product of matrices. We denote by $\mathcal{L}_s(\mathrm{M}^3_{\mathrm{sym}})$ the set of symmetric endomorphisms on $\mathrm{M}^3_{\mathrm{sym}}$. For $a, b \in \mathbb{R}^N$, $a \odot b$ stands for the symmetric matrix such that $(a \odot b)_{ij} := (a_i b_j + a_j b_i)/2$. Throughout f_1, f_2, f_3 will denote an orthonormal basis.

The sets $\mathcal{M}_b(\Omega; \mathrm{M}^3_D)$, resp. $\mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^3_D)$, will denote the spaces of finite Radon measures on Ω, resp. $\Omega \cup \Gamma_d$, with values in M^3_D. For μ in such a space, we denote its total variation by $|\mu|(\Omega)$, resp. $|\mu|(\Omega \cup \Gamma_d)$, while if f is p-integrable $(1 \le p \le \infty)$, we denote its L^p-norm by $\|f\|_p$.

If X is a normed space, we denote by $BV(a, b; X)$ and $AC(a, b; X)$ the space of functions with bounded variation and the space of absolutely continuous functions from $[a, b]$ to X, respectively. The total variation of $f \in BV(a, b; X)$ is defined as

$$\mathcal{V}(f; a, b) := \sup \left\{ \sum_{j=1}^{k} \|f(t_j) - f(t_{j-1})\|_X \, : \, a = t_0 < t_1 < \cdots < t_k = b \right\}.$$

In this paper as in previous works on elasto-plasticity the displacement field u lies in $BD(\Omega)$, the space of functions of bounded deformations. This is so because the strain tensor Eu is decomposed additively into an elastic strain e and a plastic strain p which may concentrate and thus can become a measure supported on a set of 0 (Lebsgue) measure. Thus, even if the elastic strain is smooth, the total strain tensor Eu might only be a measure.

We refer the reader to e.g. [21, Chapter II], and [2] for background material on $BD(\Omega)$. Just note that any element $u \in BD(\Omega)$ is such that its symmetrized gradient (the total strain) Eu can be decomposed as follows

$$Eu = \mathcal{E}u \; dx + (u^+ - u^-) \odot \nu \; \mathcal{H}^2 \lfloor J_u + Cu,$$

where $\mathcal{E}u$ is the Lebesgue part of Eu, $u^\pm$ are the values of u on each side of the possible jumps of u (the set J_u with normal ν), $\mathcal{H}^2$ stands for the two-dimensional Hausdorff measure (the surface measure), and Cu stands for the Cantor part of Eu which does not see the Lebesgue part of u, nor its jump part J_u, because it is supported on a set of measure 0 whose dimensionality is greater than 2.

2. Quasi-static elastoplastic evolutions

In this section, we recall the framework investigated in [7].

The reference configuration. The domain $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected set with (at least) Lipschitz boundary and exterior normal ν. Further, the Dirichlet part Γ_d of $\partial\Omega$ is a non empty open set in the relative topology of $\partial\Omega$ with relative boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ in $\partial\Omega$ and we set $\Gamma_n := \partial\Omega \setminus \bar{\Gamma}_d$. We will assume here that the relative boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ is smooth (think of a smooth closed curve on the manifold $\partial\Omega$) and refer the reader to [9, Section 6] for what we believe to be the state of the art on relative boundary regularity.

Kinematic admissibility. We take a boundary displacement w with the kind of regularity that would be sufficient for the well-posedness of the linearly elastic problem. Hence it is enough to view w as the restriction to Γ_d of an element of $H^1(\Omega; \mathbb{R}^3)$. We adopt the following

Definition 2.1 (Admissible configurations). *$\mathcal{A}(w)$, the family of admissible configurations relative to w, is the set of (kinematically admissible) triplets (u, e, p) with*

$$u \in BD(\Omega), \qquad e \in L^2(\Omega; \mathrm{M}^3_{\mathrm{sym}}), \qquad p \in \mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^3_D),$$

and such that

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w-u) \odot \nu\, \mathcal{H}^2 \lfloor \Gamma_d \quad \text{on } \Gamma_d. \tag{2.1}$$

The field e denotes the elastic (part of the) strain. Moreover, since p is assumed to take values in the space of deviatoric matrices M^3_D because plasticity only potentially activates slips, hence tangential jumps, $u^+ - u^-$ is perpendicular to ν; only this kind of plastic strain can be activated along J_u or Γ_d. We emphasize that there could actually be slips on Γ_d, the Dirichlet part of the boundary, that prevent u from reaching its desired boundary value w on Γ_d.

For simplicity, we restrict our attention to the constitutively homogenous case, although the reader should be aware that going from a homogeneous sample to a heterogeneous one is a perilous enterprise which, at present, cannot be successfully completed in full generality (see [9] on this topic).

The elasticity tensor: Hooke's law is given through an elasticity tensor in $\mathbb{C} \in \mathcal{L}_s(\mathrm{M}^3_{\mathrm{sym}})$ with

$$c_1|M|^2 \le \mathbb{C}M \cdot M \le c_2|M|^2 \text{ for every } M \in \mathrm{M}^3_{\mathrm{sym}}, \tag{2.2}$$

with $c_1, c_2 > 0$.

For every $e \in L^2(\Omega; \mathrm{M}^3_{\mathrm{sym}})$ we set

$$\mathcal{Q}(e) := \frac{1}{2}\int_\Omega \mathbb{C}e \cdot e\, dx.$$

Von Mises dissipation potential: Given $\sigma_c > 0$, the deviatoric part of the stress σ_D is constrained to satisfy

$$|\sigma_D| \le \sqrt{\frac{2}{3}}\sigma_c.$$

The so-called *dissipation potential* $H : \mathrm{M}^3_D \to [0, +\infty[$ (the convex dual of the indicatrix function of the admissible set of deviatoric stresses) is given by

$$H(\xi) := \sup\left\{\tau \cdot \xi : \tau \in \mathrm{M}^3_D,\ |\tau| \le \sqrt{\frac{2}{3}}\sigma_c\right\} = \sqrt{\frac{2}{3}}\sigma_c|\xi|.$$

For every admissible plastic strain p (which can be a measure), we want to define the dissipation functional. The classical definition, borrowed from the theory of convex functions of measures, is the following:

$$\mathcal{H}(p) := \int_{\Omega\cup\Gamma_d} H\left(\frac{p}{|p|}\right) d|p| = \sqrt{\frac{2}{3}}\sigma_c|p|(\Omega\cup\Gamma_d),$$

where $p/|p|$ denotes the Radon-Nikodym derivative (the density) of p with respect to its total variation $|p|$. Such a definition allows one to define the dissipation for measures that can have singular parts (jumps or Cantor parts for example).

If $t \mapsto p(t)$ is a map from $[0, T]$ to $\mathcal{M}_b(\Omega\cup\Gamma_d; \mathrm{M}^3_D)$, we define, for every $[a, b] \subseteq [0, T]$,

$$\mathcal{D}(0, t; p) := \sqrt{\frac{2}{3}}\sigma_c\mathcal{V}(0, t; p)$$

to be the *total dissipation* over the time interval $[a, b]$. It is precisely the amount of plastic work spent during the time interval $(0, t)$ and, provided that the plastic strain is absolutely continuous in time (which will be the case), it also satisfies

$$\mathcal{D}(0, t; p) = \int_0^t \mathcal{H}(\dot p(s))\, ds.$$

Body and traction forces: We consider external loads with associated potential

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t, x) \cdot u(x)\, dx + \int_{\Gamma_t} g(t, x) \cdot u(x)\, d\mathcal{H}^2(x),$$

where the body forces $f(t)$ and traction forces $g(t)$ on $\mathit{\Gamma}_t$ are such that

$$f \in AC(0,T;L^3(\mathit{\Omega};\mathbb{R}^3)), \quad g \in AC(0,T;L^\infty(\mathit{\Gamma}_t;\mathbb{R}^3)). \tag{2.3}$$

(The L^3-regularity of f is just so that the product of f by u be integrable, which it will be since u, being in $BD(\mathit{\Omega})$, is in $L^{3/2}(\mathit{\Omega};\mathbb{R}^3)$.)

