A CASE STUDY FOR UNIQUENESS OF ELASTO-PLASTIC EVOLUTIONS: THE BI-AXIAL TEST

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Abstract. The uniqueness of the solution to a quasi-static problem in perfect elasto-plasticity is established in the case of a bi-axial test. It is the first example known to us of uniqueness in the context of multi-dimensional elasto-plasticity.

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1. Introduction

1.1. **General considerations.** The current mathematical view of perfect elasto-plasticity finds its roots in the work of P.-M. Suquet (see e.g. [10],[11]), later completed by various works of R. Temam (see e.g. [12]) and R.V. Kohn and R. Temam (see [7]). After a long lull, that work was revived by G. Dal Maso, A. De Simone and M. G. Mora [3] within the framework of the variational theory of rate independent evolutions advocated in particular by A. Mielke (see e.g. [9]).

Those evolutions are quasi-static, that is that inertia effects are neglected. Then, the evolution can be viewed as a time-parameterized set of minimization problems for the sum of the elastic energy and of the add-dissipation; see Section 2 for details. The minimizers should also conserve energy throughout the evolution, a statement that can be seen as the result of a combination of the two principles of thermodynamics. Once such an evolution is secured, it has been shown in various recent works that that evolution satisfies the original system of equations, and in particular the so-called flow rule: whenever the (deviatoric part of the) stress reaches the boundary of its admissible set, the plastic strain should flow in the direction normal to that set.

Although the evolutions constructed in this manner share a unique stress field (see for example Theorem 2.12 below), hence a unique elastic strain field, uniqueness of the plastic strain, hence of the displacement field, is a delicate issue. As a matter of fact, to the best of our knowledge, there is *no example of uniqueness* to be found in the overwhelmingly abundant literature on elasto-plasticity, except when the problem reduces to a one-dimensional setting. In the latter setting, A. Demyanov proved in [4, Section 10.2] that uniqueness is achieved for a very specific loading process. To be fair, the beauty of his result resides elsewhere: it is in the demonstration that any measure which does not charge atoms can be attained as a plastic strain in a one-dimensional context.

In this paper, we propose to exhibit a *bona fide three-dimensional problem* for which uniqueness is generic. To do so, we have to introduce a set of boundary conditions which does not fall within the scope of the established variational theory introduced in [3]. This is because our example necessitates the consideration of mixed type boundary conditions in the following sense: on part of the boundary of the domain, only the normal component of the displacement is given, while the tangential components of the normal stress are nul. This is because our example is a bi-axial test where such boundary conditions are needed if one is to secure an explicit elastic field.

Consequently, we have to revisit the whole theory of variational evolutions in this new context. Only once this is done can we address the example of the bi-axial test.

The paper is organized as follows.

After a short Subsection (Subsection 1.2) devoted to notation and mathematical preliminaries, Section 2 revisits the variational approach to quasi-static elasto-plastic evolutions in this new context and re-establishes all needed results. We attempt to perform the task by striking a sometimes delicate balance between precision and brevity. Particular attention is paid to the kinematics of the problem. This is because admissible rigid body motions have to be factored out if coercivity is to be secured.

Section 3 addresses the bi-axial test. In a first Subsection (Subsection 3.1), the setting is detailed and a spatially homogeneous evolution is evidenced. Subsection 3.2 then investigates the possible uniqueness of the homogeneous solution. Theorem 3.2 demonstrates that uniqueness is generic.

Finally, when uniqueness does not hold, it is shown in Remark 3.4 that the multiplicator of the plastic strain satisfies a spatially hyperbolic equation with an incomplete set of boundary data. Consequently, an infinite number of solutions to the quasi-static evolution are generated. In particular, there exist solutions for which the plastic strain exhibits a Cantor part! This is to our knowledge the first truly multi-dimensional example for which the solution displacement field $u(t)$ is in BD, and not only in SBD (see, once again, [4, Section 10.2] for a one-dimensional example) and it dashes any hope for a regularity statement on solution-displacement fields of quasi-static elasto-plastic evolutions.

1.2. **Mathematical Preliminaries.** Here, we detail the mathematical notation, as well as a few mathematical remarks that will be of relevance.

Throughout the paper, we refer to e.g. [2] for background material, especially concerning finer measure theoretical points.

General notation. For $B \subseteq \mathbb{R}^N$, the symbol $A \subset\subset B$ means that the closure of A is compact and contained in B. The symbol $\lfloor_A$ stands for "restricted to A".

Matrices. We denote by $\mathrm{M}^N_{\mathrm{sym}}$ the set of $N \times N$-symmetric matrices and by M^N_D the set of trace-free elements of $\mathrm{M}^N_{\mathrm{sym}}$. The identity matrix in $\mathrm{M}^N_{\mathrm{sym}}$ is denoted by $\mathfrak{i}$. If M is an element of $\mathrm{M}^N_{\mathrm{sym}}$, then M_D denotes its deviatoric part, *i.e.*, its projection onto the subspace M^N_D of $\mathrm{M}^N_{\mathrm{sym}}$ orthogonal to $\mathfrak{i}$ for the Frobenius inner product. The symbol $\cdot$ denotes that inner product. We denote by $\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}})$ the set of symmetric endomorphisms on $\mathrm{M}^N_{\mathrm{sym}}$. For $a, b \in \mathbb{R}^N$, $a \odot b$ stands for the symmetric matrix such that $(a \odot b)_{ij} := (a_i b_j + a_j b_i)/2$.

Depending on the context, we will denote by $B(x, r)$ the open ball of center x and radius r in $\mathbb{R}^N$, or that in M^N_D.

Measures. If $E \subseteq \mathbb{R}^N$ is locally compact and Y a finite dimensional normed space, $\mathcal{M}_b(E; Y)$ will denote the space of finite Radon measures with values in Y. If $Y = \mathbb{R}$, we denote by $\mathcal{M}^+_b(E)$ the subspace of nonnegative elements of $\mathcal{M}_b(E; \mathbb{R})$. For $\mu \in \mathcal{M}_b(E; Y)$, we denote its total variation by $|\mu|(E)$, or equivalently by $\|\mu\|_1$. This is because the total variation of μ is also the norm of μ as an element of the topological dual of $C^0_0(E; Y^*)$, the set of continuous functions u from E to the vector dual Y^* of Y which "vanish at the boundary", *i.e.*, such that for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ with $|u(x)| < \varepsilon$ for $x \notin K$. Besides the associated weak-$\star$ convergence, we also use strict convergence. We say that

$$\mu_n \xrightarrow{s} \mu \quad \text{strictly in } \mathcal{M}_b(E; Y)$$

iff

$$\mu_n \xrightharpoonup{*} \mu \text{ weakly}^* \text{ in } \mathcal{M}_b(E; Y) \text{ and } |\mu_n|(E) \to |\mu|(E).$$

Functional spaces. Given $E \subseteq \mathbb{R}^N$ measurable, $1 \leq p < +\infty$, and Y a finite dimensional normed space, $L^p(E; Y)$ stands for the space of p-summable functions on E with values in Y, with associated norm denoted by $\|\cdot\|_p$. (Note that $\|f\|_1$ thus denotes both the L^1-norm if $f \in L^1(E; Y)$, and the total variation of a finite radon measure if $f \in \mathcal{M}_b(E; Y)$; no confusion should ensue.) Given $A \subseteq \mathbb{R}^N$ open, $H^1(A; Y)$ is the Sobolev space of functions in $L^2(A; Y)$ with distributional derivatives in L^2.

Finally, let X be a normed space. We denote by $BV(a, b; X)$ and $AC(a, b; X)$ the space of functions with bounded variation and the space of absolutely continuous functions from $[a, b]$ to

X, respectively. We recall that the total variation of $f \in BV(a,b;X)$ is defined as

$$\mathcal{V}_X(f;a,b) := \sup\left\{ \sum_{j=1}^{k} \|f(t_j) - f(t_{j-1})\|_X \,:\, a = t_0 < t_1 < \cdots < t_k = b \right\}.$$

The (kinematic) space BD. In this paper as in previous works on elasto-plasticity the displacement field u lies in $BD(\Omega)$, the space of functions of bounded deformations. We refer the reader to e.g. [12, Chapter II], and [1] for background material. Besides elementary properties of $BD(\Omega)$, we will use the following Korn's inequality: if Ω has a Lipschitz boundary, there exists $C > 0$, such that, for every $u \in BD(\Omega)$, there exists $r \in \mathcal{R}(\Omega)$ satisfying

$$\|u - r\|_{BD(\Omega)} \le C\|Eu\|_{\mathcal{M}_b(\Omega;\mathrm{M}^N_{\mathrm{sym}})}, \tag{1.1}$$

where Eu denotes the symmetrized gradient of u, and

$$\mathcal{R}(\Omega) := \{u = b + Ax \,:\, b \in \mathbb{R}^N, A \in \mathrm{M}^N_{\mathrm{skew}}\} \tag{1.2}$$

denotes the family of *infinitesimal rigid body motions on* Ω; see [12, Chapter II, Prop. 2.3].

We say that

$$u_n \overset{*}{\rightharpoonup} u \qquad \text{weakly}^* \text{ in } BD(\Omega)$$

iff

$$u_n \to u, \;\text{ strongly in } L^1(\Omega;\mathbb{R}^N) \text{ and } Eu_n \overset{*}{\rightharpoonup} Eu \;\text{ weakly}^* \text{ in } \mathcal{M}_b(\Omega;\mathrm{M}^N_{\mathrm{sym}}).$$

Bounded sequences in $BD(\Omega)$ always admit a weakly* converging subsequence.

The (static) space Σ. It is defined as

$$\Sigma := \left\{\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) : \mathrm{div}\,\sigma \in L^2(\Omega;\mathbb{R}^N) \text{ and } \sigma_D \in L^\infty(\Omega;\mathbb{R}^N)\right\}.$$

It is classical that, if $\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ with div $\sigma \in L^2(\Omega;\mathbb{R}^N)$, $\sigma\nu$ is well defined as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^N)$, ν being the outer normal to $\partial\Omega$.

More generally, consider an arbitrary Lipschitz subdomain $A \subset \Omega$ with outer normal ν, and $\Delta \subset \partial A$ open in the relative topology. We can define the restriction of $\sigma\nu$ "on Δ" by testing against functions in $H^{1/2}(\partial A;\mathbb{R}^N)$ with compact support in Δ. This amounts to viewing $\sigma\nu$ as an element of the dual to $H^{1/2}_{00}(\Delta;\mathbb{R}^N)$.

If $\sigma \in \Sigma$, then, in the spirit of [7, Lemma 2.4], we can define a tangential component $[\sigma\nu]_\tau$ of $\sigma\nu$ on Δ such that

$$[\sigma\nu]_\tau \in L^\infty(\Delta;\mathbb{R}^N) \qquad \text{with} \qquad \|[\sigma\nu]_\tau\|_\infty \le \|\sigma_D\|_\infty.$$

Indeed, consider any regularization $\sigma_n \in C^\infty(\bar{A};\mathrm{M}^N_{\mathrm{sym}})$ of σ on $\bar{A}$ such that

$$\begin{cases} \sigma_n \to \sigma & \text{strongly in } L^2(A;\mathrm{M}^N_{\mathrm{sym}}) \\ \mathrm{div}\sigma_n \to \mathrm{div}\sigma & \text{strongly in } L^2(A;\mathbb{R}^N) \\ \|(\sigma_n)_D\|_\infty \le \|\sigma_D\|_\infty. \end{cases}$$

Define the tangential component $[\sigma_n\nu]_\tau := (\sigma_n)\nu - ((\sigma_n)\nu\cdot\nu)\nu$. It is readily seen that $[\sigma_n\nu]_\tau = [(\sigma_n)_D\nu]_\tau$ (the tangential component of $(\sigma_n)_D$ is defined analogously). Since $x \mapsto \nu(x)$ is an $L^\infty(\Delta;\mathbb{R}^N)$-mapping, there exists an $L^\infty(\Delta;\mathbb{R}^N)$-function $[\sigma\nu]_\tau$ such that, up to a subsequence,

$$[\sigma_n\nu]_\tau \overset{*}{\rightharpoonup} [\sigma\nu]_\tau \text{ weakly}^* \text{ in } L^\infty(\Delta;\mathbb{R}^N).$$

If $\sigma_D \equiv 0$ then, clearly, $[\sigma\nu]_\tau \equiv 0$, so that $[\sigma\nu]_\tau$ is only a function of $(\sigma_n)_D$ which we will denote henceforth by $[\sigma_D\nu]_\tau$. Notice that $[\sigma_D\nu]_\tau$ may depend upon the approximation sequence σ_n (or at least upon $(\sigma_n)_D$).

If Δ is a C^2-hypersurface, then $[\sigma_D\nu]_\tau$ is uniquely determined as an element of $L^\infty(\Delta;\mathbb{R}^N)$. Indeed, for every $\varphi \in H^{1/2}(\partial A;\mathbb{R}^N)$ with support compactly contained in Δ (that is by density $\varphi \in H^{1/2}_{00}(\Delta;\mathbb{R}^N)$),

$$\int_\Delta [\sigma\nu]_\tau \cdot \varphi \, d\mathcal{H}^{N-1} = \langle \sigma\nu, \varphi\rangle - \langle (\sigma\nu)_\nu, \varphi\rangle, \tag{1.3}$$

where

$$\langle (\sigma\nu)_\nu, \varphi\rangle := \langle \sigma\nu, (\varphi\cdot\nu)\nu\rangle.$$

Since the normal component $(\varphi\cdot\nu)\nu$ of φ with respect to Δ belongs to $H^{1/2}(\partial A;\mathbb{R}^N)$ in view of the regularity of ν on Δ, the definition of $(\sigma\nu)_\nu$ is meaningful.

