The proofs of the optimal bounds for mixtures of two anisotropic conducting materials in two dimensions

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Abstract

We provide a complete proof of the result announced twenty years ago in [5], namely the characterization in two dimensions of the set of the effective conductivities obtained by mixing two anisotropic conducting materials with arbitrary orientation. We also provide a complete proof of the characterization (also already announced in [5]) of the sets of conducting materials with arbitrary orientation which are stable under H-convergence in two dimensions.

Keywords: homogenization, H-convergence, optimal bounds, conductivity, anisotropy.

1 Introduction

In the fall of 2007, Graeme Milton was awarded the prestigious *William Prager Medal* from the Society of Engineering Science. It is our pleasure and honor to write this article as a pale tribute to his impressive scientific achievements.

The Eighties witnessed a flurry of investigations – spearheaded largely by Graeme Milton's groundbreaking work – on bounds for two-phase mixtures of conducting and/or elastic materials. The effort has since subsided for want of new methods, with the notable exception of V. Nesi's two-dimensional work based on quasi-conformal mappings; see [12] and [1].

At the time, one concern was the determination of the set of all mixtures of two anisotropic conducting materials, when both the volume fractions and the orientations of the materials are arbitrary. The first result in that direction was that of A. Cherkaev and K. Lurie [6] who proposed a characterization of that set in two dimensions. Unfortunately, their paper contained a flaw and the announced result was incorrect. This prompted us to revisit the problem and to give in [5] a full characterization of the set in two dimensions. (The original paper [6] was amended in [7] at a later time.) Our paper sketched the argument but it was certainly not meant to remain celibate for so long. It actually contained several references to a more complete paper allegedly in the process of being written at that time. The completion of that companion paper has remained a pious and largely forgotten wish for twenty years, in spite of the contemporaneous use of its results in [4] and [8]. When Graeme Milton was immersed in the writing of his treatise on bounds [9], his rendering of part of the argument in Chapter 22.5 relied solely on oral expositions of that work.

We now put an end to [5]'s solitude and provide a complete, albeit brief account of the characterization. Of course, we benefit from hindsight and do not dwell on features of homogenization that should be part of the familiar of any concerned reader. The less familiar readers, or those who do not read French and therefore cannot benefit from [10], may wish to refer to its english translation [11], or to [2], Chapter 1, for a rather thorough presentation of the tools of H-convergence. Those familiar with the concept of G-convergence [13] can freely substitute 'G for H' since we deal here with symmetric matrices.

The present paper is organized as follows. Section 2 sets the framework and formulates the characterization result (Theorem 2.3). It also formulates the characterization of the sets of two-dimensional conducting materials which are stable under H-convergence, when the definition of the set is independent of the orientation of the material (Theorem 2.7), a result which is interesting in and of itself. The proofs are given in Section 3.

The following notation is used throughout.

If A and B are 2×2 symmetric matrices, $A \le B$ means that $Ae.e \le Be.e$ for all $e \in \mathbb{R}^2$.

For some fixed $0 < \alpha < \beta < \infty$, we denote by $\mathbb{M}_s(\alpha,\beta)$ the set of 2×2 symmetric matrices M with $\alpha I \le M \le \beta I$, where I is the identity 2×2 matrix.

The matrix $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ denotes the $-\pi/2$ rotation matrix, so that, if φ is a $\mathbb{R}^2$-valued field,

$$-\text{div } R\varphi = \text{curl } \varphi \quad \text{and} \quad \text{curl } R\varphi = \text{div } \varphi,$$

with curl φ defined by curl $\varphi := \partial\varphi_1/\partial x_2 - \partial\varphi_2/\partial x_1$.

2 Framework and results

The setting is two-dimensional, $0 < \alpha < \beta < \infty$ are given, and (e_1, e_2) is a fixed orthonormal basis of $\mathbb{R}^2$. Two anisotropic materials

$$\begin{cases} A = \alpha_1 e_1 \otimes e_1 + \alpha_2 e_2 \otimes e_2, \\ B = \beta_1 e_1 \otimes e_1 + \beta_2 e_2 \otimes e_2, \end{cases} \tag{2.1}$$

are considered and it is assumed, with no loss of generality, that, for some $0 < \alpha < \beta$,

$$\alpha \leq \alpha_1 \leq \alpha_2 \leq \beta, \quad \alpha \leq \beta_1 \leq \beta_2 \leq \beta, \quad \alpha_1\alpha_2 \leq \beta_1\beta_2. \tag{2.2}$$

Remark that we may as well set

$$\alpha = \inf\{\alpha_1, \beta_1\}, \quad \beta = \sup\{\alpha_2, \beta_2\}.$$

We will loosely refer to the material with conductivity A as the A-material; idem for B.

A mixture of those two materials is characterized at each point x by a marker at that point, *i.e.*, by the characteristic function $\chi(x) \in \{0, 1\}$ of, say, the A-material, together with the orientation of the material at that point, *i.e.*, by a rotation matrix $R(x) \in SO(2)$. Thus, the conductivity of the mixture at any point $x \in \mathbb{R}^2$ is

$$A(x) := R^T(x)\,(\chi(x)A + (1 - \chi(x))B)\,R(x), \tag{2.3}$$

and we further assume measurability of the matrix $A(x)$, or, equivalently of $\chi(x)$ and $R(x)$. Note that there is no loss of generality in assuming in (2.1) that the matrices A and B are diagonalizable in the same orthonormal basis (e_1, e_2). Indeed if A is diagonalizable in the orthonormal basis (e_1, e_2) while B is diagonalizable in the orthonormal basis (f_1, f_2), and if J is the rotation matrix which permits one to pass from the second to the first basis, any mixture of the form

$$A(x) = R^T(x)(\chi(x)A + (1 - \chi(x))B)R(x)$$

can be written in the form (2.1) (2.3) upon replacing $R(x)$ by $JR(x)$ whenever $\chi(x) = 0$.