We set

$$\langle \dot{\mathcal{L}}(t), u\rangle := \int_{\mathit{\Omega}} \dot{f}(t,x)\cdot u(x)\,dx + \int_{\mathit{\Gamma}_t} \dot{g}(t,x)\cdot u(x)\,d\mathcal{H}^2(x),$$

and assume the following *uniform safe load condition*:
There exist $\alpha > 0$ and $\rho \in AC(0,T;L^2(\mathit{\Omega};\mathrm{M}^3_{\mathrm{sym}}))$ with $\rho_D \in AC(0,T;L^\infty(\mathit{\Omega};\mathrm{M}^3_D))$ such that

$$\begin{cases} -\mathrm{div}\,\rho(t) = f(t) \text{ in } \mathit{\Omega},\ \rho(t)\nu = g(t) \text{ on } \mathit{\Gamma}_t \\ |\rho_D(t,x)| \le \sqrt{2/3}\,\sigma_c - \alpha, \quad \text{a.e. in } \mathit{\Omega}. \end{cases} \tag{2.4}$$

That kind of safe load condition is often used to ensure that, for given force data f and g, the set of statically admissible stresses is not empty. The uniformity of that condition (*i.e.*, the existence of α) is a mathematical refinement aimed at ensuring coercivity (see e.g. [7] for details).

Prescribed boundary displacements. The boundary displacement w on $\mathit{\Gamma}_d$ for the time interval $[0,T]$ is given by the trace on $\mathit{\Gamma}_d$ of some

$$w \in AC(0,T;H^1(\mathbb{R}^3;\mathbb{R}^3)). \tag{2.5}$$

The energetic formulation of the quasi-static evolution derived in [7] consists in two ingredients: a stability statement at each time, together with an energy conservation statement that relates the total energy of the system to the work of the loads applied to that system.

Definition 2.2 (Energetic quasi-static evolution). *The mapping*

$$t \mapsto (u(t), e(t), p(t)) \in \mathcal{A}(w(t))$$

is an energetic quasi-static evolution relative to w iff the following conditions hold for every $t \in [0,T]$:

(a) *Global stability: for every $(v,\eta,q) \in \mathcal{A}(w(t))$*

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t), u(t)\rangle \le \mathcal{Q}(\eta) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(q - p(t)). \tag{2.6}$$

(b) *Energy equality: $p \in BV\left(0,T;\mathcal{M}_b(\mathit{\Omega}\cup\mathit{\Gamma}_d;\mathrm{M}^3_D)\right)$ and*

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{D}(0,t;p) = \mathcal{Q}(e(0)) - \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \left[\int_{\mathit{\Omega}} \sigma(\tau)\cdot E\dot{w}(\tau)\,dx - \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\right] d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\,d\tau,$$

where $\sigma(t) := \mathbb{C}e(t)$.

The following result has been proved in [7, Theorem 4.5] (see also [9, Theorem 2.7] for an existence theorem which only necessitates Lipschitz regularity for the boundary $\partial\mathit{\Omega}$).

Theorem 2.3 (Existence of quasi-static evolutions). *Assume that* (2.2), (2.3), (2.4), (2.5) *are satisfied, and let $(u_0,e_0,p_0) \in \mathcal{A}(w(0))$ satisfy the global stability condition* (2.6).

Then there exists a quasi-static evolution $\{t \mapsto (u(t),e(t),p(t)), t\in[0,T]\}$ relative to the boundary displacement w such that $(u(0),e(0),p(0)) = (u_0,e_0,p_0)$. Finally the Cauchy stress

$$t \mapsto \sigma(t) := \mathbb{C}e(t)$$

is uniquely determined by the initial conditions.

Remark 2.4 (Time regularity). Further, it can be proved (see [7, Theorem 5.2]) that time derivatives of the fields $u(t), e(t), p(t)$ do exist (in some weak sense) and that $(\dot{u}(t),\dot{e}(t),\dot{p}(t)) \in \mathcal{A}(\dot{w}(t))$. Moreover, the total dissipation $\mathcal{D}(0,t;p)$ is absolutely continuous and its derivative is given by

$$\dot{\mathcal{D}}(0,t;p) = \sqrt{\frac{2}{3}}\sigma_c|\dot{p}(t)|(\mathit{\Omega}\cup\mathit{\Gamma}_d) \qquad \text{for a.e. } t\in[0,T].$$

¶

We now quickly demonstrate how one can formally recover the classical setting of elasto-plasticity from any quasi-static evolution. A completely rigorous exposition of that derivation is rather involved because of the duality problems alluded to in the introduction; the interested reader is invited to consult [7] or [9] for details.

First, test global stability with $(v,\eta,q) = (u(t)+\zeta\varphi, e(t)+\zeta E\varphi, p(t))$ where $\zeta > 0$ and φ is taken to be in $C_c^\infty(\Omega;\mathbb{R}^N)$, then in $C^\infty(\overline{\Omega};\mathbb{R}^N)$ with $\varphi \equiv 0$ on $\overline{\Gamma}_d$, and let $\zeta \searrow 0$. This is akin to the classical computation that demonstrates that potential energy minimizers in linear elasticity satisfy the equilibrium equations, together with the natural boundary conditions.

We thus obtain

$$\operatorname{div}\sigma(t) + f(t) = 0 \text{ in } \Omega, \qquad \sigma(t)\nu = g(t) \text{ on } \partial\Omega \setminus \overline{\Gamma}_d \tag{2.7}$$

as expected.

Now, fix $x \in \Omega$ to be a Lebesgue point for $\sigma_D(t)$ and test global stability with $(v,\eta,q) = (u(t), e(t) - \zeta\chi_{B(x,\delta)}\xi, p(t) + \zeta\chi_{B(x,\delta)}\xi)$ with $\xi \in \mathrm{M}^3_D$ and $\chi_{B(x,\delta)}$ the characteristic function of the open ball $B(x,\delta) \subset \Omega$ for δ small enough, and let $\zeta \searrow 0$. Because of the one-homogeneous character of H, this yields

$$\int_{B(x,\delta)} \sigma_D(t,y)\cdot\xi \; dy \le |B(x,\delta)| H(\xi).$$

Letting in turn $\delta \searrow 0$, we obtain

$$\sigma_D(t,x)\cdot\xi \le H(\xi) = \sqrt{2/3}\;\sigma_c|\xi|.$$

The arbitrary character of ξ then implies the stress admissibility

$$|\sigma_D(t,x)| \le \sqrt{2/3}\;\sigma_c, \text{a.e. in } \Omega, \tag{2.8}$$

as expected.

Remark 2.5 (Stress admissibility on Γ_d)**.** Note that, from (2.8) which, we recall, means that $\operatorname{tr}[(\sigma_D(t))^T(\sigma_D(t))] \le 2/3\;\sigma_c^2$, we can deduce that the tangential projection of the normal stress, *i.e.*, $(\sigma(t)\nu)_\tau \equiv (\sigma_D(t)\nu)_\tau$ satisfies $|(\sigma_D(t)\nu)_\tau| \le \sqrt{1/3}\;\sigma_c$ on Γ_d. This is a consequence of a useful result in linear algebra that can be found in [22, Section 79]. Of course, the meaning of that projection should be made explicit; for that see [9, Subsection 1.2]). ¶

The formal derivation of the flow rule goes as follows. Differentiate the energy equality. We obtain, in view of Remark 2.4 and since $\dot e(t) = E\dot u(t) - \dot p(t)$ on Ω,

$$\int_\Omega \sigma(t)\cdot E(\dot u(t) - \dot w(t))\; dx - \langle \mathcal{L}(t), \dot u(t) - \dot w(t)\rangle + \sqrt{2/3}\;\sigma_c|\dot p(t)|(\Omega\cup\Gamma_d) = \int_\Omega \sigma_D(t)\cdot\dot p(t)\; dx.$$

Once again, the reader is warned that the last term in the equality above should be suitably interpreted as a duality.