2. Quasi-static evolution for mixed boundary conditions

In this section, we demonstrate that the setting introduced in [3] can be extended to the case where boundary conditions are of a mixed type, that is to the case where, on a part Γ_d of the boundary of the domain Ω under investigation, the following type of boundary conditions is imposed:

$$u\cdot\nu = w \quad \text{(prescribed normal displacement)},$$

$$[\sigma\nu]_\tau = 0 \qquad \text{(zero tangential forces)}.$$

As mentioned in the introduction, such a setting does not fall squarely within the confines of the evolution discussed in [3] and revisited in [5]. Thus, a revisiting of the results obtained there is required. In what follows, we accomplish this task while relying as much as feasible upon available results.

2.1. **Mathematical framework.** The reference configuration. In all that follows $\Omega\subset\mathbb{R}^N$ is an open, bounded, connected set with (at least) Lipschitz boundary and exterior normal ν. Further, the Dirichlet part Γ_d of $\partial\Omega$ is a non empty open set in the relative topology of $\partial\Omega$ with boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ in $\partial\Omega$ and we set $\Gamma_n := \partial\Omega\setminus\bar{\Gamma}_d$. Reproducing the setting of [5, Section 6], we introduce the following

Definition 2.1 (Admissible boundaries). *We will say that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible iff, for any $\sigma\in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ with*

$$\mathrm{div}\sigma = f \ \text{ in } \Omega, \quad \sigma\nu = h \ \text{ on } \Gamma_n, \quad \sigma_D\in L^\infty(\Omega;\mathrm{M}^N_D)$$

where $f\in L^N(\Omega;\mathbb{R}^N)$ and $h\in L^\infty(\Gamma_n;\mathbb{R}^N)$, and every $p\in\mathcal{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D)$ such that there exists an associated pair $(u,e)\in BD(\Omega)\times L^{N/N-1}(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ and $g\in H^1(\Omega;\mathbb{R}^N)$ with

$$Eu = e+p \quad \text{in } \Omega, \qquad p = (g-u)\odot\nu\,\mathcal{H}^{N-1}\lfloor\Gamma_d \quad \text{on } \Gamma_d,$$

the distribution, defined for all $\varphi\in C^\infty_c(\mathbb{R}^N)$ by

$$\langle\sigma_D,p\rangle(\varphi) := -\int_\Omega \varphi\sigma\cdot(e-Eg)\,dx - \int_\Omega \varphi f\cdot(u-g)\,dx - \int_\Omega \sigma\cdot[(u-g)\odot\nabla\varphi]\,dx + \int_{\Gamma_n}\varphi h\cdot(u-g)\,d\mathcal{H}^{N-1} \tag{2.1}$$

extends to a bounded Radon measure on $\mathbb{R}^N$ with $|\langle\sigma_D,p\rangle|\le\|\sigma_D\|_\infty|p|$.

Definition 2.1 covers many "practical" settings like those of a hypercube with one of its faces standing for the Dirichlet part Γ_d; see [5, Section 6] for that and other such settings.

Remark 2.2. Expression (2.1) defines a meaningful distribution on $\mathbb{R}^N$. Indeed, according to [5, Proposition 6.1] if $\sigma\in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ is such that $\mathrm{div}\sigma\in L^N(\Omega;\mathbb{R}^N)$ and $\sigma_D\in L^\infty(\Omega;\mathrm{M}^N_D)$, then $\sigma\in L^r(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ for every $1\le r<\infty$ with

$$\|\sigma\|_r \le C_r\left(\|\sigma\|_2 + \|\mathrm{div}\sigma\|_N + \|\sigma_D\|_\infty\right)$$

for some $C_r>0$. On the other hand, $u\in L^{N/N-1}(\Omega;\mathbb{R}^N)$ in view of the embedding of $BD(\Omega)$ into $L^{N/N-1}(\Omega;\mathbb{R}^N)$. Further, u has a trace on $\partial\Omega$ which belongs to $L^1(\partial\Omega;\mathbb{R}^N)$. Finally note that, if σ is the restriction to Ω of a C^1-function and if $\mathcal{H}^{N-1}(\partial\lfloor_{\partial\Omega}\Gamma_d)=0$, then, an integration by parts in BD (see [12, Chapter 2, Theorem 2.1]) would demonstrate that the right-hand side of (2.1) coincides with the integral of φ with respect to the (well defined) measure $\sigma_D p$. ¶

We will use the following result concerning the duality pairing $\langle\sigma_D,p\rangle$: for a proof we refer to [5, Lemma 3.8 and Theorem 6.2].

Proposition 2.3 (Duality). *Let the duality $\langle \sigma_D, p\rangle \in \mathcal{M}_b(\Omega \cup \Gamma_d)$ be defined according to* (2.1). *Then the following facts hold true.*

(a) *Absolutely continuous part: we have*

$$\langle \sigma_D, p\rangle^a = \sigma_D \cdot p^a \, dx,$$

where $p^a \in L^1(\Omega; \mathrm{M}^N_D)$ is the density of the absolutely continuous part of p.

(b) *Behaviour at the boundary: if Γ_d is of class C^2, then*

$$\langle \sigma_D, p\rangle \lfloor \Gamma_d = [\sigma\nu]_\tau \cdot (g - u)\mathcal{H}^{N-1} \lfloor \Gamma_d,$$

where $[\sigma\nu]_\tau \in L^\infty(\Gamma_d; \mathbb{R}^N)$ is uniquely defined according to (1.3).

(c) *Total mass: we have*

$$\langle \sigma_D, p\rangle(\Omega \cup \Gamma_d) = -\int_\Omega \sigma \cdot (e - Eg)\, dx - \int_\Omega f \cdot (u - g)\, dx + \int_{\Gamma_n} h \cdot (u - g)\, d\mathcal{H}^{N-1}.$$

Kinematic admissibility. In this contribution we only prescribe the normal component of the displacements on Γ_d. In other words, given $w \in H^1(\Omega)$, we impose that the displacement u should satisfy

$$u \cdot \nu = w \text{ on } \Gamma_d,$$

where ν denotes the outer normal to $\partial\Omega$ and $u \cdot \nu$ is to be understood in the sense of traces. Correspondingly, we define the admissible infinitesimal rigid body motions as those with normal component vanishing on Γ_d, i.e., the elements of

$$\mathcal{R}_d(\Omega) := \{u \in \mathcal{R}(\Omega) \,:\, u \cdot \nu = 0 \text{ on } \Gamma_d\},$$

where $\mathcal{R}(\Omega)$ is defined in (1.2). Depending on the geometry of Γ_d, it may be so that $\mathcal{R}_d(\Omega)$ reduces to the null displacement.

In turn, this prompts the definition of the quotient space

$$\begin{cases} BD_d(\Omega; w) := \{u \in BD(\Omega) \,:\, u \cdot \nu = w \text{ on } \Gamma_d\}/\mathcal{R}_d(\Omega) \\ BD_d(\Omega) := BD_d(\Omega; 0). \end{cases}$$

From now onward, we denote by $[u]$ an element of the set

$$BD_d(\Omega; w) := \{u + r : u \in BD(\Omega) \text{ with } u \cdot \nu = w \text{ on } \Gamma_d,\ r \in \mathcal{R}_d(\Omega)\},$$

by Eu the common value of $E(u+r)$, $r \in \mathcal{R}_d(\Omega)$, and recall that, since $\mathcal{R}_d(\Omega)$ is finite dimensional,

$$\|[u]\|_{BD_d(\Omega;w)} = \min\left\{\|u + r\|_{BD(\Omega)} : r \in \mathcal{R}_d(\Omega)\right\}.$$

Then, we adopt the following

Definition 2.4 (Admissible configurations). *$\mathcal{A}(w)$, the family of admissible configurations relative to w, is the set of triplets $([u], e, p)$ with*

$$[u] \in BD_d(\Omega; w), \qquad e \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}), \qquad p \in \mathcal{M}_b(\Omega; \mathrm{M}^N_D), \qquad Eu = e + p \quad \textit{in } \Omega.$$

The function u denotes the displacement field on Ω, while e and p are the associated elastic and plastic strains.

We will make repeated use of the following

Proposition 2.5 (Korn). *There exists a constant $c(\Omega) > 0$ such that, for every $[u] \in BD_d(\Omega)$,*

$$\|[u]\|_{BD_d(\Omega)} \le c(\Omega)\|Eu\|_1.$$

Proof. The inequality follows if we show that $\|Eu\|_1$ is a norm on the quotient space equivalent to that induced by the BD-norm. This is a consequence of the open mapping theorem provided that we show that the space $BD_d(\Omega)$ is complete when equipped with that norm (see e.g.[12, Chapter II, Proposition 1.2] for a similar argument).

Since $\mathcal{R}(\Omega)$ is a finite dimensional vector space, we decompose $\mathcal{R}(\Omega)$ as

$$\mathcal{R}(\Omega) = \mathcal{R}_d(\Omega) \oplus \mathcal{R}_d^\perp(\Omega), \tag{2.2}$$

and this for any inner product on $\mathcal{R}(\Omega)$. Consider $([u_n])_{n\in\mathbb{N}} \subset BD_d(\Omega)$ such that $(Eu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_b(\Omega;\mathrm{M}^N_{\mathrm{sym}})$. Then, in view of (1.1), there exist $v \in BD(\Omega)$ and $r_n \in \mathcal{R}(\Omega)$ such that

$$\|u_n + r_n - v\|_{BD(\Omega)} \to 0.$$

By the continuity of the trace operator, this entails that

$$r_n \cdot \nu \to v \cdot \nu \qquad \text{strongly in } L^1(\Gamma_d).$$

Set

$$r_n = \tilde{r}_n + \hat{r}_n$$

with $\tilde{r}_n \in \mathcal{R}_d(\Omega)$ and $\hat{r}_n \in \mathcal{R}_d^{\perp}(\Omega)$ according to decomposition (2.2). Then

$$\hat{r}_n \cdot \nu \to v \cdot \nu \qquad \text{strongly in } L^1(\Gamma_d).$$

Since $\mathcal{R}_d^{\perp}(\Omega)$ is a finite dimensional space on which

$$r \mapsto \int_{\Gamma_d} |r \cdot \nu| \, d\mathcal{H}^{N-1}$$

is a norm, we may assume that, up to a subsequence,

$$\hat{r}_n \to \hat{r} \qquad \text{in } \mathcal{R}_d^{\perp}(\Omega)$$

and also strongly in $BD(\Omega)$. Then, $u := v - \hat{r}$ satisfies $u \cdot \nu = 0$ on Γ_d and

$$u_n + \tilde{r}_n \to u \qquad \text{strongly in } BD(\Omega),$$

so that $[u_n] \to [u]$, strongly in $BD_d(\Omega)$. The result follows. □

The elasticity tensor. The Hooke's law is given by an element $\mathbb{C} \in L^\infty(\Omega;\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}}))$ such that

$$c_1|M|^2 \le \mathbb{C}M \cdot M \le c_2|M|^2 \text{ for every } M \in \mathrm{M}^3_{\mathrm{sym}}, \tag{2.3}$$

with $c_1, c_2 > 0$.

For every $e \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ we set

$$\mathcal{Q}(e) := \frac{1}{2}\int_\Omega \mathbb{C}e \cdot e \, dx.$$

The set of admissible stresses. In elasto-plasticity, the deviatoric part of the stress σ is assumed to be restricted by the yield condition. We thus assume that there exists a convex compact set $K \subset \mathrm{M}^N_D$ such that $\sigma_D(x) \in K$ for a.e. $x \in \Omega$. We further assume that K cannot be too small or too large, *i.e.*, there exist $c_3, c_4 > 0$ such that

$$B(0,c_3) \subset K \subset B(0,c_4). \tag{2.4}$$

Our formulation of the problem uses the Legendre transform of $\mathbb{I}_K$, which is often referred to as the dissipation potential.