In the spirit of the H-convergence, we consider an ε-indexed sequence of conductivities of that type, *i.e.*, a sequence

$$A_\varepsilon = R_\varepsilon^T\,(\chi_\varepsilon A + (1 - \chi_\varepsilon)B)\,R_\varepsilon$$

with obvious notation. According to H-convergence [11], there exists a subsequence of $\{\varepsilon\}$, still labeled by $\{\varepsilon\}$, and a matrix $A_0 \in L^\infty(\mathbb{R}^2; \mathbb{M}_s(\alpha, \beta))$, such that

Lemma 2.1 *For every open bounded subset Ω of $\mathbb{R}^2$ and any sequence $w_\varepsilon \in L^2(\Omega;\mathbb{R}^2)$ that satisfies*

$$\begin{cases} w_\varepsilon \rightharpoonup w \text{ weakly in } L^2(\Omega;\mathbb{R}^2), \\ q_\varepsilon := A_\varepsilon w_\varepsilon \rightharpoonup q \text{ weakly in } L^2(\Omega;\mathbb{R}^2), \end{cases}$$

while

$$\begin{cases} \operatorname{curl} w_\varepsilon \text{ lies in a compact set of } H^{-1}(\Omega), \\ \operatorname{div} q_\varepsilon \text{ lies in a compact set of } H^{-1}(\Omega), \end{cases}$$

we have

$$q = A_0 w,$$

where the matrix A_0 is the H-limit of the sequence A_ε.

The matrix A_0 should be viewed as the overall, effective, or homogenized matrix associated to the (sequence of) mixtures A_ε; see e.g. [11], [2]. From now onward, we will call such a matrix an *effective* conductivity.

The bounding problem alluded to in the introduction consists in characterizing the set of all such effective conductivities, henceforth referred to as the *effective set*. More precisely, the effective set is the H-closure of the matrices of the form (2.3), or equivalently the set of those matrices $A_0 \in L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta))$ such that there exists a sequence of matrices of the form (2.3) that H-converges to A_0.

The result announced in [5], correcting an earlier result of [6], is precisely the characterization of that set. Its proof is merely sketched in [5].

We define the following two subsets $\mathcal{L}_{\text{wo}}$ and $\mathcal{L}_{\text{bo}}$ of $L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta))$, where the indices wo and bo respectively stand for "well ordered" and "badly ordered".

Definition 2.2 *If $\alpha_1\alpha_2 \neq \beta_1\beta_2$, the set L_{wo} is defined as the set of points $(\lambda_1,\lambda_2) \in \mathbb{R}^2$ that satisfy*

$$\begin{cases} \alpha_1\alpha_2 \le \lambda_1\lambda_2 \le \beta_1\beta_2, \\ \dfrac{(\beta_1-\alpha_1)\lambda_1\lambda_2 + (\beta_2-\alpha_2)\alpha_1\beta_1}{\beta_1\beta_2-\alpha_1\alpha_2} \le \inf(\lambda_1,\lambda_2) \le \\ \le \sup(\lambda_1,\lambda_2) \le \dfrac{\lambda_1\lambda_2(\beta_1\beta_2-\alpha_1\alpha_2)}{(\beta_1-\alpha_1)\lambda_1\lambda_2 + (\beta_2-\alpha_2)\alpha_1\beta_1}, \end{cases} \tag{2.4}$$

while the set L_{bo} *is defined as the set of points* $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ *that satisfy*

$$\begin{cases} \alpha_1\alpha_2 \le \lambda_1\lambda_2 \le \beta_1\beta_2, \\ \dfrac{\lambda_1\lambda_2(\beta_1\beta_2 - \alpha_1\alpha_2)}{(\beta_2-\alpha_2)\lambda_1\lambda_2 + (\beta_1-\alpha_1)\alpha_2\beta_2} \le \inf(\lambda_1,\lambda_2) \le \\ \le \sup(\lambda_1,\lambda_2) \le \dfrac{(\beta_2-\alpha_2)\lambda_1\lambda_2 + (\beta_1-\alpha_1)\alpha_2\beta_2}{\beta_1\beta_2 - \alpha_1\alpha_2}. \end{cases} \tag{2.5}$$

If $\alpha_1\alpha_2 = \beta_1\beta_2$*, the set* $L_{\text{wo}} = L_{\text{bo}}$ *is the set of points* $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ *that satisfy*

$$\begin{cases} \lambda_1\lambda_2 = \alpha_1\alpha_2 = \beta_1\beta_2, \\ \inf\{\alpha_1, \beta_1\} \le \inf(\lambda_1,\lambda_2) \le \sup(\lambda_1,\lambda_2) \le \sup\{\alpha_2, \beta_2\}. \end{cases} \tag{2.6}$$

We then define the sets $\mathcal{L}_{\text{wo}}$ *and* $\mathcal{L}_{\text{bo}}$ *by*

$$\begin{aligned} \mathcal{L}_{\text{wo}} := \big\{ & C \in L^\infty(\mathbb{R}^2; \mathbb{M}_s(\alpha,\beta)) : \\ & \text{the eigenvalues } (\lambda_1(x), \lambda_2(x)) \text{ of } C(x) \text{ belong to } L_{\text{wo}} \text{ a.e.} \big\}, \\ \mathcal{L}_{\text{bo}} := \big\{ & C \in L^\infty(\mathbb{R}^2; \mathbb{M}_s(\alpha,\beta)) : \\ & \text{the eigenvalues } (\lambda_1(x), \lambda_2(x)) \text{ of } C(x) \text{ belong to } L_{\text{bo}} \text{ a.e.} \big\}. \end{aligned}$$

The characterization of the effective set is the following

Theorem 2.3 *Under assumption* (2.2)*, if* A *and* B *are well ordered, i.e., if*

$$\alpha_1 \le \beta_1 \quad \textit{and} \quad \alpha_2 \le \beta_2,$$

then the effective set is $\mathcal{L}_{\text{wo}}$*, while if* A *and* B *are badly ordered, i.e., if either*

$$\alpha_1 \le \beta_1 \quad \textit{and} \quad \alpha_2 > \beta_2,$$

or

$$\alpha_1 > \beta_1 \quad \textit{and} \quad \alpha_2 \le \beta_2,$$

then the effective set is $\mathcal{L}_{\text{bo}}$.

As an obvious aside, note that, if $\alpha_1\alpha_2 = \beta_1\beta_2$, then, in view of (2.2), A and B are badly ordered if $A \ne B$.