Now, a simple integration by parts on the left hand-side of the previous equality yields

$$\sqrt{2/3}\;\sigma_c|\dot p(t)|(\Omega\cup\Gamma_d) = \int_\Omega \sigma_D(t)\cdot\dot p(t)\; dx + \int_{\Gamma_d} \sigma(t)\nu\cdot(\dot w(t) - \dot u(t))\; d\mathcal{H}^2.$$

But the second term of the right hand-side of this last equality is precisely $\int_{\Gamma_d}\sigma_D(t)\cdot\dot p(t)\; d\mathcal{H}^2$, according to (2.1), so that that equality also reads as

$$\sqrt{2/3}\;\sigma_c|\dot p(t)|(\Omega\cup\Gamma_d) = \int_\Omega \sigma_D(t)\cdot\dot p(t)\; dx + \int_{\Gamma_d} \sigma_D(t)\cdot\dot p(t)\; d\mathcal{H}^2,$$

with a warning similar to the previous one.

Now, in view of (2.8), $H(\dot p(t)) = \sqrt{2/3}\;\sigma_c|\dot p(t)| \ge \sigma_D(t)\cdot\dot p(t)$ on Ω while, in view of Remark 2.5, $H(\dot p(t)) = \sqrt{1/3}\;\sigma_c|\dot w(t) - \dot u(t)| \ge (\sigma_D(t)\nu)_\tau.(\dot w(t) - \dot u(t))$ on Γ_d. Consequently, if the previous equality is to hold, then all previous inequalities are actually equalities. This result is exactly the local version of Hill's maximum plastic work principle [14].

Those equalities imply in turn that, for a.e. $x \in \Omega$,

$$\begin{cases} \dot p(t,x) = 0, \text{ if } |\sigma_D(t,x)| < \sqrt{2/3}\ \sigma_c \\ \dot p(t,x) = \lambda(t,x)\sigma_D(t,x),\ \lambda \ge 0, \text{ else} \end{cases} \tag{2.9}$$

while, for $\mathcal{H}^2$-a.e. x on Γ_d,

$$\begin{cases} \dot u(t,x) = \dot w(t,x), \text{ if } |(\sigma_D(t,x)\nu)_\tau| < \sqrt{1/3}\ \sigma_c \\ \dot u(t,x) - \dot w(t,x) = \lambda(t,x)(\sigma_D(t,x)\nu)_\tau,\ \lambda(t,x) \ge 0, \text{ else.} \end{cases} \tag{2.10}$$

Relation (2.9) is the classical flow rule of plasticity and, together with (2.7), (2.8), it provides the "complete" traditional set of equations for quasi-static Von Mises elasto-plasticity. Of course, once again, we have argued as if the measure $\dot p(t)$ was purely distributed (*i.e.*, as if it had no jump or Cantor part).

However, relation (2.10) is an additional relation which expresses a flow rule on the Dirichlet part of the boundary. It states that, for a velocity jump to appear at a point on that boundary, the tangential part of the normal stress at that point must be extremal, that is that it must satisfy $|(\sigma_D(t,x)\nu)_\tau| = \sqrt{1/3}\ \sigma_c$. This equality is *not* automatically enforced, even by the bulk equality $|\sigma_D(t)| \equiv \sqrt{2/3}\ \sigma_c$ everywhere on Ω. It is thus truly an additional equation which, to our knowledge, has never been introduced in the mechanics community.

3. Slipping, or not

In this section, we focus of the possibility of a slip. As already mentioned in the introduction, it was shown in [6] that, for smooth enough data, the stress field $\sigma(t)$ is actually in $L^\infty(0,T;H^1_{loc}(\Omega;\mathrm{M}^3_{\mathrm{sym}}))$. In particular, for a.e. time $t \in (0,T)$ and for $\mathcal{H}^2$-a.e. point $x \in \Omega$, $\sigma(t,x)$ is well defined as a Lebesgue value (a limit of averages of $\sigma(t)$ over vanishing balls centered at x), whereas, barring that regularity, that would only be true for a.e. $x \in \Omega$, so that there would be no reason to assume that it would hold on a slip which has 0 (Lebesgue) measure but positive $\mathcal{H}^2$ (surface) measure.

Now, because we can define unambiguously the stress field $\sigma(t,x)$ at those points x and since the measure $\dot p(t)$ cannot see sets of zero $\mathcal{H}^2$-measure, the relation (2.9) actually holds $\dot p(t)$-a.e. (recall that the variation $|\dot p(t)|$ is a measure). In particular, $\mathcal{H}^2$-a.e. on $J_{\dot u(t)} \subset \Omega$ we have

$$\sigma_D(t)\cdot\frac{\dot p^s(t)}{|\dot p^s(t)|} = \sqrt{\frac{2}{3}}\sigma_c\frac{\dot p^s(t)}{|\dot p^s(t)|} \qquad |\dot p^s(t)|\text{-a.e.},$$

where $\dot p_s(t)$ is the part of $\dot p(t)$ which is singular with respect to (does not see) the distributed (Lebesgue) part of that measure. At the same time

$$|\sigma_D(t)| \le \sqrt{\frac{2}{3}}\sigma_c \qquad |\dot p^s(t)|\text{-a.e..}$$

Putting these two equations together yields

$$|\sigma_D(t,x)| = \sqrt{\frac{2}{3}}\sigma_c \quad \text{and} \quad \frac{\dot p^s(t)}{|\dot p^s(t)|}(x) = \frac{\sigma_D(t,x)}{|\sigma_D(t,x)|} \qquad \text{for } |\dot p^s(t)|\text{-a.e. } x \in \Omega. \tag{3.1}$$

Now, on $J_{\dot u(t)}$, we know that the time derivative of the plastic strain is of the form of a tangential jump, that is

$$\dot p(t) = [\dot u(t)] \odot \nu\mathcal{H}^2, \text{ on } J_{\dot u(t)}.$$

If we insert this information into (3.1), we obtain, for $\mathcal{H}^2$-a.e. $x \in J_{\dot u(t)}$,

$$|\sigma_D(t,x)| = \sqrt{\frac{2}{3}}\sigma_c \qquad \text{and} \qquad \frac{[\dot u(t,x)] \odot \nu_{\dot u(t)}}{|[\dot u(t,x)] \odot \nu_{\dot u(t)}|} = \frac{\sigma_D(t,x)}{|\sigma_D(t,x)|}.$$

A simple argument in linear algebra states that if $a \perp b$, then $a \odot b$ is diagonalizable in an orthonormal basis with the first and third basis vector in the plane generated by a, b and at a 45^0-angle with the directions a, b. Further the ordered eigenvalues are $-|a||b|/2, 0, |a||b|/2$ (see e.g.

[11, Appendix]). Thus, we conclude that, for $\mathcal{H}^2$-a.e. $x \in J_{\dot{u}(t)}$, there exists a basis (f_1', f_2', f_3') such that

$$\sigma_D(t,x) = \operatorname{diag}\left(-\frac{\sigma_c}{\sqrt{3}}, 0, \frac{\sigma_c}{\sqrt{3}}\right). \tag{3.2}$$

Moreover the orthogonal lines determined by $[\dot{u}(t,x)]$ and $\nu_{\dot{u}(t)}(x)$ are bisected by f_1' and $\pm f_3'$ (and viceversa).