The dissipation functional. For a.e. $x \in \Omega$, we define the dissipation to be

$$H(\xi) := \sup\{\tau \cdot \xi : \tau \in K\}. \tag{2.5}$$

Definition (2.5) produces a convex, one-homogeneous function which further satisfies

$$c_3|\xi| \le H(\xi) \le c_4|\xi|.$$

For every $p \in \mathcal{M}_b(\Omega;\mathrm{M}^N_D)$, we define the dissipation functional as

$$\mathcal{H}(p) := \int_\Omega H\left(x, \frac{p}{|p|}\right) d|p|,$$

where $p/|p|$ denotes the Radon-Nikodym derivative of p with respect to its total variation $|p|$. We have

$$c_3\|p\|_1 \le \mathcal{H}(p) \le c_4\|p\|_1. \tag{2.6}$$

The total dissipation. If $t \mapsto p(t)$ is a map from $[0,T]$ to $\mathcal{M}_b(\Omega;\mathrm{M}^N_D)$, we define, for every $[a,b] \subseteq [0,T]$,

$$\mathcal{D}(a,b;p) := \sup \left\{ \sum_{j=1}^{k} \mathcal{H}\left(p(t_j) - p(t_{j-1})\right) \,:\, a = t_0 < t_1 < \cdots < t_k = b \right\}$$

to be the *total dissipation* over the time interval $[a,b]$. Thanks to (2.6), the total dissipation satisfies for every $t \in [0,T]$

$$c_3 \mathcal{V}(0,t;p) \le \mathcal{D}(0,t;p) \le c_4 \mathcal{V}(0,t;p). \tag{2.7}$$

The loads. As mentioned at the beginning of this section, we prescribe a normal boundary displacement w on Γ_d. To that aim, we define, $w(t)$, $t \in [0,T]$, as the normal trace on Γ_d of some

$$g \in AC(0,T;H^1(\mathbb{R}^N;\mathbb{R}^N)), \tag{2.8}$$

i.e., for every $t \in [0,T]$,

$$g(t)\cdot\nu = w(t) \quad \text{on } \Gamma_d. \tag{2.9}$$

Then, in particular,

$$w \in AC(0,T;L^2(\Gamma_d)).$$

On the part Γ_n of the boundary, we prescribe a surface traction which should not produce any work against admissible rigid body motions, that is

$$h \in AC(0,T;L^\infty(\Gamma_n;\mathbb{R}^N)) \text{ with, for every } r \in \mathcal{R}_d(\Omega),\ \int_{\Gamma_n} h(t)\cdot r \; d\mathcal{H}^{N-1} = 0,\ t \in [0,T]. \tag{2.10}$$

Define $\mathcal{L}(t) \in BD'(\Omega)$ as

$$\langle \mathcal{L}(t), v\rangle := -\int_{\Gamma_n} h(t)\cdot v \; d\mathcal{H}^{N-1},\ v \in BD(\Omega).$$

Then, the work of the surface traction is given, for any element $([u],e,p)$ of $\mathcal{A}(w(t))$ by

$$\mathcal{L}(t;[u]) := \langle \mathcal{L}(t), u\rangle, \tag{2.11}$$

where, thanks to (2.10), u is any representative of $[u]$. Note that, under our assumptions on h,

$$\dot{\mathcal{L}}(t) := w^* - \lim_{s\to t} \frac{\mathcal{L}(s) - \mathcal{L}(t)}{s-t}$$

exists in $BD(\Omega)'$ for a.e. $t \in [0,T]$ (see [3, Remark 4.1]) and that it is represented by

$$\langle \dot{\mathcal{L}}(t), u\rangle = -\int_{\Gamma_n} \dot{h}(t)\cdot u \; d\mathcal{H}^{N-1}, \text{ with, for every } r \in \mathcal{R}_d(\Omega),\ \int_{\Gamma_n} \dot{h}(t)\cdot r \; d\mathcal{H}^{N-1} = 0.$$

In particular, we can define

$$\dot{\mathcal{L}}(t;[u]) := \langle \dot{\mathcal{L}}(t), u\rangle,\ u \in BD_d(\Omega;w(t)), \tag{2.12}$$

and $t \mapsto \dot{\mathcal{L}}(t;[u(t)])$ is an $L^1(0,T)$-function whenever $[u-g] \in L^\infty(0,T;BD_d(\Omega))$.

As is well known in plasticity, the traction should also be such that it satisfies a uniform safe load condition (see [3, Eqns. (2.17)-(2.18)]). We will assume henceforth that

$$\mathcal{H}^{N-1}(\partial\lfloor_{\partial\Omega}\Gamma_d) = 0 \tag{2.13}$$

and that there exists a stress field $\pi \in AC(0,T;L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}))$ with

$$\pi_D \in AC(0,T;C^0(\overline{\Omega};\mathrm{M}^N_D)) \tag{2.14}$$

$$\pi_D(t,x) + B(0,\alpha) \subset K \tag{2.15}$$

for some $\alpha > 0$, and such that, for every $t \in [0,T]$,

$$\operatorname{div}\pi(t) = 0 \text{ in } \Omega, \qquad \pi(t)\nu = h(t) \text{ on } \Gamma_n, \qquad [\pi(t)\nu]_\tau = 0 \text{ on } \Gamma_d \tag{2.16}$$

The work defined in (2.11) can in turn be rewritten as

$$\mathcal{L}(t;[u]) = -\int_\Omega \pi(t)\cdot(e - Eg(t))\,dx - \langle \pi_D(t),\tilde p\rangle(\Omega\cup\Gamma_d) - \int_{\Gamma_n} h(t)\cdot g(t)\,dt,$$

where $\tilde p \in \mathcal{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D)$ is defined as

$$\tilde p\lfloor\Omega = p \qquad\text{and}\qquad \tilde p\lfloor\Gamma_d = (g(t)-u)\odot\nu\,\mathcal{H}^{N-1}\lfloor\Gamma_d.$$

Note that $\tilde p$ has values in M^N_D since $([u],e,p)\in\mathcal{A}(w(t))$. The previous expression is obtained from (2.1) with the choice $\varphi\equiv 1$ on $\overline{\Omega}$ and the term $\langle\pi_D,\tilde p\rangle$ uses the duality. However, the regularity assumption (2.14) entails that the duality $\langle\pi_D,\tilde p\rangle$ reduces to the well defined measure $\pi_D(t)\cdot\tilde p$ (see [5, Section 6]), while, thanks to the last equality in (2.16), we may write

$$(2.17)\quad (\pi_D(t)\cdot\tilde p)(\Omega\cup\Gamma_d) = (\pi_D(t)\cdot p)(\Omega) + \int_{\Gamma_d}\pi_D(t)\cdot[(g(t)-u)\odot\nu]\,d\mathcal{H}^{N-1}$$
$$= (\pi_D(t)\cdot p)(\Omega) + \int_{\Gamma_d}[\pi_D(t)\nu]_\tau(g(t)-u)\,d\mathcal{H}^{N-1} = (\pi_D(t)\cdot p)(\Omega).$$

Using an integration by parts formula in $H^1(\Omega;\mathbb{R}^N)$, (2.9), (2.13), (2.14), (2.16), and (2.17) yield

$$(2.18)\qquad \mathcal{L}(t;[u]) = -\int_\Omega \pi(t)\cdot e\,dx - (\pi_D(t)\cdot p)(\Omega) + \int_{\Gamma_d}(\pi(t)\nu\cdot\nu)w(t)dx.$$

Remark 2.6. Following [3], we could simply require that

$$\begin{cases}\pi_D\in AC(0,T;L^\infty(\Omega;\mathrm{M}^N_D))\\ [\pi(t)\nu]_\tau = 0 \text{ on } \Gamma_d \text{ of class } C^2,\end{cases}$$

and interpret the calculations above using the stress-strain duality. However, the absence of spatial regularity of π is the only reason for appealing to the duality in that argument, because the product of π_D with p is not meaningful under the assumed regularity of π_D. ¶

In all that follows, we will have to deal with maps of the form

$$(2.19)\quad \mathscr{F}[t;w;p] : ([v],\eta,q)\in\mathcal{A}(w(t))\mapsto \mathcal{Q}(\eta)+\mathcal{H}(q-p)+\mathcal{L}(t;[v]),\quad\text{for some } p\in\mathcal{M}_b(\Omega;\mathrm{M}^N_D).$$

The following remark addresses the lower semi-continuity and coercivity properties of $\mathscr{F}[t;w;p]$.

Remark 2.7 (Lower semi-continuity and coercivity)**.** First, in view of the expression (2.18) for $\mathcal{L}(t;\cdot)$ we can write

$$(2.20)\quad \mathscr{F}[t;w;p]([v],\eta,q) = \mathcal{Q}(e) - \int_\Omega \pi(t)\cdot e\,dx$$
$$+\mathcal{H}(q-p) - (\pi_D(t)\cdot(q-p))(\Omega) + \int_{\Gamma_d}(\pi(t)\nu\cdot\nu)w(t)\,d\mathcal{H}^{N-1}.$$

The term involving e is clearly weakly lower semi-continuous on $L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$. In view of the safe load condition (2.15) on $\pi_D(t)$, H defined in (2.5) satisfies

$$(2.21)\qquad \forall\xi\in\mathrm{M}^N_D\;:\;H(\xi)-\pi_D(t)\cdot\xi\geq\alpha|\xi|.$$

Consequently, the term

$$q\mapsto\mathcal{H}(q-p)-(\pi_D(t)\cdot(q-p))(\Omega) = \int_\Omega\left[H\left(\frac{q-p}{|q-p|}\right)-\pi_D(t)\cdot\frac{q-p}{|q-p|}\right]d|q-p|$$

is weakly-$\star$ lower semicontinuous on $\mathcal{M}_b(\Omega;\mathrm{M}^N_D)$ in view of Reshetnyak's lower semi-continuity theorem (see e.g. [2, Theorem 2.38]). The remaining term in (2.20) is a t-dependent constant.

We conclude that $\mathscr{F}[t;w;p]$ is weak-$\star$ (sequentially) lower semi-continuous for the (natural) weak-$\star$ topology on $\mathcal{A}(w(t))$.

As far as coercivity is concerned, (2.3), (2.8), (2.14) and (2.21) imply that, for every $([v],\eta,q)\in\mathcal{A}(w(t))$

$$\mathscr{F}[t;w;p]([v],\eta,q)\geq\alpha\|q\|_1+\frac{c_3}{2}\|\eta\|_2+C(p),$$

where $C(p)$ is a constant depending on of $\|p\|_1$. ¶

2.2. **Quasi-static evolutions.** In what follows, the energetic formulation of the quasi-static evolution is detailed.

Definition 2.8 (Quasi-static evolution). *The mapping*

$$t \mapsto ([u(t)], e(t), p(t)) \in \mathcal{A}(w(t))$$

is a quasi-static evolution iff the following conditions hold for every $t \in [0, T]$:

(a) *Global stability: for every $([v], \eta, q) \in \mathcal{A}(w(t))$*

$$\mathcal{Q}(e(t)) + \mathcal{L}(t; [u(t)]) \leq \mathcal{Q}(\eta) + \mathcal{L}(t; [v]) + \mathcal{H}(q - p(t)). \tag{2.22}$$

(b) *Energy equality: $p \in BV\left(0, T; \mathcal{M}_b(\Omega; \mathrm{M}^N_D)\right)$ and*

$$\mathcal{Q}(e(t)) + \mathcal{L}(t; [u(t)]) + \mathcal{D}(0, t; p) = \mathcal{Q}(e(0)) + \mathcal{L}(0; [u(0)]) + \int_0^t \left\{ \int_\Omega \sigma(\tau) \cdot E\dot{g}(\tau)\, dx + \mathcal{L}(\tau; [\dot{g}(\tau)]) + \dot{\mathcal{L}}(\tau, [u(\tau)]) \right\} d\tau \tag{2.23}$$

where $\sigma(t) := \mathbb{C}e(t)$.

Remark 2.9. Note that the global stability condition for $([u], e, p) \in \mathcal{A}(w)$ (we drop the t-dependence throughout this remark), implies in particular the uniqueness of $e \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ and of $[u] \in BD_d(\Omega; w)$ once p is fixed. Indeed, the global stability is equivalent, thanks to the one-homogeneous character of $\mathcal{H}$, to the following set of inequalities

$$-\mathcal{H}(q) \leq \int_\Omega \mathbb{C}e \cdot \eta\, dx + \mathcal{L}(t; [v]) \leq \mathcal{H}(-q) \quad \text{for every } ([v], \eta, q) \in \mathcal{A}(0). \tag{2.24}$$

One implication is proved in [3, Theorem 3.4] while the other is immediate by convexity of the quadratic form $\mathcal{Q}(e)$. Then, if $([u'], e', p') \in \mathcal{A}(w)$ is an other globally stable configuration, we obtain

$$-\mathcal{H}(p' - p) \leq \int_\Omega \mathbb{C}e \cdot (e' - e)\, dx + \mathcal{L}(t; [u' - u]) \leq \mathcal{H}(p - p')$$

and

$$-\mathcal{H}(p - p') \leq \int_\Omega \mathbb{C}e' \cdot (e - e')\, dx + \mathcal{L}(t; [u - u']) \leq \mathcal{H}(p' - p)$$

so that

$$\int_\Omega \mathbb{C}(e' - e) \cdot (e' - e)\, dx \leq \mathcal{H}(p - p') + \mathcal{H}(p' - p),$$

and, appealing to (2.3), (2.6), we conclude that

$$\|e' - e\|_2 \leq C|p' - p|^{1/2}(\Omega),$$

for some $C > 0$. Concerning the displacement, thanks to Proposition 2.5,

$$\|[u' - u]\|_{BD_d(\Omega)} \leq C\|E(u' - u)\|_1 \leq C\left[|\Omega|^{1/2}\|e' - e\|_2 + |p' - p|(\Omega)\right]$$

from which the result follows. ¶

Remark 2.10 (Independence of the extension g). Whenever (2.13), (2.14) hold true, the quasi-static evolution of Definition 2.8 remains unchanged if g defined in (2.8) is replaced by any

$$g' \in AC(0, T; H^1(\mathbb{R}^N; \mathbb{R}^N))$$

with, for every $t \in [0, T]$,

$$g'(t) \cdot \nu = w(t) \quad \text{on } \Gamma_d. \tag{2.25}$$

In other words the evolution only depends on $w(t)$, not on $g(t)$, provided that (2.25) is satisfied. Indeed, in the context of Remark 2.9, take $([v], \eta, q) = ([\dot{g} - \dot{g}'], E\dot{g} - E\dot{g}', 0)$ as test function in (2.24). We get

$$\int_\Omega \mathbb{C}e \cdot (E\dot{g} - E\dot{g}')dx + \mathcal{L}(t; [\dot{g} - \dot{g}']) = 0.$$

Reintroducing the t-dependence, we thus conclude that we are at liberty to replace $\dot g(t)$ by $\dot g'(t)$ in the energy equality. Since, in view of (2.18), this expression is the only one that involves $\dot g(t)$, rather than $\dot w(t)$, in the definition of a quasi-static evolution, the conclusion is reached.