Remark 2.4 The set L_{wo} is non-empty under the sole assumption (2.2), while the set L_{bo} is non-empty under the further assumption that A and B are badly ordered. Indeed, when (2.2) holds, the left-hand side of the second line of (2.4) is smaller than the right-hand side of its third line for $\lambda_1\lambda_2 = \alpha_1\alpha_2$ and for $\lambda_1\lambda_2 = \beta_1\beta_2$. The same holds for (2.5). Also the affine functions $Ad+B$ which appear in the inequalities (2.4) and (2.5) are positive on the interval $\alpha_1\alpha_2 \le \lambda_1\lambda_2 \le \beta_1\beta_2$ since they are positive at its extremities. The non-empty character of L_{wo} then follows from the fact that the inequality $(Ad+B)^2 \le C^2 d$ holds on an interval of $\mathbb{R}^+$whenever it holds at its end points. On the other hand, the non-empty character of L_{bo} is proved by re-writing the inequality $C^2 d \le (Ad+B)^2$ in the form $C^2(e-B)/A \le e^2$ and by observing that, if $AB \le 0$, the latter inequality holds on an interval of $\mathbb{R}^+$ whenever it holds at its end points. ¶

We now graphically represent the sets L_{wo} and L_{bo}. This we will do in two different representations respectively labeled the (λ_1, λ_2) and the (d, λ) representations.

In the two figures below, we choose, for the well ordered case, $\alpha_1 = 1 \le \beta_1 = 2$ and $\alpha_2 = 3 \le \beta_2 = 4$; for the first badly ordered case, $\alpha_1 = 1 \le \beta_1 = 2$ and $\alpha_2 = 4 > \beta_2 = 3$; for the second badly ordered case, $\alpha_1 = 2 > \beta_1 = 1$ and $\alpha_2 = 3 \le \beta_2 = 8$; note that (2.2) is satisfied in the three cases.

The (λ_1, λ_2) representation is the classical representation of the sets L_{wo} and L_{bo}, where each point (λ_1, λ_2) is represented as a pair of points P and P' with respective coordinates (λ_1, λ_2) and (λ_2, λ_1) which are symmetric with respect to the line $\lambda_1 = \lambda_2$. Figure 2.1 plots the three cases detailed above in that representation.

The (d, λ) representation represents each point (λ_1, λ_2) of the sets L_{wo} and L_{bo} as a pair of points P and P' with respective coordinates $(\lambda_1\lambda_2, \inf\{\lambda_1, \lambda_2\})$ and $(\lambda_1\lambda_2, \sup\{\lambda_1, \lambda_2\})$. The line $\lambda_1 = \lambda_2$ becomes the parabola $\lambda = \sqrt{d}$ and straight vertical lines represent matrices with equal determinant. The points P and P' are mapped onto one another through the map $(d, \lambda) \mapsto (d, d/\lambda)$. Once P is plotted, P' is graphically obtained as follows: intersect the straight vertical line going through P with the straight line going through the origin and the intersection point of the horizontal line going through P with the parabola $\lambda = \sqrt{d}$. Figure 2.2 plots the three cases detailed above in that representation.

Remark 2.5 The result of Theorem 2.3 (as well as most of the results announced in [5] and proved in the present paper) deals with mixtures of two symmetric conducting materials in two dimensions. Keeping the dimension equal to two, this result was extended in [4] to the case of mixtures of an arbitrary number of conducting materials. It was also extended in [8] to the case of mixtures of two non-symmetric materials. To this effect, Graeme Milton remarks that H-convergence is stable under the transformation $A \to (aA + b\,R^T)(cI + d\,R^T\,A)^{-1}$ where a, b, c and d are real

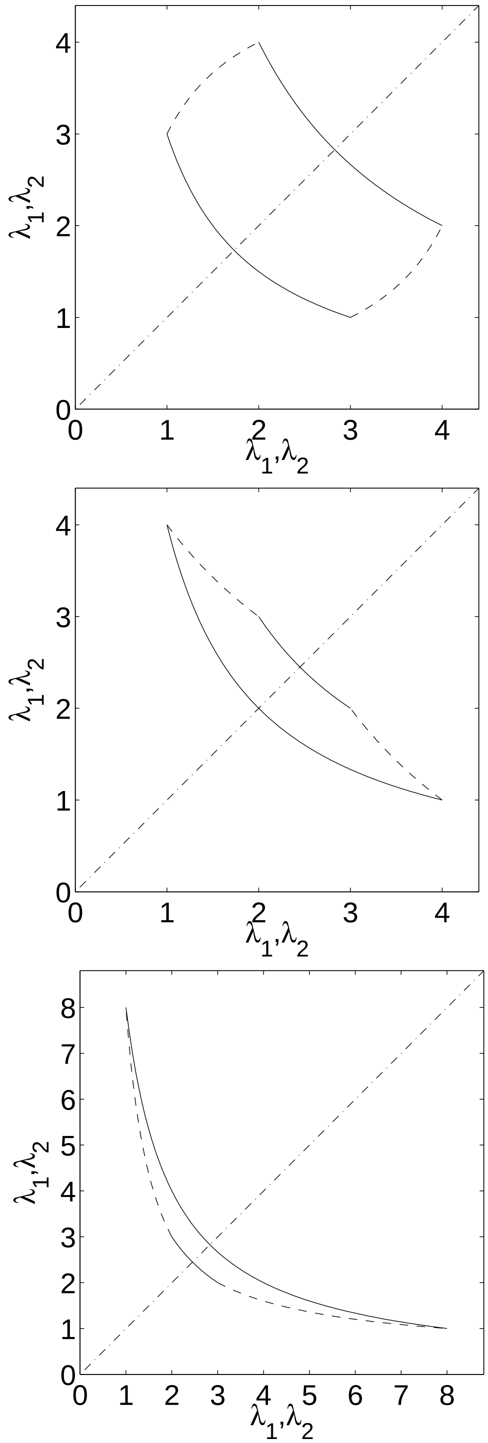

Figure 2.1: (λ_1, λ_2) representation: well-ordered case; badly ordered case $\alpha_2 > \beta_2$; badly ordered case $\alpha_1 > \beta_1$.