This is a very strong restriction on the form of the (uniquely determined) stress tensor on $J_{\dot{u}(t)}$. In particular, if this condition is not satisfied at any point in a subregion of Ω for some time interval (t_1, t_2), then we can easily conclude that there will not be any additional plastic slips in that region before t_2 (see [11, Section 4]). So, in particular, if the condition is not satisfied from the initial time onward, no plastic slip can ever be triggered in that region.

As a consequence, if a particular solution to a problem of elasto-plastic evolution is found, so that we know, by virtue of the uniqueness of the stress field, that the associated stress is *the* stress field for the problem, then a mere investigation of the eigenvalues and eigendirections of that stress field could be sufficient to bar the onset of plastic slippage. We will see such an example below.

The form (3.2) of the stress on a slip (or, more precisely, on a jump set for the velocities) had been derived by B. Halphen and J. Salençon in [13, 12] upon postulating that there was a flow rule on plastic slips (or, precisely, on surfaces of discontinuities for the velocity field). Our analysis grounds this result – which, by the way, seemed to have been completely ignored in the plastic literature – in a firm mathematical setting.

Remark 3.1. A similar line of argumentation would actually also apply to possible Cantor parts of the plastic strain if we only knew that these are also symmetrized rank-one deviatoric matrices. Unfortunately this is one of the remaining mysteries in the structure of $BD(\Omega)$. In the diagonal case, that is when $BD(\Omega)$ can be substituted with $[BV(\Omega)]^3$, the corresponding result, namely the rank-one structure of the Cantor part of Du has been established by G. Alberti [1]. ¶

4. Being unique, or not

In this last section, we discuss two three-dimensional problem for which uniqueness and non-uniqueness can be precisely quantified. As mentioned in the introduction, it is, to our knowledge, the first such results in a three-dimensional setting. In both settings the main tool for producing uniqueness will be strain compatibility. However, in the second example, the jump conditions derived in Section 3 above will prove instrumental in establishing uniqueness.

4.1. **The bi-axial test.**

We refer the reader to [10] for a mathematically detailed exposition in a more general case.

The example in question is a bi-axial test on a cylindrical sample of rectangular cross section. The domain is

$$\Omega = (-d/2, d/2) \times (-\ell/2, \ell/2) \times (0, \ell).$$

The elasticity tensor $\mathbb{C}$ is assumed to be homogeneous and isotropic with Young's modulus $\boldsymbol{E} > 0$, while, for simplicity, we will assume here that the Poisson's ratio $\boldsymbol{\nu}$ is 0 (see however [10] for the case $\boldsymbol{\nu} \neq 0$). Then

$$\mathbb{C}^{-1}\sigma = \frac{1}{\boldsymbol{E}}\sigma, \ \sigma \in \mathrm{M}^3_{\mathrm{sym}}.$$

At initial time, the initial conditions are solutions to a traction problem in the x_2-direction, the imposed traction being $\overline{\sigma}_2 f_2$. In other words, the boundary conditions are

$$\begin{cases} \sigma_0 f_1 = 0 & \text{on} \quad x_1 = \pm d/2 \\ \sigma_0 f_2 = \overline{\sigma}_2 f_2 & \text{on} \quad x_2 = \pm \ell/2 \\ (\sigma_0)_{13} = (\sigma_0)_{23} = 0 & \text{on} \quad x_3 = 0, \ell \\ u_3 = 0 & \text{on} \quad x_3 = 0, \ell, \end{cases}$$

and there are no body loads.

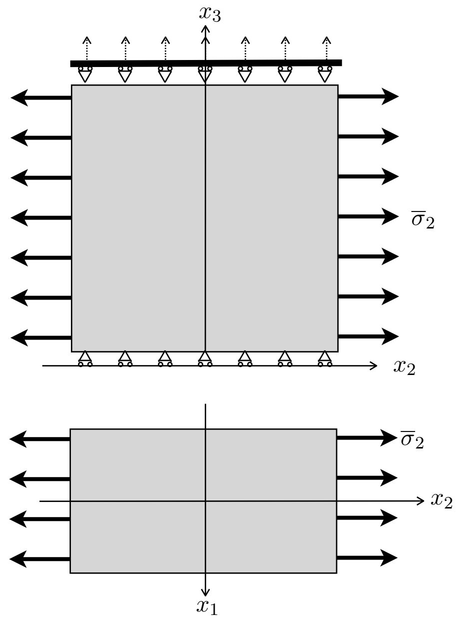

FIGURE 1. Bi-axial test.

The elastic solution (u_0, e_0, σ_0) unique, up to compatible infinitesimal rigid body motions, is given by

$$\begin{cases} u_0 := \dfrac{1}{\boldsymbol{E}}\overline{\sigma}_2 x_2 f_2 \\ e_0 = \dfrac{\overline{\sigma}_2}{\boldsymbol{E}} f_2 \otimes f_2 \\ \sigma_0 = \overline{\sigma}_2 f_2 \otimes f_2. \end{cases} \tag{4.1}$$

As long as

$$0 \le \overline{\sigma}_2 < \sigma_c, \tag{4.2}$$

the associated stress satisfies

$$|(\sigma_0)_D| \le \sqrt{2/3}\ \sigma_c,$$

so that the corresponding initial state is $\sigma(t=0) = \sigma_0$, $e(t=0) = e_0$, $p(t=0) = p_0 = 0$. At all later times, we simply rev up the displacement boundary condition at $x_3 = \ell$, setting

$$u_3 = 0 \text{ on } x_3 = 0, \quad u_3 = t\ell \text{ on } x_3 = \ell. \tag{4.3}$$

In other words, the stress $\overline{\sigma}_2$ is maintained constant in direction 2 while the sample is stretched in direction 3 (see Figure 1).

The quasi-static evolution with initial configuration (u_0, e_0, p_0) admits a *homogeneous solution*, by which we mean that both $e(t)$ and $p(t)$ are absolutely continuous functions independent of the spatial variable x. Set

$$t_c := \frac{1}{2\boldsymbol{E}}\left(\overline{\sigma}_2 + \sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2}\right), \tag{4.4}$$

and note that $4\sigma_c^2 - 3\overline{\sigma}_2^2 > 0$ in view of (4.2).

When $t \le t_c$, that solution can then be checked to be

$$\begin{cases} \sigma_h(t) = \sigma_0 + t\boldsymbol{E} f_3 \otimes f_3, \\ e_h(t) = e_0 + t f_3 \otimes f_3, \end{cases}$$

which corresponds to

$$u_h(t) = u_0 + t x_3 f_3$$

with u_0 given by (4.1)

Starting at the end of the elastic phase, *i.e.*, when $t \geq t_c$, the stress and elastic strain states remain constant and respectively equal to

$$\begin{cases} \sigma_h(t_c) = \overline{\sigma}_2 f_2 \otimes f_2 + \overline{\sigma}_3 f_3 \otimes f_3, \\ e_h(t_c) = e_0 + t_c f_3 \otimes f_3, \end{cases}$$

with

$$\overline{\sigma}_3 := \frac{1}{2}\left(\overline{\sigma}_2 + \sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2}\right) \tag{4.5}$$

while the plastic strain is given by

$$p_h(t) = (t - t_c)\Big(- \frac{(\overline{\sigma}_2 + \overline{\sigma}_3)}{2\overline{\sigma}_3 - \overline{\sigma}_2} f_1 \otimes f_1 + \frac{2\overline{\sigma}_2 - \overline{\sigma}_3}{2\overline{\sigma}_3 - \overline{\sigma}_2} f_2 \otimes f_2 + f_3 \otimes f_3\Big). \tag{4.6}$$

The displacement field $u(t)$ is determined from the boundary conditions, together with $Eu(t) = e(t) + p(t)$. It is precisely

$$u_h(t) = -\frac{(\overline{\sigma}_2 + \overline{\sigma}_3)}{2\overline{\sigma}_3 - \overline{\sigma}_2}(t - t_c)x_1 f_1 + \left\{\frac{\overline{\sigma}_2}{\boldsymbol{E}} + \frac{2\overline{\sigma}_2 - \overline{\sigma}_3}{2\overline{\sigma}_3 - \overline{\sigma}_2}(t - t_c)\right\} x_2 f_2 + t x_3 f_3.$$

As already mentioned, the uniqueness of the stress field, hence of the elastic strain field, is a given, so that the elastic phase is also unique as long as $t < t_c$ because the yield stress has not been reached.