Note that a similar remark would also apply to the general setting first introduced in [3]. To the best of our knowledge, this has not previously been explicitly observed. ¶

The following theorem is a generalization of [3, Theorem 4.5] to our setting.

Theorem 2.11 (Existence of quasi-static evolutions). *Assume that* (2.3)*,* (2.4)*,* (2.8)–(2.10) *and* (2.13)–(2.16) *are satisfied. Let* $([u_0], e_0, p_0) \in \mathcal{A}(w(0))$ *satisfy the global stability condition* (2.22)*. Then there exists a quasi-static evolution* $t \mapsto ([u(t)], e(t), p(t)) \in \mathcal{A}(w(t)), t \in [0,T]$*, such that* $([u(0)], e(0), p(0)) = ([u_0], e_0, p_0)$.

Proof. The proof proceeds as usual by discretization in time, solving incremental minimum problems depending on $\mathcal{Q}, \mathcal{H}$ and letting the time step discretization going to zero.

Step 1: Existence of the incremental configurations. Let $t_i^k := (iT)/k$ for $k \geq 1$. Let us show that for every $i = 0, ..., k$, we can find a triplet $([u_i^k], e_i^k, p_i^k) \in \mathcal{A}(w(t_i^k))$ such that, for $i > 0$,

$$([u_i^k], e_i^k, p_i^k) \in \operatorname{Argmin} \Big\{ \mathscr{F}[t_i^k; w(t_i^k); p_{i-1}^k](v, \eta, q) = \mathcal{Q}(\eta) + \mathcal{H}(q - p_{i-1}^k) + \mathcal{L}(t; [v]) : ([v], \eta, q) \in \mathcal{A}(w(t_i^k)) \Big\} \tag{2.26}$$

and $([u_0^k], e_0^k, p_0^k) = ([u_0], e_0, p_0)$.

We can proceed as follows. Let $([u_n], e_n, p_n) \in \mathcal{A}(w(t_i^k))$ be a minimizing sequence. Comparing with $([g(t_i^k)], Eg(t_i^k), 0)$ and appealing to the coercivity in Remark 2.7 yields, for some constant C,

$$\|e_n\|_2 + \|p_n\|_1 \leq C,$$

so that we can assume that, up to a subsequence,

$$e_n \rightharpoonup e \qquad \text{weakly in } L^2(\Omega; \mathbb{M}^N_{\text{sym}}),$$

and

$$p_n \rightharpoonup p \qquad \text{weakly-}\star \text{ in } \mathcal{M}_b(\Omega; \mathbb{M}^N_D).$$

The lower semi-continuity of $\mathscr{F}[t_i^k; w(t_i^k); p_{i-1}^k]$ is ensured through Remark 2.7. In order to conclude, we simply need to find $[u] \in BD_d(\Omega; w(t_i^k))$ such that $([u], e, p) \in \mathcal{A}(w(t_i^k))$. Then, $([u], e, p)$ will be a minimum.

Thanks to Proposition 2.5

$$\|[u_n - g(t_i^k)]\|_{BD_d(\Omega)} \leq c,$$

for some constant c. Consequently, there exists $r_n \in \mathcal{R}_d(\Omega)$ such that $u_n + r_n$ is bounded in $BD(\Omega)$. Let $\Omega' \subseteq \mathbb{R}^N$ be open, bounded and such that $\Omega \cup \Gamma_d = \overline{\Omega} \cap \Omega'$. Let us consider

$$v_n := \begin{cases} u_n + r_n & \text{on } \Omega \\ g(t_i^k) & \text{on } \Omega' \setminus \overline{\Omega}. \end{cases}$$

$(v_n)_{n \in \mathbb{N}}$ is a bounded sequence in $BD(\Omega')$: indeed, the jumps created across Γ_d are of the form

$$(u_n + r_n - g(t_i^k)) \odot \nu \mathcal{H}^{N-1} \lfloor \Gamma_d,$$

so that their mass is dominated by $\|u_n + r_n - g(t_i^k)\|_{L^1(\Gamma_d; \mathbb{R}^N)}$ which is bounded. Up to a subsequence we get

$$v_n \overset{*}{\rightharpoonup} v \qquad \text{weakly-}\star \text{ in } BD(\Omega').$$

Let u be the restriction of v to Ω. Since

$$Ev_n = \tilde e_n + \tilde p_n$$

where

$$\tilde e_n = \begin{cases} e_n & \text{in } \Omega \\ Eg(t_i^k) & \text{in } \Omega' \setminus \overline{\Omega} \end{cases}$$

and

$$\tilde{p}_n := \begin{cases} p_n & \text{in } \Omega \\ (u_n + r_n - g(t_i^k)) \odot \nu \mathcal{H}^{N-1} \lfloor \Gamma_d & \text{on } \Gamma_d \\ 0 & \text{on } \Omega' \setminus \overline{\Omega}, \end{cases}$$

and since $\tilde{p}_n$ is M^N_D-valued, we deduce that the singular part of Ev is also M^N_D-valued. But

$$Ev\lfloor\Gamma_d = (g(t_i^k) - u) \odot \nu \mathcal{H}^{N-1}\lfloor\Gamma_d,$$

so that $u \cdot \nu = g(t_i^k) \cdot \nu = w(t_i^k)$, which entails that $([u], e, p) \in \mathcal{A}(w(t_i^k))$.

Step 2: Piecewise constant interpolation in time and conclusion. With Step 1 at our disposal, we can adapt easily the arguments of [5, Theorem 2.7] to our setting.

Define $([u^k(t)], e^k(t), p^k(t))$ to be the right-continuous and piecewise in time constant interpolation of the $([u_i^k], e_i^k, p_i^k)$'s. Then, upon testing the minimality of $([u_i^k], e_i^k, p_i^k)$ in (2.26) with $([u_{i-1}^k + g(t_i^k) - g(t_{i-1}^k)], e_{i-1}^k + Eg(t_i^k) - Eg(t_{i-1}^k), p_{i-1}^k) \in \mathcal{A}(w(t_i^k))$ and upon iterating, we can write, using the regularity properties of g,

$$\mathcal{Q}(e^k(t)) + \mathcal{D}(0,t;p^k) + \mathcal{L}(t;[u_k(t)]) \leq \mathcal{Q}(e_0) + \int_0^{t^k_{i_k(t)}} \left[\int_\Omega \mathbb{C}e^k(s) \cdot E\dot{g}(s)\, dx + \mathcal{L}(s;[\dot{g}(s)]) \right] ds + \int_0^{t^k_{i_k(t)}} \dot{\mathcal{L}}(s,[u_k(s)])\, ds + \delta_k, \tag{2.27}$$

where $i_k(t)$ is the largest index i such that $t \in [t_i^k, t_{i+1}^k)$ and $\delta_k \overset{k}{\searrow} 0$. We deduce that there exists $C > 0$ independent of k such that

$$\sup_{t\in[0,T]} \|e^k(t)\|_2 + \mathcal{D}(0,T;p^k) \leq C. \tag{2.28}$$

In view of (2.7), a generalized version of Helly's theorem (see [8, Theorem 3.2]) implies the existence of a subsequence of $(p^k)_{k\in\mathbb{N}}$, still indexed by k, such that, for all $t \in [0,T]$,

$$p^k(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega;\mathrm{M}^N_D)$$

for a suitable $p \in BV(0,T;\mathcal{M}_b(\Omega;\mathrm{M}^N_D))$. By (2.28) and by the arguments of Step 1, there exists a t-dependent subsequence $((u^{k_t}(t), e^{k_t}(t)))_{n_t\in\mathbb{N}}$ and $r_{k_t} \in \mathcal{R}_d(\Omega)$ such that

$$\begin{aligned} u^{k_t}(t) - r_{k_t} &\overset{*}{\rightharpoonup} u(t) \qquad \text{weakly}^* \text{ in } BD(\Omega), \\ e^{k_t}(t) &\rightharpoonup e(t) \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}), \end{aligned}$$

with $([u(t)], e(t), p(t)) \in \mathcal{A}(w(t))$. The global stability of $([u^k(t)], e^k(t), p^k(t))$ easily implies that of $([u(t)], e(t), p(t))$. In view of Remark 2.9, there is no need to pass to a t-dependent subsequence. From now on, the arguments of [5, Theorem 2.7] can be followed word for word: we can pass to the limit in (2.27) obtaining

$$\mathcal{Q}(e(t)) + \mathcal{D}(0,t;p) + \mathcal{L}(t;[u(t)]) \leq \mathcal{Q}(e_0) + \int_0^t \left[\int_\Omega \mathbb{C}e(s) \cdot E\dot{g}(s) dx + \mathcal{L}(s;[\dot{g}(s)]) + \dot{\mathcal{L}}(s;[u(s)]) \right] ds,$$

while the opposite inequality is a consequence of the global stability of $([u(t)], e(t), p(t))$. We conclude that $([u(t)], e(t), p(t)) \in \mathcal{A}(w(t))$ is a quasi-static evolution, and the result follows. □

Theorem 2.12 (Uniqueness of the stress). *Let $\{t \mapsto ([u(t)], e(t), p(t)), t \in [0,T]\}$ be a quasi-static evolution according to Definition 2.8. Then, the stress $\sigma(t) := \mathbb{C}e(t)$ is uniquely determined.*

Proof. Indeed, let

$$t \mapsto ([\tilde{u}(t)], \tilde{e}(t), \tilde{p}(t)) \in \mathcal{A}(w(t)), \qquad \tilde{\sigma}(t) := \mathbb{C}\tilde{e}(t),$$

be an other quasi-static evolution relative to w with $\tilde{e}(0) = e_0$. Consider the evolution

$$t \mapsto ([u'(t)], e'(t), p'(t)) := \frac{1}{2}\Big([u(t) + \tilde{u}(t)], e(t) + \tilde{e}(t), p(t) + \tilde{p}(t) \Big) \in \mathcal{A}(w(t)).$$

In view of (2.24), the configuration $([u'(t)], e'(t), p'(t))$ is globally stable for every $t \in [0,T]$. Consequently, arguing as in last part of the proof of Theorem 2.11,

$$\begin{aligned}(2.29)\quad & \mathcal{Q}(e'(t)) + \mathcal{D}(0,t;p') + \mathcal{L}(t;[u'(t)]) \\ & \geq \mathcal{Q}(e_0) + \int_0^t \left[\int_\Omega \sigma'(\tau)\cdot E\dot g(\tau)\,dx + \mathcal{L}(s;[\dot g(s)]) + \dot{\mathcal{L}}(s;[u'(s)])\right] ds,\end{aligned}$$

where $\sigma'(\tau) := \mathbb{C}e'(\tau)$. On the other hand, in view of the energy equality satisfied by the evolutions, for every $t \in [0,T]$,

$$\begin{aligned}\mathcal{Q}(e'(t)) + \mathcal{D}(0,t;p'(t)) + \mathcal{L}(t;[u'(t)]) &= \mathcal{Q}\left(\frac{e(t)+\tilde e(t)}{2}\right) + \mathcal{D}\left(0,t;\frac{p(t)+\tilde p(t)}{2}\right) + \mathcal{L}\left(t;\left[\frac{u(t)+\tilde u(t)}{2}\right]\right) \\ &\leq \frac12\Big(\mathcal{Q}(e(t)) + \mathcal{Q}(\tilde e(t)) + \mathcal{D}(0,t;p(t)) + \mathcal{D}(0,t;\tilde p(t)) + \mathcal{L}(t;[u(t)]) + \mathcal{L}(t;[\tilde u(t)])\Big) \\ &= \mathcal{Q}(e_0) + \int_0^t \left[\int_\Omega \sigma'(\tau)\cdot E\dot g(\tau)\,dx + \mathcal{L}(s;[\dot g(s)]) + \dot{\mathcal{L}}(s;[u'(s)])\right] ds.\end{aligned}$$

But (2.29) then turns the previous string into equalities, so that, for every $t \in [0,T]$,

$$\mathcal{Q}\left(\frac12 (e(t)+\tilde e(t))\right) = \frac12 (\mathcal{Q}(e(t)) + \mathcal{Q}(\tilde e(t))),$$

hence, $\mathcal{Q}$ being strictly convex, $e(t) = \tilde e(t)$ and $\sigma(t) = \tilde\sigma(t)$. □

The following result holds true.