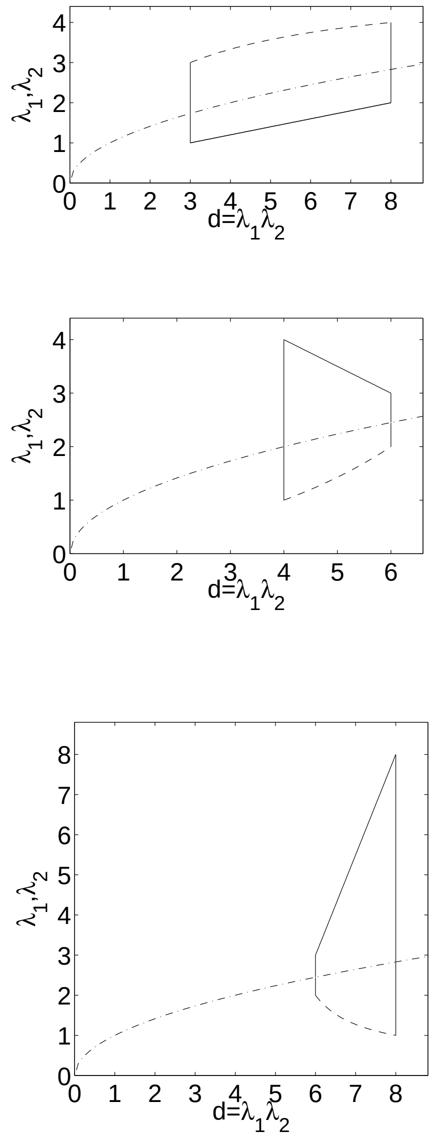

Figure 2.2: (d, λ) representation: well-ordered case; badly ordered case $\alpha_2 > \beta_2$; badly ordered case $\alpha_1 > \beta_1$.

numbers. Then a convenient choice of those parameters allows him to transform simultaneously two non-symmetric matrices into symmetric ones, to which he can apply Theorem 2.3 above. ¶

We conclude this Section by a result that will be used in the proof of Theorem 2.3 but which is also interesting in and of itself, namely the characterization of the sets of two-dimensional conducting materials which are stable under H-convergence, when the definition of the set is independent of the orientation of the material (or in other terms only depends on the eigenvalues of the material). This characterization will use the (d, λ) representation.

We first define the notion of stability under H-convergence through the following

Definition 2.6 *A subset $\mathcal{S}$ of $L^\infty(\mathbb{R}^2; \mathbb{M}_s(\alpha, \beta))$ is H-stable if and only if all H-limits of H-converging sequences of elements of $\mathcal{S}$ belong to $\mathcal{S}$.*

Consider γ and δ with $\alpha^2 \le \gamma \le \delta \le \beta^2$ and a positive and bounded function φ which is continuously differentiable on $[\gamma, \delta]$. We define the sets K and $\mathcal{K}$ by

$$K := \{(\lambda_1, \lambda_2) \in [\alpha, \beta]^2 : \gamma \le \lambda_1 \lambda_2 \le \delta, \frac{\lambda_1 \lambda_2}{\varphi(\lambda_1 \lambda_2)} \le \inf(\lambda_1 \lambda_2) \le \sup(\lambda_1 \lambda_2) \le \varphi(\lambda_1 \lambda_2)\}, \tag{2.7}$$

and

$$\mathcal{K} := \big\{C \in L^\infty(\mathbb{R}^2; \mathbb{M}_s(\alpha, \beta)) : \text{the eigenvalues } (\lambda_1(x), \lambda_2(x)) \text{ of } C(x) \text{ belong to } K \text{ a.e.}\big\}. \tag{2.8}$$

The following result characterizes the sets of this form which are stable under H-convergence.

Theorem 2.7 *The set $\mathcal{K}$ is H-stable if and only if*

$$\begin{cases} \textit{the function } d \in [\gamma, \delta] \to \varphi(d) \textit{ is concave,} \\ \textit{the function } d \in [\gamma, \delta] \to d/\varphi(d) \textit{ is convex.} \end{cases} \tag{2.9}$$

Remark 2.8 In the statement of Theorem 2.7, there is no loss of generality in assuming that the set K is of the form (2.7), when the definition of the set $\mathcal{K}$ is given by (2.8) for a set K which is sufficiently smooth, and whose definition is independent of the orientation of the material (or in other terms only depends on the eigenvalues of the material).

Indeed any subset K of $\mathbb{M}_s(\alpha,\beta)$ whose definition is given in terms of the eigenvalues can equivalently be represented as a set of pairs of points in the (d,λ) representation. Defining the functions ψ and φ by $\psi(d) := \inf\{\lambda_1,\lambda_2\}$ and $\varphi(d) := \sup\{\lambda_1,\lambda_2\}$ for all λ_1 and λ_2 in K with $\lambda_1\lambda_2 = d$, one necessarily has $\psi(d)\,\varphi(d) = d$ for every d, since, for every λ_1 and λ_2, $\inf\{\lambda_1,\lambda_2\}\sup\{\lambda_1,\lambda_2\} = \lambda_1\lambda_2$. Moreover, when K is a subset of $\mathbb{M}_s(\alpha,\beta)$, the functions ψ and φ are bounded from below by $\alpha > 0$ and from above by $\beta < \infty$. Finally let us assume for the sake of simplicity that the set of the d's for which there exists λ_1 and λ_2 in K with $\lambda_1\lambda_2 = d$ is the interval (γ,δ) with $\alpha^2 \le \gamma \le \delta \le \beta^2$ (this is what we mean by "a set K which is sufficiently smooth"; of course, more complex situations can be easily handled).