When $t \geq t_c$, the deviatoric part of the stress field is given by

$$(\sigma_h)_D(t) = \check{\sigma} := \check{\sigma}_1 f_1 \otimes f_1 + \check{\sigma}_2 f_2 \otimes f_2 + \check{\sigma}_3 f_3 \otimes f_3, \text{ with} \tag{4.7}$$

$$\check{\sigma}_1 := -(\overline{\sigma}_2 + \overline{\sigma}_3)/3 = -1/2\, \overline{\sigma}_2 - 1/6\sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2},$$

$$\check{\sigma}_2 := (2\overline{\sigma}_2 - \overline{\sigma}_3)/3 = 1/2\, \overline{\sigma}_2 - 1/6\sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2},$$

$$\check{\sigma}_3 := (2\overline{\sigma}_3 - \overline{\sigma}_2)/3 = 1/3\sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2}.$$

In particular, it is indeed the case that $\check{\sigma}_1 + \check{\sigma}_2 + \check{\sigma}_3 = 0$, $\check{\sigma}_1^2 + \check{\sigma}_2^2 + \check{\sigma}_3^2 = 2/3\, \sigma_c^2$ as expected.

Now, in the spirit of Section 3, we conclude, in view of (3.2), that no plastic slips can arise if $\check{\sigma}_2 \neq 0$, or still, if $\overline{\sigma}_2 \neq \sigma_c/\sqrt{3}$. Further, if we are to believe the statement in Remark 3.1, the same condition would also bar the onset of Cantor-like plastic strains. Of course, it could still be so that uniqueness is defeated because the integrable part of the plastic strain is not uniquely determined.

This will not be the case, as demonstrated below with the help of the Saint-Venant kinematic compatibility conditions. The use of those conditions is not new in plasticity and it is the basis of what used to be called slip line analysis (see once again [15]). However slip lines are a two-dimensional notion whereas our focus here is squarely three-dimensional and, even in a two-dimensional case, we have not been able to locate a clear articulation of the connection between slip-lines and uniqueness of the plastic strain.

Here is how it goes. The plastic strain rate must be of the form $\dot{p}(t) = \eta(t)\check{\sigma},\ t \geq t_c$ with $\check{\sigma}$ given in (4.7). Consequently, the total strain $Eu(t)$ is given by

$$\begin{aligned} (Eu(t))_{11} &= -\eta(t)(\overline{\sigma}_2 + \overline{\sigma}_3), & (Eu(t))_{12} &= 0, \\ (Eu(t))_{22} &= \frac{\overline{\sigma}_2}{\boldsymbol{E}} + \eta(t)(2\overline{\sigma}_2 - \overline{\sigma}_3), & (Eu(t))_{23} &= 0, \\ (Eu(t))_{33} &= \frac{\overline{\sigma}_3}{\boldsymbol{E}} + \eta(t)(2\overline{\sigma}_3 - \overline{\sigma}_2), & (Eu(t))_{31} &= 0. \end{aligned}$$

The compatibility equations consist in writing that

$$(Eu(t))_{ij,kl} + (Eu(t))_{kl,ij} = (Eu(t))_{ik,jl} + (Eu(t))_{jl,ik},\ 1 \leq i, j, k, l \leq 3,$$

which yields

$$
\begin{aligned}
0 &= (2\overline{\sigma}_2 - \overline{\sigma}_3)\eta_{,11}(t) - (\overline{\sigma}_2 + \overline{\sigma}_3)\eta_{,22}(t), & 0 &= (\overline{\sigma}_2 + \overline{\sigma}_3)\eta_{,23}(t),\\
0 &= (2\overline{\sigma}_2 - \overline{\sigma}_3)\eta_{,33}(t) + (2\overline{\sigma}_3 - \overline{\sigma}_2)\eta_{,22}(t), & 0 &= (2\overline{\sigma}_2 - \overline{\sigma}_3)\eta_{,31}(t),\\
0 &= (2\overline{\sigma}_3 - \overline{\sigma}_2)\eta_{,11}(t) - (\overline{\sigma}_2 + \overline{\sigma}_3)\eta_{,33}(t), & 0 &= (2\overline{\sigma}_3 - \overline{\sigma}_2)\eta_{,12}(t).
\end{aligned} \tag{4.8}
$$

For the mathematically inclined reader, we emphasize that the system above holds true as soon as $\eta(t)$ is a measure. Uniqueness will be achieved if we prove that $\eta(t) = (t - t_c)/(2\overline{\sigma}_3 - \overline{\sigma}_2)$ (see(4.6)).

First note that $2\overline{\sigma}_3 \neq \overline{\sigma}_2$ and $\overline{\sigma}_2 + \overline{\sigma}_3 \neq 0$.

If now $\overline{\sigma}_2 \neq \sigma_c/\sqrt{3}$, then $2\overline{\sigma}_2 \neq \overline{\sigma}_3$. We immediately conclude that, in such a case, the only solution to the linear system (4.8) is $\eta_{,ij}(t) = 0\ \forall i, j \in \{1, 2, 3\}$. Thus η is an affine function of x,

$$\eta(x, t) = \eta_0(t) + \sum_{i=1}^{3} \eta_i(t)x_i.$$

Then,

$$u_{3,3}(x, t) = \frac{\overline{\sigma}_3}{\boldsymbol{E}} + (2\overline{\sigma}_3 - \overline{\sigma}_2)\left(\eta_0(t) + \sum_{i=1}^{3} \eta_i(t)x_i\right).$$

Since $u_3(t) = 0$ at $x_3 = 0$, we obtain

$$u_3(x, t) = \frac{\overline{\sigma}_3}{\boldsymbol{E}}x_3 + (2\overline{\sigma}_3 - \overline{\sigma}_2)\left(\eta_0(t)x_3 + \eta_1(t)x_1x_3 + \eta_2(t)x_2x_3 + \frac{1}{2}\eta_3(t)x_3^2\right).$$

Since $u_3(t) = t\ell$ at $x_3 = \ell$, this yields in turn, thanks to the expressions (4.4),(4.5) for t_c and $\overline{\sigma}_3$,

$$\eta_1(t) = \eta_2(t) = 0, \qquad \eta_0(t) + \frac{1}{2}\eta_3(t)\ell = \frac{t - t_c}{2\overline{\sigma}_3 - \overline{\sigma}_2}.$$

Since $u_{3,2}(t) = 0$ and $(Eu(t))_{23} = 0$, $u(t)_{2,3} = 0$, hence $(Eu(t))_{22,3} = u(t)_{2,23} = 0$, which implies that $\eta_{,3}(t) = \eta_3(t) = 0$. The conclusion is reached.

Thus *uniqueness of the whole elasto-plastic evolution holds true whenever* $\overline{\sigma}_2 \neq \sigma_c/\sqrt{3}$. This result is, as already stressed in the introduction, an authentic three-dimensional uniqueness result. A quick and admittedly incomplete perusal of the existing literature has not produced any result of a similar ilk.