Proposition 2.13 (Regularity in time). *Let $\{t \mapsto ([u(t)], e(t), p(t)), t \in [0,T]\}$ be a quasi-static evolution according to Definition 2.8. Then,*

$$(2.30)\qquad (e,p) \in AC\left(0,T; L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega;\mathrm{M}^N_D)\right)$$

and the map $t \mapsto [u(t)]$ is absolutely continuous on $[0,T]$ with values in the quotient space $BD(\Omega)/\mathcal{R}_d(\Omega)$. Moreover for a.e. $t \in [0,T]$ the following limits exist

$$\begin{aligned}[\dot u(t)] &:= \lim_{s\to t} \frac{[u(s)-u(t)]}{s-t} && \text{weakly* in } BD(\Omega)/\mathcal{R}_d(\Omega),\\ \dot e(t) &:= \lim_{s\to t} \frac{e(s)-e(t)}{s-t} && \text{strongly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}),\\ \dot p(t) &:= \lim_{s\to t} \frac{p(s)-p(t)}{s-t} && \text{strictly in } \mathcal{M}_b(\Omega;\mathrm{M}^N_D),\end{aligned}$$

with $([\dot u(t)], \dot e(t), \dot p(t)) \in \mathcal{A}(\dot w(t))$. Finally, $\mathcal{D}(0,t;p) \in AC(0,T)$ and, for a.e. $t \in [0,T]$,

$$(2.31)\qquad \dot{\mathcal{D}}(0,t;p) = -\int_\Omega \sigma(t)\cdot(\dot e(t) - E\dot g(t))\,dx + \int_{\Gamma_n} h(t)(\dot u(t) - \dot g(t))\,d\mathcal{H}^{N-1}.$$

Proof. Property (2.30) follows by a straightforward adaptation of [3, Theorem 5.2]. Concerning the displacements, the absolute continuity follows in view of Proposition 2.5. The existence of the limits of the difference quotients for e and p follow from [3, Section 5.1]; that for the displacement follows again by Proposition 2.5; further, $([\dot u(t)], \dot e(t), \dot p(t)) \in \mathcal{A}(\dot w(t))$. Finally (2.31) is obtained through differentiation of the energy equality. □

2.3. **Equilibrium conditions and flow rules.** In this subsection we derive, in a weak form compatible with the mathematical framework explained above, the classical equilibrium condition and yield constraint for the Cauchy stress, together with the flow rule satisfied by the plastic strain.

Concerning the stress we have the following result.

Theorem 2.14 (Equilibrium and stress admissibility)**.** *Let $\{t \mapsto ([u(t)], e(t), p(t)), t \in [0,T]\}$ be a quasi-static evolution according to Definition 2.8. Then, for every $t \in [0,T]$, $\sigma(t) = \mathbb{C}e(t)$ satisfies the balance equations*

$$\begin{cases} \operatorname{div} \sigma(t) = 0 & \text{in } \Omega \\ \sigma(t) \cdot \nu = h(t) & \text{on } \partial\Omega \setminus \overline{\Gamma}_d \end{cases} \tag{2.32}$$

and the admissibility constraint

$$\sigma_D(t,x) \in K \text{ for a.e. } x \in \Omega. \tag{2.33}$$

Finally, assuming that Γ_d is of class C^2, then

$$[\sigma(t)\nu]_\tau = 0 \qquad \text{a.e. on } \Gamma_d, \tag{2.34}$$

where $[\sigma(t)\nu]_\tau \in L^\infty(\Gamma_d; \mathbb{R}^N)$ is defined according to (1.3)*.*

Proof. As mentioned in Remark 2.9, the global stability of $(u(t), e(t), p(t))$ is equivalent to the relation

$$-\mathcal{H}(q) \le \int_\Omega \mathbb{C}e(t) \cdot \eta \, dx + \mathcal{L}(t; [v]) \le \mathcal{H}(-q) \quad \text{for every } ([v], \eta, q) \in \mathcal{A}(0). \tag{2.35}$$

Choose $([v], \eta, q)$ to be $([\varphi], E\varphi, 0)$ with $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$, then with $\varphi \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ s.t. $\varphi \equiv 0$ on $\overline{\Gamma}_d$. We obtain

$$\text{div}\sigma(t) = 0 \text{ in } \Omega, \qquad \sigma(t)\nu = h \text{ on } \partial\Omega \setminus \overline{\Gamma}_d.$$

Finally, choose $([v], \eta, q)$ to be $([0], \chi_B \xi, -\chi_B \xi)$ with $\xi \in \mathbb{M}_D^N$ and χ_B the characteristic function of an arbitrary Borel subset B of Ω. Letting ξ vary first in a countable and dense set in $\mathbb{M}_D^N$, and then using the continuity of $\xi \mapsto H(\xi)$ for a.e. $x \in \Omega$, we obtain that

$$-H(-\xi) \le \sigma_D(t,x) \cdot \xi \le H(\xi), \text{ a.e. in } \Omega,$$

so that, in view of (2.5),

$$\sigma_D(t,x) \in K, \text{ a.e. in } \Omega.$$

Let us come to (2.34). Because of the C^2-regularity of Γ_d, we know that $[\sigma\nu]_\tau$ is well defined as

$$\int_{\Gamma_d} [\sigma\nu]_\tau \cdot \varphi \, d\mathcal{H}^{N-1} = \langle \sigma\nu, \varphi \rangle - \langle (\sigma\nu)_\nu, \varphi \rangle, \tag{2.36}$$

for every $\varphi \in H^{1/2}(\partial\Omega; \mathbb{R}^N)$ with support compactly contained in Γ_d (that is by density $\varphi \in H_{00}^{1/2}(\Gamma_d; \mathbb{R}^N)$), where

$$\langle (\sigma\nu)_\nu, \varphi \rangle := \langle \sigma\nu, (\varphi \cdot \nu)\nu \rangle. \tag{2.37}$$

Furthermore, as observed in Section 1.2, $[\sigma\nu]_\tau$ is also an element of $L^\infty(\Gamma_d; \mathbb{R}^N)$.

Now choose $([v], e, q)$ of the form $([\varphi], E\varphi, 0)$ where $\varphi \in C^2(\overline{\Omega}; \mathbb{R}^N)$ is such that $\varphi_{|\partial\Omega}$ has support compactly contained in Γ_d, with $\varphi \cdot \nu = 0$ on Γ_d. Using (2.35) again, we get, in view of (2.36) and (2.37),

$$0 = \int_\Omega \sigma(t) \cdot E\varphi \, dx = \langle \sigma\nu, \varphi \rangle = \int_{\Gamma_d} [\sigma\nu]_\tau \cdot \varphi \, d\mathcal{H}^{N-1},$$

from which (2.34) follows. □

Remark 2.15. The equilibrium conditions (2.32), (2.33) and (2.34) for the Cauchy stress $\sigma(t)$ are indeed equivalent to the global stability of the configuration $([u(t)], e(t), p(t)) \in \mathcal{A}(w(t))$ provided that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible and that Γ_d is of class C^2. Thanks to (2.32), (2.33), [5, Proposition 3.9] applies and yields in particular

$$\langle \sigma_D(t), q \rangle \le H\left(\frac{q}{|q|}\right)|q| \qquad \text{as measures on } \Omega,$$

for every $([v], \eta, q) \in \mathcal{A}(0)$. Here $q/|q|$ stands for the Radon-Nykodim derivative of q with respect to $|q|$. In particular we deduce, upon taking the masses,

$$-\mathcal{H}(-q) \le \langle \sigma_D(t), q \rangle(\Omega) \le \mathcal{H}(q).$$

In view of the abmissibility of $\partial\lfloor_{\partial\Omega}\Gamma_d$, and thanks to (2.34), we get invoking Proposition 2.3 (points (b) and (c))

$$\langle\sigma_D(t),q\rangle(\Omega\cup\Gamma_d)=\langle\sigma_D(t),q\rangle(\Omega)=-\int_\Omega \sigma\cdot\eta\,dx-\mathcal{L}(t,[v])$$

so that

$$-\mathcal{H}(q)\le\int_\Omega \sigma\cdot\eta\,dx+\mathcal{L}(t,[v])\le\mathcal{H}(-q),$$

which is equivalent to global stability by convexity. ¶

A first step towards a flow rule is given by the following result.

Proposition 2.16. *Let $\{t\mapsto([u(t)],e(t),p(t)),t\in[0,T]\}$ be a quasi-static evolution according to Definition 2.8. Assume further that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible and that Γ_d is of class C^2. Then, for a.e. $t\in[0,T]$,*

$$H\left(\frac{\dot p(t)}{|\dot p(t)|}\right)|\dot p(t)|=\langle\sigma_D(t),\dot p(t)\rangle\qquad \textit{as measures on } \Omega.$$

Proof. Let $t\in[0,T]$ be such that equality (2.31) holds true. Thanks to (2.32), (2.33), [5, Proposition 3.9] applies and yields in particular

$$H\left(\frac{\dot p(t)}{|\dot p(t)|}\right)|\dot p(t)|\ge\langle\sigma_D(t),\dot p(t)\rangle\qquad \text{as measures on } \Omega. \tag{2.38}$$

Since, in view of the regularity of p, $\dot{\mathcal{D}}(0,t;p)=\mathcal{H}(\dot p(t))$ (see [3, Theorem 7.1]), we deduce that

$$\mathcal{H}(\dot p(t))=-\int_\Omega\sigma(t)\cdot(\dot e(t)-E\dot g(t))\,dx+\int_{\Gamma_n}h(t)(\dot u(t)-\dot g(t))\,d\mathcal{H}^{N-1}.$$

The left hand side is the total mass of the measure $H\left(\dot p(t)/|\dot p(t)|\right)|\dot p(t)|$, while, in view of the admissibility of $\partial\lfloor_{\partial\Omega}\Gamma_d$, the right hand side is the mass of the duality pairing $\langle\sigma_D(t),\dot p(t)\rangle$ as a measure on $\Omega\cup\Gamma_d$ (apply (2.1) with the choice $\varphi\equiv 1$ on $\Omega\cup\Gamma_d$). Note however that such a duality vanishes on Γ_d according to item (b) in Proposition 2.3 and to (2.34). Then, inequality (2.38) immediately implies the desired equality. □

We are now in a position to recover the flow rule for a variational quasi-static evolution.

Theorem 2.17 (Flow rule)**.** *Let $\{t\mapsto([u(t)],e(t),p(t)),t\in[0,T]\}$ be a quasi-static evolution according to Definition 2.8. Assume further that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible and that Γ_d is of class C^2. Set*

$$\dot p(t)=\dot p^a(t)\,dx+\dot p^s(t),$$

where $\dot p^a(t)$ (resp. $\dot p^s(t)$) are the absolutely continuous (resp. singular) part of the measure $\dot p(t)$. Then the following items hold true:

(a) *For a.e. $t\in[0,T]$,*

$$\frac{\dot p^a(t,x)}{|\dot p^a(t,x)|}\in N_K(\sigma_D(t,x))\quad \textit{for } \mathcal{L}^N \textit{ a.e. } x\in\{|\dot p^a(t|>0\}.$$

(b) *If, for every $r>0$ and $x\in\Omega$, we set*

$$\sigma^r(t,x):=\frac{1}{|B_r(x)\cap\Omega|}\int_{B_r(x)\cap\Omega}\sigma(t,y)\,dy,$$

then, there exists $r_n\to 0^+$ such that

$$\sigma_D^{r_n}(t)\overset{*}{\rightharpoonup}\hat\sigma_D(t)\qquad \textit{weakly-}\star \textit{ in } L^\infty_{|\dot p^s(t)|}(\Omega;\mathrm{M}^N_D), \tag{2.39}$$

with

$$\frac{\dot p^s(t)}{|\dot p^s(t)|}(x)\in N_{\hat\sigma_D(t,x)}(K)\qquad \textit{for } |\dot p^s(t)|\textit{-a.e. } x\in\Omega. \tag{2.40}$$

In (2.40), $\dot p^s(t)/|\dot p^s(t)|$ denotes the Radon-Nikodym derivative of $\dot p^s(t)$ with respect to its total variation.

Proof. Let $t \in [0,T]$ be such that the equality of Proposition 2.16 holds true. Considering the absolutely continuous parts of the measures in the equality of Proposition 2.16 we get

$$H(\dot p^a(t,x)) = \sigma_D(t,x)\cdot \dot p^a(t,x) \qquad \text{for a.e. in } \Omega$$

so that item (a) follows in view of the admissibility constraint (2.33) for $\sigma_D(t)$.