We now consider for every fixed $d \in (\gamma,\delta)$ the two constant materials $\mu_1 f_1\otimes f_1 + \mu_2 f_2 \otimes f_2$ and $\mu_2 f_1 \otimes f_1 + \mu_1 f_2\otimes f_2$, with $\mu_1 = \psi(d)$, $\mu_2 = \varphi(d)$ (so that $\mu_1\mu_2 = d$), and (f_1,f_2) an orthonormal basis of $\mathbb{R}^2$. It results from the first paragraph of Subsection 3.1 that the rank-one layering in direction f_1 (or in direction f_2) of those two materials with volume fraction θ $(0 \le \theta \le 1)$ of the first material produces, when θ varies, all the materials with constant effective conductivity $\lambda_1 f_1\otimes f_1 + \lambda_2 f_2 \otimes f_2$ where $\lambda_1\lambda_2 = d$ and $\psi(d) \le \inf\{\lambda_1,\lambda_2\} \le \sup\{\lambda_1,\lambda_2\} \le \varphi(d)$. Passing from constant materials to variable measurable materials with arbitrary orientation is a classical argument. Thus, if the set $\mathcal{K}$ defined by (2.8) from the set K is H-stable, the set $\mathcal{K}$ necessarily contains the set $\hat{\mathcal{K}}$ defined by

$$\hat{\mathcal{K}} := \Big\{ C \in L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta)) : \text{the eigenvalues } (\lambda_1(x),\lambda_2(x)) \text{ of } C(x) \text{ belong to } \hat{K} \text{ a.e.} \Big\},$$

where the set $\hat{K}$ is defined by

$$\hat{K} := \{(\lambda_1,\lambda_2) \in [\alpha,\beta]^2 : \gamma \le \lambda_1\lambda_2 \le \delta, \psi(\lambda_1\lambda_2) \le \inf(\lambda_1\lambda_2) \le \sup(\lambda_1\lambda_2) \le \varphi(\lambda_1\lambda_2)\}.$$

Since the result of Theorem 2.7 is about H-stable sets, there is therefore no loss of generality in assuming that the set K is of the form (2.7). ¶

Remark 2.9 We assumed that the positive bounded function φ is continously differentiable for the sake of simplicity. Actually, the proof given in Subsection 3.3 shows that if φ is continuous (or even measurable) and if the set $\mathcal{K}$ is H-stable, then the fonction $\varphi(d)$ is concave and the fonction $d/\varphi(d)$ is convex, while conversely assuming that the fonction $\varphi(d)$ is concave and that the fonction $d/\varphi(d)$ is convex suffices to prove that the set $\mathcal{K}$ is H-stable. ¶

3 Proofs

3.1 Layering and attainability

This Subsection provides a rapid proof of the attainability of the sets $\mathcal{L}_{\text{wo}}$ and $\mathcal{L}_{\text{bo}}$ introduced in Definition 2.2.

Lemma 3.1 *Under assumption* (2.2), *every element of the sets* $\mathcal{L}_{\text{wo}}$ *and* $\mathcal{L}_{\text{bo}}$ *is the H-limit of a sequence of conductivities of the form* (2.3).

Proof. Recall (see e.g. [15]) that, if (f_1, f_2) is an orthonormal basis of $\mathbb{R}^2$, the rank-one layering of $\gamma_1 f_1 \otimes f_1 + \gamma_2 f_2 \otimes f_2$ with $\delta_1 f_1 \otimes f_1 + \delta_2 f_2 \otimes f_2$ in direction f_1 with volume fraction θ of the first material produces the effective conductivity

$$(\theta/\gamma_1 + (1-\theta)/\delta_1)^{-1} f_1 \otimes f_1 + (\theta\gamma_2 + (1-\theta)\delta_2) f_2 \otimes f_2,$$

while the rank-one layering in direction f_2 with volume fraction θ of the first material produces the effective conductivity

$$(\theta\gamma_1 + (1-\theta)\delta_1) f_1 \otimes f_1 + (\theta/\gamma_2 + (1-\theta)/\delta_2)^{-1} f_2 \otimes f_2.$$

Note that, whenever $\gamma_1\gamma_2 = \delta_1\delta_2$, then the effective conductivities produced above also share that common value for their determinant.

Now, take C to be a constant element of $\mathcal{L}_{\text{wo}}$ (resp. $\mathcal{L}_{\text{bo}}$), with eigenvectors (f_1, f_2) and eigenvalues (λ_1, λ_2) satisfying the equalities in the first inequality of the second line and in the last inequality of the third line of (2.4) (resp. (2.5)) (and which therefore belong to the boundary of L_{wo} (resp. L_{bo})) (see Figures 2.1 and 2.2 for a pictorial representation of those boundaries). It is easily checked that C is the effective conductivity associated to the rank-one layering, for some volume fraction $\theta \in [0, 1]$, of $\alpha_1 f_1 \otimes f_1 + \alpha_2 f_2 \otimes f_2$ with $\beta_1 f_1 \otimes f_1 + \beta_2 f_2 \otimes f_2$ in direction f_1 (resp. f_2). Note that, as θ varies between 0 and 1, the determinant of the effective conductivity resulting from this layering varies continuously between $\alpha_1\alpha_2$ and $\beta_1\beta_2$.

Then, take C to be a constant element of $\mathcal{L}_{\text{wo}}$ (resp. $\mathcal{L}_{\text{bo}}$), with eigenvalues (λ_1, λ_2) in L_{wo} (resp. L_{bo}) and associated eigenvectors (f_1, f_2). Then its determinant $\lambda_1\lambda_2$ lies between $\alpha_1\alpha_2$ and $\beta_1\beta_2$ and thus, the hyperbola $xy = \lambda_1\lambda_2$ intersects the boundary of L_{wo} (resp. L_{bo}) at two points (μ_1, μ_2), and (μ_2, μ_1) with $\mu_1\mu_2 = \lambda_1\lambda_2$ (see once again Figures 2.1 and 2.2). It then suffices to layer $\mu_1 f_1 \otimes f_1 + \mu_2 f_2 \otimes f_2$ with $\mu_2 f_1 \otimes f_1 + \mu_1 f_2 \otimes f_2$ in direction f_1 or f_2 to generate all conductivities that are diagonal in the basis (f_1, f_2) with eigenvalues on the hyperbola $xy = \lambda_1\lambda_2$ inside L_{wo} (resp. L_{bo}).