If however $\overline{\sigma}_2 = \sigma_c/\sqrt{3}$, then the previous result collapses because the linear system (4.8) may admit a non trivial solution. Looking at (4.8), we immediately obtain

$$0 = \eta_{,12}(t) = \eta_{,22}(t) = \eta_{,23}(t)$$

and

$$0 = \eta_{,11}(t) - \eta_{,33}(t). \tag{4.9}$$

Thus, $\nabla(\eta_{,2}(t)) = 0$ and, consequently, $\eta(t) = \mathring{\eta}(t) + \beta(t)x_2$, with $\mathring{\eta}(t)$ a measure independent of x_2 and $\beta(t)$ a constant. Since $\eta(t)$ satisfies the spatial wave equation (4.9), we conclude that

$$\mathring{\eta}(t) = \zeta_-(t)(x_1 - x_3) + \zeta_+(t)(x_1 + x_3)$$

where $\zeta_\pm$ are nonnegative measures on $\mathbb{R}$.

From this, *assuming further that* $d < \ell$, we can seek – taking e.g. $\beta(t) \equiv 0$ – a solution displacement field of the form $u(t) = u(t_c) + \bar{u}(t)$ with

$$\bar{u}(t) = -(Z_-(t, x_1 - x_3) + Z_+(t, x_1 + x_3))f_1 + (-Z_-(t, x_1 - x_3) + Z_+(t, x_1 + x_3))f_3,$$

where $Z_\pm$ is a primitive of $\zeta_\pm$.

Recalling the boundary conditions (4.3) on u_3 at $x_3 = 0, \ell$, we obtain

$$Z_-(t, s) = Z_+(t, s),\ s \in (-d/2, d/2), \quad Z_-(t, s) + (t - t_c)\ell = Z_+(t, s + 2\ell),\ s \in (-d/2 - \ell, d/2 - \ell).$$

For example, any pair

$$\begin{cases} Z_+(t,s) = (t-t_c)\ell, \\ Z_-(t,s) = (t-t_c)\ell \int_{d/2-\ell}^{s} d\zeta \end{cases} \tag{4.10}$$

where ζ is any probability measure with support $[d/2-\ell, -d/2]$ is a solution.

Thus, not only can one generate an infinity of solutions in the case where $d < \ell$, but those can be as smooth or unsmooth as one desires. By choosing ζ_- to have a smooth density, one gets a smooth plastic strain. But one could as easily generate solutions with jumps. For example, take $\zeta_- = \delta_{-\ell/2}$ (the Dirac mass at the point $-\ell/2$). Then

$$Z_-(t,s) = \begin{cases} 0, \ s \le -\ell/2, \\ (t-t_c)\ell, \ \text{else} \end{cases}$$

so that $u(t)$ experiences a jump on the line $x_3 - x_1 = \ell/2$.

Even worse, the choice of ζ_- as measure supported on a Cantor set will generate a solution for which the plastic strain is purely Cantor-like; the corresponding displacement field $u(t)$ would not jump, yet have 0 derivative almost everywhere.

Finally note that the plastic strain does not have to be a linear function of $t - t_c$. Indeed, in lieu of (4.10), it suffices to take, for an arbitrary $t^* > t_c$,

$$\begin{cases} Z_+(t,s) = (t-t_c)\ell, \\ Z_-(t,s) = (t-t_c)\ell \int_{d/2-\ell}^{s} d\zeta, & t \le t^*, \\ Z_-(t,s) = \begin{cases} (t^*-t_c)\ell \int_{d/2-\ell}^{s} d\zeta, \ s \le -d/2, \\ (t-t_c)\ell, \ s > -d/2, \end{cases} & t \ge t^*. \end{cases}$$

4.2. **The spherical cavity under internal pressure.** In this subsection we consider a spherical shell Ω with interior radius a subject to a a time increasing internal pressure t – we identify time and pressure throughout this subsection – at $r = a$, while its external surface with radius b is free of forces (see Figure 2). The elasticity tensor $\mathbb{C}$ is assumed to be homogeneous and isotropic with Young's modulus $\boldsymbol{E} > 0$ and Poisson's ratio $-1 < \boldsymbol{\nu} < 1/2$. At the initial time $t = 0$, the shell is undeformed.

This problem is a textbook case of elasto-plastic evolution (see e.g. [16, Section 25]). The classical solution exhibits spherical symmetry. It is described as follows.

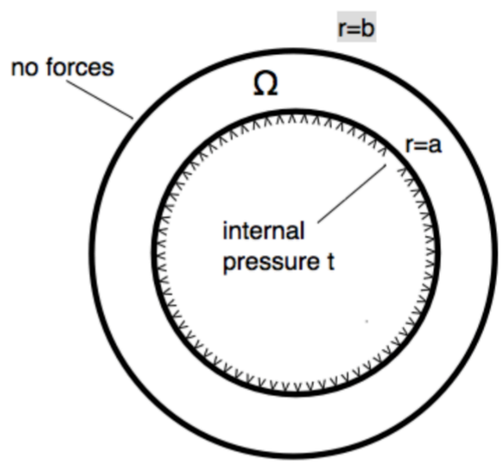

FIGURE 2. The spherical cavity.

As long as $t \le t_e$ defined below, the response is purely elastic, the displacement field is radial and depends only on r and t. Specifically, denoting by f_r, f_θ, f_ϕ the three orthonormal basis vectors associated with spherical coordinates,

$$u(t,x) = u_r(t,r)f_r, \qquad p(t,x) = 0, \qquad \sigma(t,x) = \sigma_r(t,r)f_r \otimes f_r + \sigma_\theta(t,r)(f_\theta \otimes f_\theta + f_\phi \otimes f_\phi)$$

with

$$u_r(t,r) = \frac{ta^3}{b^3 - a^3}\left(\frac{(1-2\boldsymbol{\nu})r}{\boldsymbol{E}} + \frac{(1+\boldsymbol{\nu})b^3}{2\boldsymbol{E}r^2}\right)$$

and

$$\sigma_r(t,r) = \frac{ta^3}{b^3 - a^3}\left(1 - \frac{b^3}{r^3}\right), \qquad \sigma_\theta(t,r) = \frac{ta^3}{b^3 - a^3}\left(1 + \frac{b^3}{2r^3}\right). \tag{4.11}$$

The Von Mises criterion reduces to $|\sigma_\theta - \sigma_r| \le \sigma_c$, so that it is satisfied throughout the spherical shell as long as

$$t \le t_e := \frac{2}{3}\left(1 - \frac{a^3}{b^3}\right)\sigma_c.$$

When $t > t_e$, plastification occurs on a spherical lying between $r = a$ and $r = c(t) \overset{t}{\nearrow}$ with $c(t_e) = a$. At a time $t = t_\ell$ which will be determined below, $c(t) = b$. Consider a time t such that $t_e < t < t_\ell$.