Let us come to item (b). Let us consider $A \subset\subset \Omega$. Since for r small enough, $\sigma^r(t)$ is continuous with a continuous divergence on A (thanks to the equilibrium condition (2.32)), we have that

$$\langle \sigma^r_D(t), \dot p(t)\rangle = \sigma^r_D(t)\cdot \frac{\dot p(t)}{|\dot p(t)|}|\dot p(t)| \qquad \text{on } A. \tag{2.41}$$

Moreover, since

$$\sigma^r(t) \to \sigma(t) \qquad \text{strongly in } L^2(A; \mathrm{M}^N_{\mathrm{sym}})$$

and

$$\mathrm{div}\ \sigma^r(t) \to \mathrm{div}\ \sigma(t) \qquad \text{strongly in } L^N(A;\mathbb{R}^N)$$

we deduce that

$$\langle \sigma^r_D(t), \dot p(t)\rangle \overset{*}{\rightharpoonup} \langle \sigma_D(t), \dot p(t)\rangle \qquad \text{weakly-}\star \text{ in } \mathcal{M}_b(A).$$

In view of the stress admissibility condition (2.33),

$$\sigma^r_D(t,x) \in K \qquad \text{for every } x \in A,$$

so that, up to a subsequence in r,

$$\sigma^r_D(t) \overset{*}{\rightharpoonup} \hat\sigma_D(t) \qquad \text{weakly-}\star \text{ in } L^\infty_{|\dot p^s(t)|}(A;\mathrm{M}^N_D).$$

with

$$\hat\sigma_D(t,x) \in K \qquad \text{for } |\dot p^s(t)|\text{-a.e. } x \in A. \tag{2.42}$$

In the light of (2.41) and of the equality in Proposition 2.16 we conclude that

$$H\left(\frac{\dot p^s(t)}{|\dot p^s(t)|}\right) = \hat\sigma_D(t)\cdot \frac{\dot p^s(t)}{|\dot p^s(t)|} \qquad |\dot p^s(t)|\text{-a.e. on } A.$$

The previous equality and (2.42) entail the flow rule (2.40) on A: since A is arbitrary, and $\sigma^r_D(t)$ is uniformly bounded on Ω, the previous results can be extended to Ω, which completes the proof. □

Remark 2.18. If K is strictly convex, then, as detailed in [3, Theorem 6.6], the stress $\hat\sigma_D(t)$ is uniquely determined in $L^\infty_{|\dot p^s(t)|}(\Omega;\mathrm{M}^N_D)$. In particular convergence (2.39) holds for $r \to 0$, and it can be shown to occur strongly in $L^1_{|\dot p^s(t)|}(\Omega;\mathrm{M}^N_D)$. ¶

Remark 2.19. In this remark, we specialize the results of Theorem 2.17 and of Remark 2.18 to the Von Mises case, that is to the case where the dimension is $N = 3$ and, for some $\sigma_c > 0$,

$$K := \{\tau \in \mathrm{M}^3_D : |\tau| \le \sqrt{2/3}\ \sigma_c\}.$$

In such a setting $H(\xi) = \sqrt{2/3}\sigma_c|\xi|$, $\xi \in \mathrm{M}^3_D$, and the statement of the theorem becomes

(1) $|\sigma_D(t,x)| = \sqrt{\frac{2}{3}}\sigma_c$ and $\frac{\dot p^a(t,x)}{|\dot p^a(t,x)|} = \frac{\sigma_D(t,x)}{|\sigma_D(t,x)|}$ for $\mathcal{L}^3$-a.e. $x \in \{|\dot p^a(t,x)| > 0\}$;

(2) $\sigma^r_D(t) \overset{r\to 0^+}{\to} \hat\sigma_D(t)$ strongly in $L^1_{|\dot p^s(t)|}(\Omega;\mathrm{M}^3_D)$, where

$$|\hat\sigma_D(t,x)| = \sqrt{\frac{2}{3}}\sigma_c \quad \text{and} \quad \frac{\dot p^s(t)}{|\dot p^s(t)|}(x) = \frac{\hat\sigma_D(t,x)}{|\hat\sigma_D(t,x)|} \qquad \text{for } |\dot p^s(t)|\text{-a.e. } x \in \Omega.$$

¶

3. An application to the bi-axial test

In this section, we propose to discuss a *bona fide* three-dimensional problem for which the results of Section 2 will permit to attain *a uniqueness result for the quasi-static evolution.* As mentioned in the introduction, it is, to our knowledge, the first such result in a three-dimensional setting.

3.1. **Setting of the bi-axial test.** Throughout this section, the elasticity tensor $\mathbb{C}$ is assumed to be homogeneous and isotropic with Young's modulus $\boldsymbol{E} > 0$ and Poisson's ratio $-1 < \boldsymbol{\nu} < 1/2$, *i.e.*,

$$\mathbb{C}^{-1}\sigma = -\frac{\boldsymbol{\nu}}{\boldsymbol{E}}\operatorname{tr}\sigma\, \mathfrak{i} + \frac{1+\boldsymbol{\nu}}{\boldsymbol{E}}\sigma,\ \sigma \in \mathrm{M}^3_{\mathrm{sym}}.$$

Further (f_1, f_2, f_3) is a fixed orthonormal basis in $\mathbb{R}^3$.

The domain is

$$\Omega = (-d/2, d/2) \times (-\ell/2, \ell/2) \times (0, \ell),\ d < \ell.$$

At initial time, we solve the mixed boundary value problem

$$\begin{cases} \operatorname{div}\sigma_0 = 0 & \text{in } \Omega \\ \sigma_0 = \mathbb{C}e_0 & \text{in } \Omega \\ e_0 = Eu_0 & \text{in } \Omega \end{cases}$$

with the following boundary conditions

$$\begin{cases} \sigma_0 f_1 = 0 & \text{on} \quad x_1 = \pm d/2 \\ \sigma_0 f_2 = \overline{\sigma}_2 f_2 & \text{on} \quad x_2 = \pm \ell/2 \\ (\sigma_0)_{13} = (\sigma_0)_{23} = 0 & \text{on} \quad x_3 = 0, \ell \\ u_3 = 0 & \text{on} \quad x_3 = 0, \ell. \end{cases}$$

The elastic solution (u_0, e_0, σ_0) is unique, up to the possible addition to the displacement field u_0 of infinitesimal rigid body motions belonging to

$$\mathcal{R}_d(\Omega) = \{r(x) = (a - \omega x_2) f_1 + (b + \omega x_1) f_2,\ a, b, \omega \in \mathbb{R}\}.$$

It is given by

$$\begin{cases} u_0 := -\dfrac{\boldsymbol{\nu}(1+\boldsymbol{\nu})}{\boldsymbol{E}}\overline{\sigma}_2 x_1 f_1 + \dfrac{(1-\boldsymbol{\nu}^2)}{\boldsymbol{E}}\overline{\sigma}_2 x_2 f_2 \\ e_0 = -\boldsymbol{\nu}(1+\boldsymbol{\nu})\dfrac{\overline{\sigma}_2}{\boldsymbol{E}} f_1 \otimes f_1 + (1-\boldsymbol{\nu}^2)\dfrac{\overline{\sigma}_2}{\boldsymbol{E}} f_2 \otimes f_2 \\ \sigma_0 = \overline{\sigma}_2 f_2 \otimes f_2 + \boldsymbol{\nu}\overline{\sigma}_2 f_3 \otimes f_3. \end{cases} \tag{3.1}$$

As long as

$$0 \le \overline{\sigma}_2 < \frac{\sigma_c}{\sqrt{1 - \boldsymbol{\nu} + \boldsymbol{\nu}^2}}, \tag{3.2}$$

the associated stress is also such that, for some $\alpha > 0$,

$$(\sigma_0)_D + B(0, \alpha) \in K_{\mathrm{vm}} := \{\tau \in \mathrm{M}^3_D : |\tau| \le \sqrt{2/3}\ \sigma_c\}, \tag{3.3}$$

where K_{vm} is the Von Mises yield region. We will assume that (3.2) holds true throughout the rest of this section.

The corresponding initial state is

$$\begin{cases} \sigma(t=0) := \sigma_0 = \overline{\sigma}_2 f_2 \otimes f_2 + \boldsymbol{\nu}\overline{\sigma}_2 f_3 \otimes f_3, \\ e(t=0) := e_0 = -\boldsymbol{\nu}(1+\boldsymbol{\nu})\dfrac{\overline{\sigma}_2}{\boldsymbol{E}} f_1 \otimes f_1 + (1-\boldsymbol{\nu}^2)\dfrac{\overline{\sigma}_2}{\boldsymbol{E}} f_2 \otimes f_2 \\ p(t=0) = p_0 = 0. \end{cases} \tag{3.4}$$

At all later times, the following boundary conditions are imposed:

$$\begin{cases} \sigma f_1 = 0 & \text{on} \quad x_1 = \pm d/2 \\ \sigma f_2 = \overline{\sigma}_2 f_2 & \text{on} \quad x_2 = \pm \ell/2 \\ \sigma_{13} = \sigma_{23} = 0 & \text{on} \quad x_3 = 0, \ell, \end{cases} \tag{3.5}$$

and

$$u_3 = 0 \text{ on } x_3 = 0, \quad u_3 = t\ell \text{ on } x_3 = \ell. \tag{3.6}$$

In other words, the stress $\overline{\sigma}_2$ is maintained constant in direction 2 while the sample is stretched in direction 3 (see Figure 1).

Note that, in the framework of Section 2,

$$\varGamma_d = (-d/2, d/2) \times (-\ell/2, \ell/2) \times \{0,\ell\} \text{ is of class } C^2,$$

while the associated relative boundary of $\varGamma_d$ is admissible, thanks to [5, Section 6, Example 2]. The tractions on $\varGamma_n = \partial\varOmega \setminus \bar{\varGamma}_d$ are independent of time and are given by

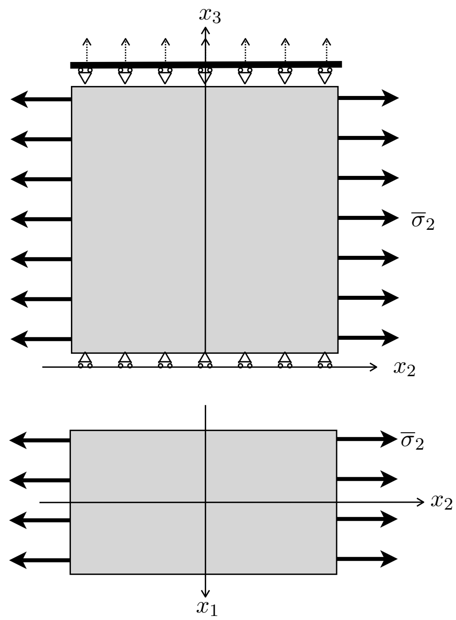

FIGURE 1. Bi-axial test.

$$h(x) := \begin{cases} \pm\bar{\sigma}_2 f_2 & \text{for } x_2 = \pm\ell/2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular $h(x)$ is associated to the homogeneous stress σ_0 which satisfies

$$\operatorname{div} \sigma_0 = 0, \qquad \sigma_0\nu = h \text{ on } \varGamma_n, \qquad [\sigma_0\nu]_\tau = 0 \text{ on } \varGamma_d,$$

together with $(\sigma_0)_D + B(0,\alpha) \in K_{\rm vm}$ for some $\alpha > 0$ (see (3.3)). Finally, the imposed normal displacement is given by

$$w(t) := tx_3 \qquad \text{on } \varGamma_d.$$

We can then take

$$g(t) := tx_3 f_3$$

to be a suitable extension giving the normal trace $w(t)$.

In the notation of Subsection 2.1, it is immediately checked that the triplet $([u_0], e_0, p_0) \in \mathcal{A}(0)$, so that, in view of the properties of σ_0 detailed above and of (3.4), Remark 2.15 implies that $([u_0], e_0, p_0)$ is a global minimizer for (2.22) at $t = 0$.

The quasi-static evolution with initial configuration $([u_0], e_0, p_0)$ admits a *homogeneous solution*, by which we mean that both $e(t)$ and $p(t)$ are $W^{1,1}(0,\infty)$-functions, hence independent of the spatial variable x.Noting that $4\sigma_c^2 - 3\overline{\sigma}_2^2 > 0$ in view of (3.2) and defining

$$t_c := \frac{1}{2\boldsymbol{E}} \left((1 - 2\boldsymbol{\nu})\overline{\sigma}_2 + \sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2} \right), \tag{3.7}$$

that solution is found as follows.

1. Elastic Phase, $0 \le t < t_c$: Then, the response is purely elastic. We obtain

$$\begin{cases} \sigma(t) = \sigma_0 + t\boldsymbol{E} f_3 \otimes f_3, \\ e(t) = e_0 + t\big(-\boldsymbol{\nu} f_1 \otimes f_1 - \boldsymbol{\nu} f_2 \otimes f_2 + f_3 \otimes f_3 \big), \end{cases} \tag{3.8}$$

which corresponds to

$$u(t) = u_0 + t(-\boldsymbol{\nu} x_1 f_1 - \boldsymbol{\nu} x_2 f_2 + x_3 f_3), \tag{3.9}$$

with u_0 given by (3.1).

2. Plastification Phase: $t \ge t_c$: At the end of the elastic phase, *i.e.*, when $t = t_c$, the stress state is

$$\sigma(t_c) = \overline{\sigma}_2 f_2 \otimes f_2 + \overline{\sigma}_3 f_3 \otimes f_3 \text{ with } \overline{\sigma}_3 := \frac{1}{2}\left(\overline{\sigma}_2 + \sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2} \right). \tag{3.10}$$

From thereon, the stress field $\sigma(t)$ remains constant and equal to $\sigma(t_c)$. Consequently, according to item (a) in Theorem 2.17, the absolutely continuous part of the plastic strain rate $\dot{p}$ has a set direction (that of $\sigma_D(t_c)$). Thus, *since we are only seeking a homogeneous solution*, the plastic strain will be of the form

$$p(t) = \eta(t)\Big(-(\overline{\sigma}_2 + \overline{\sigma}_3) f_1 \otimes f_1 + (2\overline{\sigma}_2 - \overline{\sigma}_3) f_2 \otimes f_2 + (2\overline{\sigma}_3 - \overline{\sigma}_2) f_3 \otimes f_3 \Big), \tag{3.11}$$

with $\eta \in W^{1,1}(0,T)$.