In conclusion, a rank-2 lamination (see e.g. [16]) permits one to recover all constant elements of $\mathcal{L}_{\text{wo}}$ and of $\mathcal{L}_{\text{bo}}$ as effective conductivities associated to a mixture of the A-material with the B-material. Passing from constant elements to arbitrary elements of $\mathcal{L}_{\text{WO}}$ and of $\mathcal{L}_{\text{bo}}$ is by now a classical argument based on the local and metrizable character of H-convergence; we refer the interested reader to e.g. [16]. ¶

3.2 First results

The proofs below will appeal to a few known results that we briefly collect in

Lemma 3.2 *Let A_ε and B_ε be two sequences in $L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta))$ which respectively H-converge to A_0 and B_0 (also in $L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta))$). Then*

1. *if $A_\varepsilon \le B_\varepsilon$ a.e., then $A_0 \le B_0$ a.e. (see e.g. [11]);*
2. *if $A_\varepsilon \rightharpoonup \overline{A}$ weakly-$\star$ in $L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta))$, then $A_0 \le \overline{A}$ a.e., while, if $A_\varepsilon^{-1} \rightharpoonup \underline{A}^{-1}$ weakly-$\star$ in $L^\infty(\mathbb{R}^2;\mathbb{M}_s(\beta^{-1},\alpha^{-1}))$, then $\underline{A} \le A_0$ a.e. (see e.g. [11]);*
3. *for every open bounded subset Ω of $\mathbb{R}^2$, if $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \ge 0$, and $v_\varepsilon \rightharpoonup v$ weakly in $H^1(\Omega)$, then [3]*
$$\liminf_{\varepsilon} \int_\Omega \varphi A_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon dx \ge \int_\Omega \varphi A_0 \nabla v . \nabla v dx;$$
4. *$A_\varepsilon / \det A_\varepsilon \overset{H}{\rightharpoonup} A_0 / \det A_0$ (this famous result is generally attributed to J.B. Keller or A.M. Dykhne; a proof can be found in e.g. [5]);*
5. *if $\det A_\varepsilon \ge \gamma$ (resp. $\det A_\varepsilon \le \gamma$), then $\det A_0 \ge \gamma$ (resp. $\det A_0 \le \gamma$) (this results from items 1 and 4 above).*

We then prove preliminary results, namely the stability by H-convergence of special sets.

For a and b in $\mathbb{R}$, we introduce the following sets (note that they are restrictions on a and b in order for those sets to be non-empty)

$$L^{\ge}(a,b) := \left\{ (\lambda_1,\lambda_2) \in [\alpha,\beta]^2 : \inf\{\lambda_1,\lambda_2\} \ge a\lambda_1\lambda_2 + b \right\},$$
$$L^{\le}(a,b) := \left\{ (\lambda_1,\lambda_2) \in [\alpha,\beta]^2 : \sup\{\lambda_1,\lambda_2\} \le a\lambda_1\lambda_2 + b \right\}.$$

Then we define

$$\mathcal{L}^{\ge}(a,b) := \big\{ C \in L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta)) : \text{the eigenvalues } (\lambda_1(x),\lambda_2(x)) \text{ of } C(x) \text{ belong to } L^{\ge}(a,b) \text{ a.e.} \big\},$$

$$\mathcal{L}^{\le}(a,b) = \big\{ C \in L^\infty(\mathbb{R}^2;\mathbb{M}_s(\alpha,\beta)) : \text{the eigenvalues } (\lambda_1(x),\lambda_2(x)) \text{ of } C(x) \text{ belongs to } L^{\le}(a,b) \text{ a.e.} \big\}.$$

Remark 3.3 Since the set $L^{\geq}(a,b)$ is defined by $1 \geq a\lambda_1 + b/\lambda_2$ and $1 \geq a\lambda_2 + b/\lambda_1$, the set $\mathcal{L}^{\geq}(a,b)$ is equivalently defined as

$$\mathcal{L}^{\geq}(a,b) = \left\{ A \in L^\infty(\mathbb{R}^2; \mathbb{M}_s(\alpha,\beta)) : I \geq aA(x) + bA(x)/\det A(x) \text{ a.e.} \right\}.$$

Similarly, the set $\mathcal{L}^{\leq}(a,b)$ is equivalently defined as

$$\mathcal{L}^{\leq}(a,b) = \left\{ A \in L^\infty(\mathbb{R}^2; \mathbb{M}_s(\alpha,\beta)) : I \leq aA(x) + bA(x)/\det A(x) \text{ a.e.} \right\}.$$

¶

Then

Lemma 3.4 *If $a \geq 0$ and $b \geq 0$, then $\mathcal{L}^{\geq}(a,b)$ is H-stable, while, if $ab \leq 0$, $\mathcal{L}^{\leq}(a,b)$ is H-stable.*

Proof. Throughout this proof we set $C_\varepsilon := A_\varepsilon / \det A_\varepsilon$.

Consider first the case where a and b are both non-negative. Then, according to Remark 3.3, together with item 4 of Lemma 3.2, it suffices to show that the relation

$$I \geq aA_\varepsilon + b\, C_\varepsilon \tag{3.1}$$

is preserved by H-convergence. But, passing to the weak-$\star$ limit in (3.1), we obtain

$$I \geq a\overline{A} + b\, \overline{C}$$

in the notation of item 2 of Lemma 3.2. Since, according to that same item and to item 4,

$$\overline{A} \geq A_0, \quad \overline{C} \geq C_0 = A_0/\det A_0,$$

the H-stability of $\mathcal{L}^{\geq}(a,b)$ is established when $a \geq 0$ and $b \geq 0$.

Consider now the case where $ab \leq 0$. As in the previous proof, it suffices to show that the relation

$$I \leq aA_\varepsilon + bC_\varepsilon \tag{3.2}$$

is preserved by H-convergence. Since A_ε and C_ε play symmetric roles (indeed $C_\varepsilon/\det C_\varepsilon = A_\varepsilon$), we may as well investigate only the case $a \geq 0$, $b \leq 0$. Consider for every open bounded subset Ω of $\mathbb{R}^2$ a sequence $u_\varepsilon \in H^1(\Omega)$ satisfying, for some $\lambda \in \mathbb{R}^2$,

$$\begin{cases} \nabla u_\varepsilon \rightharpoonup \lambda \text{ weakly in } L^2(\Omega;\mathbb{R}^2), \\ A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0\lambda \text{ weakly in } L^2(\Omega;\mathbb{R}^2), \\ \text{div } A_\varepsilon \nabla u_\varepsilon \text{ lies in a compact set of } H^{-1}(\Omega); \end{cases}$$

the existence of such a sequence (called a corrector sequence) is well-known; see e.g. [11].