In the domain $\Omega_p(t) = \{x : |x| \in (a, c(t)]\}$ the plasticity threshold is reached while in its complement $\Omega_e(t) = \{x : |x| \in (c(t), b)\}$ it is not. Assuming spherical symmetry, the two non-zero components σ_r et σ_θ of the stress field are obtained in $\Omega_p(t)$ from the equilibrium equations, the criterion and the boundary condition at $r = a$, that is

$$\sigma_{r,r}(t,r) + \frac{2}{r}(\sigma_r(t,r) - \sigma_\theta(t,r)) = 0, \quad \sigma_\theta(t,r) - \sigma_r(t,r) = \sigma_c, \quad \sigma_r(t,a) = -t.$$

Thus,

$$\sigma_r(t,r) = -t + 2\sigma_c \ln\frac{r}{a}, \quad \sigma_\theta(t,r) = -t + \sigma_c + 2\sigma_c \ln\frac{r}{a} \qquad a < r \le c(t). \tag{4.12}$$

Remark that the deviatoric part of the stress is

$$\sigma_D(t,x) = \frac{\sigma_c}{3}\left(-2f_r \otimes f_r + f_\theta \otimes f_\theta + f_\phi \otimes f_\phi\right). \tag{4.13}$$

The flow rule implies that

$$p(t,x) = \eta(t,r)\left(-2f_r \otimes f_r + f_\theta \otimes f_\theta + f_\phi \otimes f_\phi\right)$$

with $\dot\eta \ge 0$. Assuming displacements of the form $u(t,x) = u_r(t,r)f_r$, we deduce that η et u_r satisfy

$$3\eta = \frac{u_r}{r} - u_{r,r} - \frac{(1+\boldsymbol{\nu})\sigma_c}{E}, \qquad (r^2u_r)_{,r} = \frac{(2\sigma_c - 3t)(1-2\boldsymbol{\nu})}{\boldsymbol{E}}r^2 + \frac{6\sigma_c(1-2\boldsymbol{\nu})}{\boldsymbol{E}}r^2\ln\frac{r}{a}. \tag{4.14}$$

Thus they are determined, up to a function of t that will be obtained upon imposing the continuity of the displacement at $r = c(t)$.

In $\Omega_e(t)$ there is no plastic strain. Still under the assumption of spherical symmetry, that is $u(t,x) = u_r(t,r)f_r$, Hooke's law, the equilibrium equations and the boundary condition at $r = b$ imply that

$$u_r(t,r) = q(t)\left(\frac{r(1-2\boldsymbol{\nu})}{\boldsymbol{E}} + \frac{(1+\boldsymbol{\nu})b^3}{2\boldsymbol{E}r^2}\right), \qquad c(t) < r < b, \tag{4.15}$$

$$\sigma_r(t,r) = q(t)\left(1 - \frac{b^3}{r^3}\right), \quad \sigma_\theta(t,r) = q(t)\left(1 + \frac{b^3}{2r^3}\right), \qquad c(t) < r < b. \tag{4.16}$$

The pressure $q(t)$ and the radius $c(t)$ are obtained upon invoking the continuity of the normal stress and the criterion at $r = c(t)$. Thus

$$q(t) = \frac{2c(t)^3}{3b^3}\sigma_c, \tag{4.17}$$

whereas $c(t)$ is given through

$$t = 2\sigma_c \ln \frac{c(t)}{a} + \frac{2}{3}\sigma_c \left(1 - \frac{c(t)^3}{b^3}\right) \tag{4.18}$$

which admits a unique solution $c(t) \in (a, b)$ as long as $t \in (t_e, t_\ell)$ with

$$t_\ell := 2\sigma_c \ln \frac{b}{a}.$$

The continuity of the normal displacement at $r = c(t)$ completes the determination of both the displacement field and the plastic strain in $\Omega_p(t)$ through the use of (4.14). We finally obtain

$$\begin{cases} \eta(t,r) = (1-\boldsymbol{\nu})\dfrac{\sigma_c}{\boldsymbol{E}}\left(\dfrac{c(t)^3}{r^3} - 1\right), \\ u_r(t,r) = \dfrac{2}{3}(1-2\boldsymbol{\nu})\dfrac{\sigma_c}{\boldsymbol{E}}\left(3r\ln\dfrac{r}{c(t)} + \dfrac{c(t)^3}{b^3}r - r\right) + (1-\boldsymbol{\nu})\dfrac{\sigma_c}{\boldsymbol{E}}\dfrac{c(t)^3}{r^2}, \end{cases} \qquad a < r \le c(t).$$

Using (4.17) and (4.18) to compute $\dot q(t)$, it is easily checked, through differentiation of the previous expression, that $\dot\eta > 0$. Specifically,

$$\dot\eta(t,r) = (1-\boldsymbol{\nu})\frac{3\dot q(t)}{2\boldsymbol{E}}\frac{b^3}{r^3}, \qquad \dot q(t) = \frac{c(t)^3}{b^3 - c(t)^3}. \tag{4.19}$$

Finally when $t = t_\ell$, the whole shell is plastified. The stress field is

$$\sigma_r(t_\ell, r) = 2\sigma_c \ln \frac{r}{b}, \quad \sigma_\theta(t,r) = \sigma_c + 2\sigma_c \ln\frac{r}{b}, \qquad a < r < b. \tag{4.20}$$

The solution fields are no longer unique. Those with spherical symmetry are of the form

$$\begin{cases} \eta(t_\ell,r) = (1-\boldsymbol{\nu})\dfrac{\sigma_c}{\boldsymbol{E}}\left((1+\kappa)\dfrac{b^3}{r^3} - 1\right) \\ \\ u_r(t_\ell,r) = (1-\boldsymbol{\nu})\dfrac{\sigma_c}{\boldsymbol{E}}(1+\kappa)\dfrac{b^3}{r^2} + 2(1-2\boldsymbol{\nu})\dfrac{\sigma_c}{\boldsymbol{E}} r\ln\dfrac{r}{b} \end{cases}, \qquad a < r < b$$

where κ is an arbitrary non negative constant.

We label the above derived elasto-plastic evolution the *evolution with spherical symmetry.*

We now address the issue of uniqueness of that evolution.

The unique stress field is given by (4.11) when $t \in [0, t_e]$, (4.12) and (4.16) with $q(t)$ given by (4.17) and $c(t)$ given by (4.18) when $t \in (t_e, t_\ell)$, and by (4.20) when $t = t_\ell$.

In the rest of this subsection we propose to establish that *the evolution with spherical symmetry is the unique evolution as long as* $t < t_\ell$. We also construct all possible solutions when $t = t_\ell$.

First, note that the displacement field is only defined up to an infinitesimal rigid body motion, so that uniqueness (above and below) only takes place up to rigid body motions.

As long as $t \le t_e$, uniqueness obviously holds since the threshold is not reached anywhere in the shell. When $t \in (t_e, t_\ell)$, the stress field is given by (4.12) and (4.16), with $q(t)$ given by (4.17) and $c(t)$ by (4.18).

In $\Omega_e(t)$, the plastic strain rate $\dot p$ is null, hence $E\dot u = \mathbb{C}^{-1}\dot\sigma$. Consequently $\dot u$ exhibits spherical symmetry; it is thus given by (see (4.15))

$$\dot u(t,x) = \dot q(t)\left(\frac{r(1-2\boldsymbol{\nu})}{\boldsymbol{E}} + \frac{(1+\boldsymbol{\nu})b^3}{2\boldsymbol{E}r^2}\right) f_r, \qquad x \in \Omega_e(t). \tag{4.21}$$

In $\Omega_p(t)$, (4.13) implies that

$$\dot p(t,x) = \dot\eta(t,x)\left(-2f_r \otimes f_r + f_\theta \otimes f_\theta + f_\phi \otimes f_\phi\right), \quad \dot\eta \ge 0 \tag{4.22}$$

while, in view of (4.12), the elastic strain rate is

$$\dot e = \mathbb{C}^{-1}\dot\sigma = -\frac{1-2\boldsymbol{\nu}}{\boldsymbol{E}}\mathfrak{i}. \tag{4.23}$$

That strain rate is automatically compatible (*i.e.*, it satisfies kinematic compatibility) since it is that associated with a velocity field of the form $-(1-2\boldsymbol{\nu})/\boldsymbol{E}\ r f_r$. Thus $\dot{p}$ must be compatible. We will show below that this requires that $\dot{\eta}$ be of the form

$$\dot{\eta}(t,x) = \frac{\alpha_1}{r^3} + \frac{1}{r^3}(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3) \qquad \text{in } \Omega_p(t), \tag{4.24}$$

with $\alpha_1, \kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$. In particular $\dot{u}$ is a smooth function in $\Omega_p(t)$.