The plastic multiplier $\eta(t)$ can be derived as follows. Since $p \in W^{1,1}(0,\infty)$, then, for a.e $t \in (0,T)$, $u_3(t)$ is an affine function of x. But, from (3.8), (3.11),

$$(Eu(t))_{33} = t_c + \eta(t)(2\overline{\sigma}_3 - \overline{\sigma}_2),$$

so that

$$u_3(t) = (t_c + \eta(t)(2\overline{\sigma}_3 - \overline{\sigma}_2))x_3 + U(t, x_1, x_2),$$

with U affine in x_1, x_2. In view of (3.6), we conclude that we can set $U \equiv 0$, so that

$$t_c + \eta(t)(2\overline{\sigma}_3 - \overline{\sigma}_2) = t.$$

Since $2\overline{\sigma}_3 - \overline{\sigma}_2 > 0$,

$$\eta(t) = \frac{1}{2\overline{\sigma}_3 - \overline{\sigma}_2}(t - t_c), \tag{3.12}$$

thus

$$p(t) = (t - t_c)\Big(-\frac{(\overline{\sigma}_2 + \overline{\sigma}_3)}{2\overline{\sigma}_3 - \overline{\sigma}_2} f_1 \otimes f_1 + \frac{2\overline{\sigma}_2 - \overline{\sigma}_3}{2\overline{\sigma}_3 - \overline{\sigma}_2} f_2 \otimes f_2 + f_3 \otimes f_3 \Big). \tag{3.13}$$

The elastic strain $e(t)$ is obtained upon setting $t = t_c$ in (3.8) while the displacement field $u(t)$ is determined from the boundary conditions, together with $Eu(t) = e(t) + p(t)$. It is precisely

$$\begin{aligned} u(t) = & \left\{ -\boldsymbol{\nu}(1+\boldsymbol{\nu})\frac{\overline{\sigma}_2}{\boldsymbol{E}} - \boldsymbol{\nu} t_c - \frac{(\overline{\sigma}_2 + \overline{\sigma}_3)}{2\overline{\sigma}_3 - \overline{\sigma}_2}(t - t_c) \right\} x_1 f_1 \\ & + \left\{ (1 - \boldsymbol{\nu}^2)\frac{\overline{\sigma}_2}{\boldsymbol{E}} - \boldsymbol{\nu} t_c + \frac{2\overline{\sigma}_2 - \overline{\sigma}_3}{2\overline{\sigma}_3 - \overline{\sigma}_2}(t - t_c) \right\} x_2 f_2 + t x_3 f_3. \end{aligned} \tag{3.14}$$

Then, we can establish the following

Proposition 3.1. *The homogeneous solution*

$$t \mapsto ([u(t)], e(t), p(t)) \in \mathcal{A}(w(t))$$

produced in (3.8)-(3.10), (3.13), (3.14) *is a quasi-static evolution according to Definition 2.8 relative to the initial condition* $(u_0, e_0, 0) \in \mathcal{A}(0)$ *(defined in* (3.4), (3.1)*), with respect to the mixed boundary conditions* (3.5) *and* (3.6). *In particular the Cauchy stress* $\sigma(t)$ *given through* (3.8),(3.10) *is uniquely determined.*

Proof. Since $\sigma(t)$ satisfies is divergence free, matches the boundary conditions (3.5) and $\sigma_D(t)$ belongs to the yield region $K_{\rm vm}$, Remark 2.15 implies that the homogeneous solution satisfies the global minimality condition in Definition 2.8. A direct computation shows that the following items hold true:

(a) Elastic energy:

$$\mathcal{Q}(e(t)) = \mathcal{Q}(e_0) + \tfrac{1}{2}t^2\boldsymbol{E}|\Omega|, \quad \text{for } 0 \le t \le t_c$$
$$\mathcal{Q}(e(t)) = \mathcal{Q}(e(t_c)), \quad \text{for } t \ge t_c.$$

(b) Traction potential:

$$\mathcal{L}(t,[u(t)]) = \mathcal{L}(0,[u(0)]) + t\bar\sigma_2\boldsymbol{\nu}|\Omega|, \quad \text{for } 0 \le t \le t_c$$
$$\mathcal{L}(t,[u(t)]) = \mathcal{L}(t_c,[u(t_c)]) - (t-t_c)\bar\sigma_2\frac{2\bar\sigma_2-\bar\sigma_3}{2\bar\sigma_3-\bar\sigma_2}|\Omega|, \quad \text{for } t \ge t_c.$$

(c) Dissipation:

$$\mathcal{D}(0,t;p) = \int_0^t \mathcal{H}(\dot p(\tau))\,d\tau = \frac{2\sigma_c^2}{2\bar\sigma_3-\bar\sigma_2}|\Omega|(t-t_c)^+.$$

(d) External work: for $0 \le t \le t_c$

$$\int_0^t \left[\int_\Omega \sigma(\tau)\cdot E\dot g(\tau)\,dx + \mathcal{L}(\tau,[\dot g(\tau)]) + \dot{\mathcal{L}}(\tau,[u(\tau)])\right] = \left(\boldsymbol{\nu}\bar\sigma_2 + \frac{t}{2}\boldsymbol{E}\right)|\Omega|t$$

and for $t \ge t_c$

$$\int_0^t \left[\int_\Omega \sigma(\tau)\cdot E\dot g(\tau)\,dx + \mathcal{L}(\tau,[\dot g(\tau)]) + \dot{\mathcal{L}}(\tau,[u(\tau)])\right]$$
$$= \left(\boldsymbol{\nu}\bar\sigma_2 + \frac{t_c}{2}\boldsymbol{E}\right)|\Omega|t_c + (\boldsymbol{\nu}\bar\sigma_2 + t_c\boldsymbol{E})|\Omega|(t-t_c) = -\frac{t_c^2}{2}\boldsymbol{E}|\Omega| + \bar\sigma_3|\Omega|t.$$

From the various expressions above together with (3.7), (3.10), it is easily checked that the energy equality holds true. □

We have thus determined a possible elasto-plastic evolution. In the next subsection we will argue that this is the only possible evolution, provided that $\overline{\sigma}_2$ remains strictly below its maximal value in (3.2).

3.2. **About uniqueness.** In the previous subsection, a spatially homogeneous quasi-static evolution for the bi-axial test has been evidenced. It uniqueness is debated in the present subsection.

First, in view of the uniqueness of the stress field (see Theorem 2.12), the elastic phase is also unique as long as $t < t_c$, because the yield stress has not been reached.

When $t \ge t_c$, we distinguish two cases. Note that, since $-1 < \boldsymbol{\nu} < 1/2$, then,

$$2\frac{\sigma_c}{\sqrt{3}} > \frac{\sigma_c}{\sqrt{1-\boldsymbol{\nu}+\boldsymbol{\nu}^2}} > \frac{\sigma_c}{\sqrt{3}}, \tag{3.15}$$

so that, by virtue of (3.2), it is always possible to set $\overline{\sigma}_2 = \sigma_c/\sqrt{3}$. Also note that, in view of (3.10), the deviatoric part of the stress field is given for $t \ge t_c$ by

$$\begin{cases} \sigma_D(t) = \breve\sigma := \breve\sigma_1 f_1\otimes f_1 + \breve\sigma_2 f_2\otimes f_2 + \breve\sigma_3 f_3\otimes f_3 \\ \breve\sigma_1 := -\frac{1}{2}\,\overline{\sigma}_2 - \frac{1}{6}\sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2} \\ \breve\sigma_2 := \frac{1}{2}\,\overline{\sigma}_2 - \frac{1}{6}\sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2} \\ \breve\sigma_3 := \frac{1}{3}\sqrt{4\sigma_c^2 - 3\overline{\sigma}_2^2}, \end{cases} \tag{3.16}$$

so that $\breve\sigma_1 + \breve\sigma_2 + \breve\sigma_3 = 0,\ \breve\sigma_1^2 + \breve\sigma_2^2 + \breve\sigma_3^2 = 2/3\ \sigma_c^2$.

Further, note that

$$\breve\sigma = \mathrm{diag}\left(-\frac{\sigma_c}{\sqrt{3}}, 0, \frac{\sigma_c}{\sqrt{3}}\right) \text{ iff } \overline{\sigma}_2 = \sigma_c/\sqrt{3}. \tag{3.17}$$

The first case is described in the following

Theorem 3.2 (Uniqueness of the solution). *If $\overline{\sigma}_2$ satisfies (3.2) and $\overline{\sigma}_2 \neq \sigma_c/\sqrt{3}$, the homogeneous elasto-plastic evolution derived in Subsection 3.1 is the only possible quasi-static evolution for the bi-axial test.*

Proof. The spatially homogeneous character of $\sigma_D(t)$ in (3.16) implies that $\hat{\sigma}_D(t) = \check{\sigma}$ in item (2) of Remark 2.19, so that the plastic strains are of the same form as in (3.11), except that, for a.e. $t \geq t_c$, $\eta(t)$ may be an element of $\mathcal{M}_b^+(\Omega)$ instead of a constant.

Uniqueness will be achieved if we prove that $\eta(t)$ is spatially homogeneous and given through (3.12). To that effect, we observe that

$$\begin{aligned}
(Eu(t))_{11} &= -\frac{\boldsymbol{\nu}}{\boldsymbol{E}}(\overline{\sigma}_2 + \overline{\sigma}_3) - \eta(t)(\overline{\sigma}_2 + \overline{\sigma}_3) & \qquad (Eu(t))_{12} &= 0\\
(Eu(t))_{22} &= \frac{1}{\boldsymbol{E}}(\overline{\sigma}_2 - \boldsymbol{\nu}\overline{\sigma}_3) + \eta(t)(2\overline{\sigma}_2 - \overline{\sigma}_3) & (Eu(t))_{23} &= 0\\
(Eu(t))_{33} &= \frac{1}{\boldsymbol{E}}(\overline{\sigma}_3 - \boldsymbol{\nu}\overline{\sigma}_2) + \eta(t)(2\overline{\sigma}_3 - \overline{\sigma}_2) & (Eu(t))_{31} &= 0.
\end{aligned}$$

The classical geometric compatibility equations for an element $\varepsilon \in \mathcal{D}'(\Omega; \mathrm{M}^3_{\mathrm{sym}})$ to be the distributional symmetrized gradient of a $\mathbb{R}^3$-valued distribution, namely $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$, $1 \leq i,j,k,l \leq 3$, yield

$$\begin{aligned}
0 &= (2\overline{\sigma}_2 - \overline{\sigma}_3)\eta_{,11}(t) - (\overline{\sigma}_2 + \overline{\sigma}_3)\eta_{,22}(t) & \qquad 0 &= (\overline{\sigma}_2 + \overline{\sigma}_3)\eta_{,23}(t)\\
0 &= (2\overline{\sigma}_2 - \overline{\sigma}_3)\eta_{,33}(t) + (2\overline{\sigma}_3 - \overline{\sigma}_2)\eta_{,22}(t) & 0 &= (2\overline{\sigma}_2 - \overline{\sigma}_3)\eta_{,31}(t)\\
0 &= (2\overline{\sigma}_3 - \overline{\sigma}_2)\eta_{,11}(t) - (\overline{\sigma}_2 + \overline{\sigma}_3)\eta_{,33}(t) & 0 &= (2\overline{\sigma}_3 - \overline{\sigma}_2)\eta_{,12}(t),
\end{aligned}$$

where the various derivatives of η are to be viewed as distributional derivatives. By virtue of (3.15), $2\overline{\sigma}_2 \neq \overline{\sigma}_3$, $2\overline{\sigma}_3 \neq \overline{\sigma}_2$ and $\overline{\sigma}_2 + \overline{\sigma}_3 \neq 0$, so that $\eta_{,ij}(t) = 0$ $\forall i,j \in \{1,2,3\}$. Thus η is actually an affine function of x, *i.e.*,

$$\eta(x,t) = \eta_0(t) + \sum_{i=1}^{3} \eta_i(t) x_i.$$

Then,

$$u_{3,3}(x,t) = \frac{1}{\boldsymbol{E}}(\overline{\sigma}_3 - \boldsymbol{\nu}\overline{\sigma}_2) + (2\overline{\sigma}_3 - \overline{\sigma}_2)\left(\eta_0(t) + \sum_{i=1}^{3} \eta_i(t) x_i\right).$$

Since $u_3(t) = 0$ at $x_3 = 0$, we obtain

$$u_3(x,t) = \frac{1}{\boldsymbol{E}}(\overline{\sigma}_3 - \boldsymbol{\nu}\overline{\sigma}_2)x_3 + (2\overline{\sigma}_3 - \overline{\sigma}_2)\left(\eta_0(t)x_3 + \eta_1(t)x_1x_3 + \eta_2(t)x_2x_3 + \frac{1}{2}\eta_3(t)x_3^2\right).$$

Since $u_3(t) = t\ell$ at $x_3 = \ell$, this yields in turn, thanks to the expressions (3.7),(3.10) for t_c and $\overline{\sigma}_3$ which yield in particular that $\boldsymbol{E}t_c = \bar{\sigma}_3 - \boldsymbol{\nu}\bar{\sigma}_2$,

$$\eta_1(t) = \eta_2(t) = 0, \qquad \eta_0(t) + \frac{1}{2}\eta_3(t)\ell = \frac{t - t_c}{2\overline{\sigma}_3 - \overline{\sigma}_2}.$$

Since $u_{3,2}(t) = 0$ and $(Eu(t))_{23} = 0$, $u(t)_{2,3} = 0$, hence $([Eu(t))_{22}]_{,3} = u(t)_{2,23} = 0$, which implies that $\eta_{,3}(t) = \eta_3(t) = 0$. Finally, we conclude that $\eta(x,t) = \eta(t)$ given by (3.12), hence that $p(x,t) = p(t)$ given by (3.13). □

The second case occurs when $\overline{\sigma}_2 = \sigma_c/\sqrt{3}$. Then, for $t \geq t_c$,

$$\begin{cases}
\sigma(t) = \dfrac{\sigma_c}{\sqrt{3}}\big(f_2 \otimes f_2 + 2f_3 \otimes f_3\big)\\
e(t) = \dfrac{\sigma_c}{\boldsymbol{E}}\left(-\boldsymbol{\nu}\sqrt{3}f_1 \otimes f_1 + \dfrac{1-2\boldsymbol{\nu}}{\sqrt{3}}f_2 \otimes f_2 + \dfrac{2-\boldsymbol{\nu}}{\sqrt{3}}f_3 \otimes f_3\right)\\
\sigma_D(t) = \mathring{\sigma}_D := \dfrac{\sigma_c}{\sqrt{3}}\big(-f_1 \otimes f_1 + f_3 \otimes f_3\big)
\end{cases} \tag{3.18}$$

which corresponds to the point $\mathbf{B}^*$ on Figure 2 (this is the point on the ellipse for which σ_3 reaches its maximal value).