Then, for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$, (3.2) implies that

$$\int_\Omega \varphi |\nabla u_\varepsilon|^2 dx - b \int_\Omega \varphi\, C_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon dx \leq a \int_\Omega \varphi A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon dx. \tag{3.3}$$

But the lower semi-continuity for the first term, the assumption that $b \leq 0$, together with items 3 and 4 of Lemma 3.2 for the second term, and finally the div-curl lemma (see [14]) for the third term yield

$$\int \varphi \lambda^2 dx - b \int \varphi C_0 \lambda . \lambda dx \leq \int \varphi A_0 \lambda . \lambda dx.$$

In view of item 4 of Lemma 3.2, the arbitrariness of Ω, φ and λ permits us to conclude to the H-stability of $\mathcal{L}^{\leq}(a, b)$ when $ab \leq 0$. ¶

3.3 Proof of Theorem 2.7

Define the function $\hat{\varphi}$ by $\hat{\varphi}(d) = d/\varphi(d)$.

Let us first prove that the set $\mathcal{K}$ is H-stable when (2.9) holds. Under this assumption, there exists three sequences of real numbers $(d_n, z_n, \hat{z}_n)$ such that

$$\begin{cases} d_n \in [\gamma, \delta], \\ \hat{z}_n \varphi(d_n) + z_n \hat{\varphi}(d_n) = 1, \\ (z_n, \hat{z}_n) \notin (-\infty, 0)^2, \\ \varphi(d) = \inf_n \{ z_n d + \hat{z}_n \varphi^2(d_n) \},\ \forall d \in [\gamma, \delta], \\ \hat{\varphi}(d) = \sup_n \{ \hat{z}_n d + z_n \hat{\varphi}^2(d_n) \},\ \forall d \in [\gamma, \delta]. \end{cases} \tag{3.4}$$

Indeed, define

$$z_n := \varphi'(d_n),\ \hat{z}_n := \hat{\varphi}'(d_n).$$

Since $\hat{\varphi}(d)\varphi(d) = d$,

$$\hat{\varphi}'(d_n)\varphi(d_n) + \hat{\varphi}(d_n)\varphi'(d_n) = 1,$$

hence

$$\begin{cases} \hat{\varphi}'(d_n)\varphi^2(d_n) + d_n \varphi'(d_n) = \varphi(d_n), \\ \varphi'(d_n)\hat{\varphi}^2(d_n) + d_n \hat{\varphi}'(d_n) = \hat{\varphi}(d_n). \end{cases}$$

This implies that

$$\begin{cases} z_n d + \hat{z}_n \varphi^2(d_n) = \varphi(d_n) + \varphi'(d_n)(d - d_n), \\ \hat{z}_n d + z_n \hat{\varphi}^2(d_n) = \hat{\varphi}(d_n) + \hat{\varphi}'(d_n)(d - d_n). \end{cases}$$

So, the functions wich appear in the infimum and in the supremum in the last two lines of (3.4) are in fact the tangent line to $\varphi(d)$ passing through the point $(d_n, \varphi(d_n))$ and the tangent line to $\hat{\varphi}(d)$ passing through the point $(d_n, \hat{\varphi}(d_n))$. But, since the function φ is assumed to be concave, while the function $\hat{\varphi}$ is assumed to be convex, we can choose a countable set of points $d_n \in [\gamma, \delta]$ such that φ is the infimum of its tangent lines through the points $(d_n, \varphi(d_n))$, while $\hat{\varphi}$ is the supremum of its tangent lines through the points $(d_n, \varphi(d_n))$. Finally, z_n and $\hat{z}_n$ cannot be both negative, since $\hat{z}_n\varphi(d_n) + z_n\hat{\varphi}(d_n) = 1$ while φ and $\hat{\varphi}$ are positive.

In view of (3.4), of the definition of the function $\hat{\varphi}$ and of the definitions of the sets $L^{\geq}(a,b)$ and $L^{\leq}(a,b)$, the set K defined in (2.7) is equivalently defined as

$$K = \left\{(\lambda_1, \lambda_2) \in [\alpha, \beta]^2 : \gamma \leq \lambda_1\lambda_2 \leq \delta\right\} \bigcap_n \left\{L^{\geq}(\hat{z}_n, z_n\hat{\varphi}^2(d_n)) \cap\ L^{\leq}(z_n, \hat{z}_n\varphi^2(d_n))\right\}. \tag{3.5}$$

According to item 5 of Lemma 3.2, the first set in (3.5) is H-stable. In view of Lemma 3.4 and of the third line of (3.4), the H-stability of the remaining sets in (3.5) will be ensured, provided that we show that

(i) if $z_n\hat{z}_n \leq 0$, then $L^{\leq}(z_n, \hat{z}_n\varphi^2(d_n)) \subset L^{\geq}(\hat{z}_n, z_n\hat{\varphi}^2(d_n))$;

(ii) if $z_n \geq 0$ and $\hat{z}_n \geq 0$, then $L^{\geq}(\hat{z}_n, z_n\hat{\varphi}^2(d_n)) \subset L^{\leq}(z_n, \hat{z}_n\varphi^2(d_n))$.

To this effect, we first consider the case where $\lambda_1\lambda_2 = d_n$. In this case, assuming, with no loss of generality, that $\lambda_1 \leq \lambda_2$, we have

$$\lambda_1 \geq \hat{\varphi}(d_n) \quad \text{if and only if} \quad \lambda_2 \leq \varphi(d_n),$$

but in view of the second line of (3.4),

$$\begin{cases} \hat{\varphi}(d_n) = \hat{z}_n d_n + z_n\hat{\varphi}^2(d_n) = \hat{z}_n\lambda_1\lambda_2 + z_n\hat{\varphi}^2(d_n), \\ \varphi(d_n) = z_n d_n + \hat{z}_n\varphi^2(d_n) = z_n\lambda_1\lambda_2 + \hat{z}_n\varphi^2(d_n), \end{cases}$$

so that

$$\lambda_1 \geq \hat{z}_n\lambda_1\lambda_2 + z_n\hat{\varphi}^2(d_n) \quad \text{if and only if} \quad \lambda_2 \leq z_n\lambda_1\lambda_2 + \hat{z}_n\varphi^2(d_n).$$

In other words, when $\lambda_1\lambda_2 = d_n$, the equality

$$L^{\geq}(\hat{z}_n, z_n\hat{\varphi}(d_n)) = L^{\leq}(z_n, \hat{z}_n\varphi^2(d_n))$$

holds independently of the signs of z_n and $\hat{z}_n$. Thus, assertions (i) and (ii) are proved when $\lambda_1\lambda_2 = d_n$.