We now apply the results of Section 3; since, in view of the form (4.13) of the deviatoric stress, (3.2) is never satisfied at any point of the domain, we conclude to the absence of plastic slips throughout any possible evolution. In particular, the velocity field $\dot{u}$ *must be continuous* across the interface $r = c(t)$. As a consequence, the tangential strain rate $(E\dot{u})_{\theta\theta}$ must also be continuous across $r = c(t)$ since

$$(E\dot{u})_{\theta\theta} = \frac{1}{r}\left(\frac{\partial \dot{u}_\theta}{\partial \theta} + \dot{u}_r\right).$$

Thanks to (4.17), (4.21), (4.22) and (4.23) we infer that

$$\dot{\eta}(t,x) = \dot{q}(t)\left(\frac{(1-2\boldsymbol{\nu})}{\boldsymbol{E}} + \frac{(1+\boldsymbol{\nu})b^3}{2\boldsymbol{E}c(t)^3}\right) + \frac{1-2\boldsymbol{\nu}}{\boldsymbol{E}} = \frac{3(1-\boldsymbol{\nu})}{2\boldsymbol{E}}\frac{b^3}{b^3 - c(t)^3} \quad \text{on} \quad r = c(t).$$

In view of (4.24), we deduce that $\kappa_1 = \kappa_2 = \kappa_3 = 0$, with

$$\alpha_1 = \frac{3(1-\boldsymbol{\nu})}{2\boldsymbol{E}}\frac{b^3 c(t)^3}{b^3 - c(t)^3},$$

which is precisely expression (4.19) for $\dot{\eta}$.

The uniqueness of the evolution thus follows as long as $t < t_\ell$.

When $t = t_\ell$, we still have that

$$\dot{\eta}(x) = \frac{\alpha_1}{r^3} + \frac{1}{r^3}(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3) \geq 0, \quad \forall x \ : \ |x| \in (a,b)$$

but this time no boundary condition need be enforced because of the absence of a non-plastified region. Only non negativity remains so that the coefficients α_1 et $\{\kappa_i\}_{i=1,2,3}$ are solely constrained through

$$\alpha_1 \geq b\sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}.$$

This establishes the possibility of the existence of non-spherical plastic strains which will however remain smooth.

In order to conclude, we need to show that expression (4.24) for $\dot{\eta}$ is a consequence of compatibility for $\dot{p}$ in $\Omega_p(t)$. In the case of Subsection 4.1, writing compatibility was a straightforward task. Such is not the case here because compatibility must be expressed in spherical coordinates. The only reference that we could find on this topic is [5].

We detail the computations below. Introduce the basis

$$g_r = f_r, \quad g_\theta = r\cos\phi f_\theta, \quad g_\phi = r f_\phi.$$

Then, the non-zero covariant (resp. contravariant) entries of the metric tensor are

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2\cos^2\phi, \quad g_{\phi\phi} = r^2, \qquad g^{rr} = 1, \quad g^{\theta\theta} = \frac{1}{r^2\cos^2\phi}, \quad g^{\phi\phi} = \frac{1}{r^2}.$$

The non-zero covariant (resp. contravariant) entries of $\dot{p}$ given by (4.22) are

$$\dot{p}_{rr} = -2\dot{\eta}, \quad \dot{p}_{\theta\theta} = \dot{\eta}r^2\cos^2\phi, \quad \dot{p}_{\phi\phi} = \dot{\eta}r^2, \qquad \dot{p}^{rr} = -2\dot{\eta}, \quad \dot{p}^{\theta\theta} = \frac{\dot{\eta}}{r^2\cos^2\phi}, \quad \dot{p}^{\phi\phi} = \frac{\dot{\eta}}{r^2}.$$

Introduce

$$\Gamma_{ijk} := \frac{1}{2}\Big(g_{jk,i} + g_{ki,j} - g_{ij,k}\Big), \qquad G_{ijk} := \dot{p}_{jk,i} + \dot{p}_{ki,j} - \dot{p}_{ij,k};$$

the compatibility equations then read as

$$
\begin{aligned}
0 &= \dot p_{ij,kl} + \dot p_{kl,ij} - \dot p_{ik,jl} - \dot p_{jl,ik} - 2\sum_{p,q} \dot p^{pq}\Big(\mathit{\Gamma}_{ijp}\mathit{\Gamma}_{klq} - \mathit{\Gamma}_{ikp}\mathit{\Gamma}_{jlq}\Big) \\
&+ \sum_{p,q} g^{pq}\Big(\mathit{\Gamma}_{ijp}G_{klq} + \mathit{\Gamma}_{klp}G_{ijq} - \mathit{\Gamma}_{ikp}G_{jlq} - \mathit{\Gamma}_{jlp}G_{ikq}\Big).
\end{aligned}
\tag{4.25}
$$

Six non trivial equations are obtained; they correspond to the following quadruplets in $ijkl$: $rr\theta\phi$, $\theta\theta\phi r$, $\phi\phi r\theta$, $rr\theta\theta$, $\theta\theta\phi\phi$, $\phi\phi rr$. The first three reduce to

$$
0 = \dot\eta_{,\theta\phi} + \tan\phi\ \dot\eta_{,\theta}, \quad 0 = \cos^2\phi\ (r^2\dot\eta)_{,r\phi}, \quad 0 = (r^2\dot\eta)_{,r\theta} \tag{4.26}
$$

which yields

$$
r^2\dot\eta(t,r,\theta,\phi) = \alpha(r) + \beta(\theta)\cos\phi + \kappa(\phi), \tag{4.27}
$$

the three functions $\alpha(r)$, $\beta(\theta)$ and $\kappa(\phi)$ being arbitrary. That involving $\phi\phi rr$ reads as

$$
0 = (r^2\dot\eta)_{,rr} - 2\dot\eta_{,\phi\phi} - 2\dot\eta,
$$

which, together with (4.27), yields

$$
r^2\dot\eta(t,r,\theta,\phi) = \frac{\alpha_1}{r} + \alpha_2 r^2 + \beta(\theta)\cos\phi + \kappa_3\sin\phi. \tag{4.28}
$$

At this stage, the coefficients α_1, α_2, κ_3 and the function $\beta(\theta)$ remain arbitrary. The $rr\theta\theta$ equation reads as

$$
0 = (r^2\dot\eta)_{,rr}\cos^2\phi - 2\dot\eta_{,\theta\theta} - 2\dot\eta\cos^2\phi + 2\dot\eta_{,\phi}\sin\phi\cos\phi
$$

which, together with (4.28) yields

$$
r^2\dot\eta(t,r,\theta,\phi) = \frac{\alpha_1}{r} + \alpha_2 r^2 + \kappa_1\cos\theta\cos\phi + \kappa_2\sin\theta\cos\phi + \kappa_3\sin\phi. \tag{4.29}
$$

At this stage, the coefficients α_1, α_2, κ_1 , κ_2 et κ_3 remain arbitrary. Finally, the equation involving $\theta\theta\phi\phi$ reads as

$$
\begin{aligned}
0 &= r^2(\dot\eta\cos^2\phi)_{,\phi\phi} + r^2\dot\eta_{,\theta\theta} + 4r^2\dot\eta\cos^2\phi + 2r^2\dot\eta\sin^2\phi + 2r(r^2\dot\eta)_{,r}\cos^2\phi \\
&+ r^2\dot\eta_{,\phi}\sin\phi\cos\phi + 2r^2(\dot\eta\cos^2\phi)_{,\phi}\tan\phi,
\end{aligned}
$$

which, together with (4.29) yields $\alpha_2 = 0$.

Summing up, compatibility requires that $\dot\eta$ be of the form

$$
\dot\eta(t,x) = \frac{\alpha_1}{r^3} + \frac{1}{r^3}(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3).
$$

so that (4.24) follows.

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