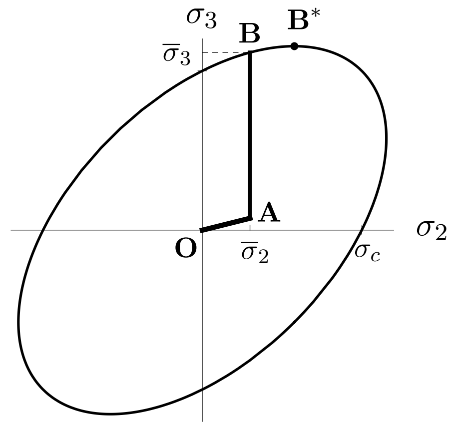

FIGURE 2. Stress path in the bi-axial test: [**OA**) corresponds to the prestress, [**AB**) to the elastic phase. During the plastification phase, stresses stay at **B**. The ellipse is the Von Mises ellipse $\sigma_2^2 + \sigma_3^2 - \sigma_2\sigma_3 = \sigma_c^2$ in the plane (σ_2, σ_3) since $\sigma_1 = 0$.

In such a case, for $t \geq t_c$, consider the following displacement field:

$$u(t,x) = \frac{\sigma_c}{\boldsymbol{E}}\left(-\boldsymbol{\nu}\sqrt{3}x_1 f_1 + \frac{1-2\boldsymbol{\nu}}{\sqrt{3}}x_2 f_2 + \frac{2-\boldsymbol{\nu}}{\sqrt{3}}x_3 f_3\right) + \bar{u}(x,t) \tag{3.19}$$

with

$$\bar{u}(t,x) = \begin{cases} 0 & \text{if } x_3 - x_1 < \ell/2 \\ (t-t_c)\ell(f_1+f_3) & \text{otherwise.} \end{cases}$$

The field u jumps across the plane $\mathit{\Gamma} := \{x_3 - x_1 = \ell/2\}$. Consider the associated plastic strain

$$p(t) = (t-t_c)\frac{\ell}{\sqrt{2}}(-f_1 \otimes f_1 + f_3 \otimes f_3)\,\mathcal{H}^2 \lfloor \mathit{\Gamma} \quad \text{ on } \mathit{\Gamma}. \tag{3.20}$$

Proposition 3.3. *The mapping $t \to (u(t), e(t), p(t))$ given by (3.18), (3.19), (3.20) is a quasi-static evolution different from the homogeneous evolution derived in Subsection 3.1.*

Proof. Notice that $([u(t)], e(t), p(t)) \in \mathcal{A}(w(t))$. Indeed clearly

$$Eu(t) = e(t) + p(t) \qquad \text{in } \mathit{\Omega}$$

since the plastic strain in (3.20) takes into account the jump occurring across $\mathit{\Gamma}$. Concerning the boundary conditions for the displacement, since $d < \ell$,

$$\bar{u}(t)_{|x_3=0} \equiv 0 \qquad \text{and} \qquad \bar{u}(t)_{|x_3=\ell} = (t-t_c)\ell(f_1+f_3).$$

Then, the displacement boundary conditions (3.6) are clearly satisfied at $x_3 = 0$. According to (3.7), $t_c = (2-\boldsymbol{\nu})\sigma_c/(\sqrt{3}\boldsymbol{E})$, so that, at $x_3 = \ell$,

$$u_3(t)_{|x_3=\ell} = \left((2-\boldsymbol{\nu})\sigma_c/(\sqrt{3}\boldsymbol{E}) + t - t_c\right)\ell = t\ell.$$

In view of Remark 2.15, global stability is a consequence of the fact that the stress field $\sigma(t)$ defined in (3.18) is divergence free and satisfies the boundary conditions (3.5) and the yield condition $\sigma(t) \in K_{\mathrm{vm}}$.

Finally, in order to check the energy equality, it suffices to notice that, for $t \geq t_c$, the left hand side of (2.23) increases by the quantity

$$\mathcal{D}(0,t;p) = \int_0^t \mathcal{H}(\dot{p}(\tau))\,d\tau = \sqrt{\frac{2}{3}}\sigma_c\ell(t-t_c)\mathcal{H}^2\lfloor(\mathit{\Gamma}\cap\mathit{\Omega})$$

(there is no additional contribution to the potential of the traction forces), while the right hand side increases by the quantity

$$(\boldsymbol{\nu}\bar{\sigma}_2 + t_c\boldsymbol{E})|\mathit{\Omega}|(t-t_c).$$

Those two quantities are readily seen to be equal for $\bar{\sigma}_2 = \sigma_c/\sqrt{3}$. □

Remark 3.4. In the case $\overline{\sigma}_2 = \sigma_c/\sqrt{3}$, an infinite number of quasi-static evolutions can be constructed. Indeed, since $d < \ell$, the slip surface Γ can be translated vertically without altering the arguments outlined above because the plastic slip can take place along the plane

$$\Gamma_a := \left\{ x + af_3 \,:\, x \in \Gamma, |a| < \frac{\ell - d}{2} \right\}. \tag{3.21}$$

We further elaborate on non-uniqueness. Since $\hat{\sigma}_D(t) = \mathring{\sigma}_D$ in (3.18), item (2) of Remark 2.19 entails $|p_s(t)|$-a.e. the following expression for the plastic strains:

$$p(t,x) = \eta(t)(-f_1 \otimes f_1 + f_3 \otimes f_3), \qquad \eta \in W^{1,1}([0,\infty); \mathcal{M}_b^+(\Omega)).$$

Thus the strain tensor reads as

$$\begin{aligned}
(Eu(t))_{11} &= -\frac{\boldsymbol{\nu}}{\sqrt{3}}\,\sigma_c/\boldsymbol{E} - \eta(t) & \qquad (Eu(t))_{12} &= 0\\
(Eu(t))_{22} &= \frac{(1-2\boldsymbol{\nu})}{\sqrt{3}}\,\sigma_c/\boldsymbol{E} & \qquad (Eu(t))_{23} &= 0\\
(Eu(t))_{33} &= \frac{(2-\boldsymbol{\nu})}{\sqrt{3}}\,\sigma_c/\boldsymbol{E} + \eta(t) & \qquad (Eu(t))_{31} &= 0.
\end{aligned}$$

Geometric compatibility implies in turn that

$$0 = \eta_{,12}(t) = \eta_{,22}(t) = \eta_{,23}(t), \quad 0 = \eta_{,11}(t) - \eta_{33}(t).$$

The three first compatibility equations yield that $\nabla(\eta_{,2}(t)) = 0$, thus that $\eta(t) = \mathring{\eta}(t) + \beta(t)x_2$, with $\mathring{\eta}(t)$ a bounded Radon measure independent of x_2 and $\beta(t)$ a constant. The last one becomes

$$\mathring{\eta}_{,11}(t) - \mathring{\eta}_{,33}(t) = 0,$$

which is a (spatial) wave equation. Thus

$$\mathring{\eta}(t) = \zeta_-(t)(x_1 - x_3) + \zeta_+(t)(x_1 + x_3), \qquad \zeta_-(t), \zeta_+(t) \in \mathcal{M}(\mathbb{R}).$$

In view of the preceding relations, we set $\beta(t) \equiv 0$ and look for solutions of the form

$$\eta(t) = (t - t_c)\left[\zeta_-(x_1 - x_3) + \zeta_+(x_1 + x_3)\right], \qquad \zeta_-, \zeta_+ \in \mathcal{M}^+(\mathbb{R}).$$

By $\zeta_-(x_1 - x_3)$ we mean the two dimensional Radon measure defined by

$$\langle \zeta_-(x_1 - x_3), \varphi\rangle := \frac{1}{2}\int\left[\int \varphi\left(\frac{u+v}{2}, \frac{-u+v}{2}\right) d\zeta_-(u)\right] dv, \qquad \varphi \in C_c^\infty(\mathbb{R}^2).$$

Similarly, $\zeta_+(x_1 + x_3)$ is defined by

$$\langle \zeta_+(x_1 + x_3), \varphi\rangle := \frac{1}{2}\int\left[\int \varphi\left(\frac{u+v}{2}, \frac{u-v}{2}\right) d\zeta_+(u)\right] dv, \qquad \varphi \in C_c^\infty(\mathbb{R}^2).$$

If we denote by $G_\pm$ a primitive of $\zeta_\pm$, we infer that the displacement $u(t)$ is given for $t \geq t_c$

$$\begin{aligned}
u(t) = u(t_c) - (t - t_c)(G_-(x_1 - x_3) + G_+(x_1 + x_3) + K)f_1\\
+ (t - t_c)(-G_-(x_1 - x_3) + G_+(x_1 + x_3) + H)f_3,
\end{aligned}$$

for some $K, H \in \mathbb{R}$. Let us set $H = K = 0$. Imposing the boundary conditions (3.6) yields the following conditions:

- For $u_3(t) = 0$ for $x_3 = 0$ to hold, we need

$$G_-(x_1) = G_+(x_1) \qquad \text{for } x_1 \in [-d/2, d/2]. \tag{3.22}$$

- For $u_3(t) = t\ell$ for $x_3 = \ell$ to hold, we need

$$t_c + (t - t_c)(-G_-(x_1 - \ell) + G_+(x_1 + \ell)) = t, \qquad \text{for } x_1 \in [-d/2, d/2],$$

  or still

$$G_-(x_1) = G_+(x_1 + 2\ell) + 1 \qquad \text{for } x_1 \in \left[-\frac{d}{2} - \ell, \frac{d}{2} - \ell\right]. \tag{3.23}$$

Since $d < \ell$, relations (3.22) and (3.23) are not sufficient to determine uniquely $G_\pm$. For example, we see that, if ζ_- is any probability measure supported on the interval $[d/2-\ell, -d/2]$, we can choose $\zeta_+ \equiv 0$ with $G_+ \equiv 1$ and G_- as the primitive of ζ_- which vanishes at $d/2-\ell$.

We conclude that the quasi-static evolution problem admits infinitely many solutions, for which the plastic strain can also be of Cantor type. This is to our knowledge the first such example of the possible appearance of a Cantor part in a plastic strain associated with a quasi-static elasto-plastic evolution.

We finally note that the solution with jump discontinuities proposed in (3.19) is recovered upon choosing ζ_- in the form of a Dirac delta concentrated at $-\ell/2$, *i.e.*, $G_-(x_1) = H_e(x_1+\ell/2)$ (where H_e is the Heaviside function) and $\zeta_+ \equiv 0$, $G_+ \equiv 0$. A similar argument applies to the solution with jumps on $\varGamma_a$ (see (3.21)). ¶

Remark 3.5. As demonstrated in [6], the condition (3.17) is not surprising. It is precisely that which allows for jumps to appear during a quasi-static evolution. For more details, we refer the reader to that reference. ¶

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References

[1] L. Ambrosio, A. Coscia, and G. Dal Maso. Fine properties of functions with bounded deformations. *Arch. Rat. Mech. Anal.*, 139:201–238, 1997.
[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.
[3] G. Dal Maso, A. DeSimone, and M. G. Mora. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Ration. Mech. Anal.*, 180(2):237–291, 2006.
[4] A. Demyanov. Regularity of stresses in Prandtl-Reuss perfect plasticity. *Calc. Var. Partial Differential Equations*, 34(1):23–72, 2009.
[5] G. A. Francfort and A. Giacomini. Small strain heterogeneous elastoplasticity revisited. *Comm. Pure Appl. Math.*, 65(9):1185–1241, 2012.
[6] G. A. Francfort, A. Giacomini, and J.-J. Marigo. The taming of plastic slips in Von Mises elasto-plasticity. *To appear.*
[7] R.V. Kohn and R. Temam. Dual spaces of stresses and strains, with applications to Hencky plasticity. *Appl. Math. Optim.*, 10(1):1–35, 1983.
[8] A. Mainik and A. Mielke. Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differential Equations*, 22(1):73–99, 2005.
[9] Alexander Mielke. Evolution of rate-independent systems. In A. Dafermos and E. Feireisl, editors, *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 461–559. Elsevier/North-Holland, Amsterdam, 2005.
[10] Pierre-M. Suquet. Un espace fonctionnel pour les équations de la plasticité. *Ann. Fac. Sci. Toulouse Math. (5)*, 1(1):77–87, 1979.
[11] Pierre-M. Suquet. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mécanique*, 20(1):3–39, 1981.
[12] Roger Temam. *Problèmes mathématiques en plasticité*, volume 12 of *Méthodes Mathématiques de l'Informatique*. Gauthier-Villars, Montrouge, 1983.

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