We then pass to the case where $\lambda_1\lambda_2 = d \neq d_n$. In view of the second line of (3.4),

$$\left(z_n d + \hat{z}_n \varphi^2(d_n)\right)\left(\hat{z}_n d + z_n \hat{\varphi}^2(d_n)\right) = d + z_n \hat{z}_n (d - d_n)^2,$$

and therefore, when $d \neq d_n$,

$$\left(z_n d + \hat{z}_n \varphi^2(d_n)\right)\left(\hat{z}_n d + z_n \hat{\varphi}^2(d_n)\right) \leq d \quad \text{if and only if} \quad z_n \hat{z}_n \leq 0. \tag{3.6}$$

We first consider the case (i) where $z_n \hat{z}_n \leq 0$, and we assume, with no loss of generality, that $0 < \lambda_1 \leq \lambda_2$. If $(\lambda_1, \lambda_2) \in L^{\leq}(z_n, \hat{z}_n \varphi^2(d_n))$, then

$$\lambda_2 \leq z_n \lambda_1 \lambda_2 + \hat{z}_n \varphi^2(d_n),$$

and, by virtue of (3.6),

$$\left(z_n \lambda_1 \lambda_2 + \hat{z}_n \varphi^2(d_n)\right)\left(\hat{z}_n \lambda_1 \lambda_2 + z_n \hat{\varphi}^2(d_n)\right) \leq \lambda_1 \lambda_2 \leq \lambda_1 \left(z_n \lambda_1 \lambda_2 + \hat{z}_n \varphi^2(d_n)\right).$$

Dividing by $(z_n \lambda_1 \lambda_2 + \hat{z}_n \varphi^2(d_n))$, which is positive since $0 < \lambda_1 \leq \lambda_2 \leq z_n \lambda_1 \lambda_2 + \hat{z}_n \varphi^2(d_n)$, we obtain

$$\hat{z}_n \lambda_1 \lambda_2 + z_n \hat{\varphi}^2(d_n) \leq \lambda_1,$$

or in other words, since $\lambda_2 \geq \lambda_1$,

$$(\lambda_1, \lambda_2) \in L^{\geq}(\hat{z}_n, z_n \hat{\varphi}^2(d_n)).$$

Thus, assertion (i) is proved when $\lambda_1 \lambda_2 \neq d_n$.

Assertion (ii) is proved in a similar manner when $\lambda_1 \lambda_2 \neq d_n$.

We conclude that $\mathcal{K}$ is H-stable as a countable intersection of H-stable sets when (2.9) holds.

Conversely let us prove that the function $\varphi(d)$ is concave and that the function $d/\varphi(d)$ is convex if the set $\mathcal{K}$ is H-stable.

To this effect we consider two constant materials $\gamma_1 f_1 \otimes f_1 + \gamma_2 f_2 \otimes f_2$ and $\delta_1 f_1 \otimes f_1 + \delta_2 f_2 \otimes f_2$, where (f_1, f_2) is an orthonormal basis of $\mathbb{R}^2$, and we set $c = \gamma_1 \gamma_2$ and $d = \delta_1 \delta_2$. As recalled in Subsection 3.1, the rank-one layering in direction f_1 of those two materials with volume fraction θ ($0 \leq \theta \leq 1$) of the first material produces the material with effective conductivity $\mu_1 f_1 \otimes f_1 + \mu_2 f_2 \otimes f_2$, where

$$1/\mu_1 := 1/\left(\theta/\gamma_1 + (1-\theta)/\delta_1\right), \ \mu_2 := \theta \gamma_2 + (1-\theta)\delta_2.$$

We set $m = \mu_1 \mu_2$. Note that when θ varies between 0 and 1, these formulas imply that m varies between d and c, while μ_1 is an affine function of m.

If the two materials belong to $\mathcal{K}$, *i.e.*, if (γ_1, γ_2) and (δ_1, δ_2) belong to K, and if the set $\mathcal{K}$ is H-stable, the effective material defined above should belong to $\mathcal{K}$, *i.e.*, (μ_1, μ_2) should belong to K. Therefore, in the (d, λ) representation (see Figure 2.2), the line segment which joins the points (c, γ_1) and (d, δ_1) should lie between the curves $\varphi(m)$ and $m/\varphi(m)$ on the interval $c \leq m \leq d$. Taking $\gamma_1 = \varphi(c)$ and $\delta_1 = \varphi(d)$, and varying c and d between γ and δ, this implies that the fonction $\varphi(m)$ is concave on the interval $\gamma \leq m \leq \delta$. Similarly taking $\gamma_1 = c/\varphi(c)$ and $\delta_1 = d/\varphi(d)$ implies that the fonction $m/\varphi(m)$ is convex on the interval $\gamma \leq m \leq \delta$.

The proof of Theorem 2.7 is now complete.

3.4 Proof of Theorem 2.3

Theorem 2.3 is an immediate consequence of Lemma 3.1 and of Theorem 2.7. Indeed, Lemma 3.1 asserts that the sets $\mathcal{L}_{\text{wo}}$ and $\mathcal{L}_{\text{bo}}$ are subsets of the effective set of mixtures of the A-material with the B-material – in others words of the H-closure of conductivites of the form (2.3) – that contain the original materials A and B, while Theorem 2.7 allows one to easily show that the set $\mathcal{L}_{\text{wo}}$ defined through inequalities (2.4) (resp. $\mathcal{L}_{\text{bo}}$ defined through inequalities (2.5)) is H-stable when A and B are well ordered (resp. when A and B are badly ordered).

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