

# ON CONSERVATION LAWS AND NECESSARY CONDITIONS IN THE CALCULUS OF VARIATIONS

**G. Francfort**

L.P.M.T.M., Université Paris 13,  
93430 Villetaneuse, France.

**J. Sivaloganathan**

Department of Mathematical Sciences,  
University of Bath,  
Bath BA2 7AY, UK.

ABSTRACT. It is well known from the work of Noether that every variational symmetry of an integral functional gives rise to a corresponding conservation law. In this paper we prove that each such conservation law arises directly as the Euler-Lagrange equation for the functional on taking suitable variations around a minimiser.

## 1. INTRODUCTION

Classically, since the pioneering work of E. NOETHER [9], conservation laws associated to stationary points of an integral functional are obtained upon writing the invariance of that functional under a suitable one-parameter group of transformations and combining the result with the Euler-Lagrange equations that express stationarity (see e.g. [10]). Such an approach implicitly assumes that the stationary points of the functional are smooth enough for all chain rules to be licit; it also presupposes that the Euler-Lagrange equations are indeed satisfied.

More recently, C.B. MORREY (see e.g. [8]) introduced the notion of quasiconvexity which was later developed by J.M. BALL [2] to obtain a variational approach to static hyperelasticity. A difficulty in that variational approach is the current inability to prove that minimizers of the potential energy satisfy the associated Euler-Lagrange equations <sup>1</sup>. A major difficulty is that the usually adopted restriction that kinematically admissible displacement fields  $\mathbf{u}$  should satisfy

$$\det \nabla \mathbf{u} > 0 \quad a.e.$$

prohibits the usual smooth variations  $\mathbf{u}_0 + t\phi$ ,  $t \in \mathbb{R}$ ,  $\phi \in C_0^1(\Omega)$ , around a minimizer  $\mathbf{u}_0 \in W^{1,1}(\Omega)$ .

In spite of that obstacle, J.M. BALL showed in [1] that conservation of the energy-momentum tensor - a seminal conservation law associated to the homogeneous character of the energy density - could be obtained for minimizers of the potential energy, even though the Euler-Lagrange equations might not be satisfied.

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<sup>1</sup>Partial regularity results for quasiconvex integrands have been obtained by Evans [7], but under hypotheses which are incompatible with the usual growth requirement (4.3) in nonlinear elasticity.

Ball obtained this result by using “inner” variations around  $\mathbf{u}_0$ , that is variations of the form  $\mathbf{u}_0(\mathbf{x} + t\phi(\mathbf{x}))$ ,  $t \in \mathbb{R}$ ,  $\phi \in C_0^1(\Omega)$ .

In this short paper, we propose to generalize the above quoted result and to show that a suitable choice of more general one-parameter groups of transformations will directly generate a host of conservation laws (in weak form) for minimizers of the energy, the Euler-Lagrange equations notwithstanding.

In section 2, we recall the classical Noetherian approach to conservation laws. In Section 3, we demonstrate how more general one-parameter groups of transformations permit us to circumvent the Euler-Lagrange equations and directly derive conservation laws for energy minimizers. In Section 4, we apply this approach to the specific setting of hyperelasticity and recover in particular J.M. BALL’s results. The final section is very short and points to a few possible extensions of our method.

Finally, we should stress a strong bias in favor of conciseness. We thus chose not to spell out all smoothness and/or growth assumptions on the energy densities, since these may vary with the specific kind of transformations under consideration. The reader is invited to consult the relevant references for a description of the precise assumptions on those densities, given a particular invariance.

For further background and references on conservation laws and applications, we refer to the comprehensive text of Olver [10]. In this paper we adopt the notation used in [10] and will also refer to key calculations contained therein.

## 2. VARIATIONAL SYMMETRIES

Let  $\Omega \subset \mathbb{R}^m$  be a domain and consider the integral functional

$$(2.1) \quad E(\mathbf{u}; \Omega') = \int_{\Omega'} L(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \, dx$$

where  $L : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$  is a given Caratheodory integrand,  $\Omega'$  is an arbitrary open subdomain of  $\Omega$ , and  $E$  is defined on maps  $\mathbf{u} : \Omega' \rightarrow \mathbb{R}^n$  which are (at least) in  $W^{1,1}(\Omega'; \mathbb{R}^n)$ .

Consider a smooth one-parameter group of transformations of independent and dependent variables given by

$$(2.2) \quad (\mathbf{x}, \mathbf{u}) \rightarrow (\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = (\Xi_t(\mathbf{x}, \mathbf{u}), \Phi_t(\mathbf{x}, \mathbf{u}))$$

where  $t \in \mathbb{R}$  is the group parameter and  $t = 0$  corresponds to the identity transformation (so that  $\Xi_0(\mathbf{x}, \mathbf{u}) \equiv \mathbf{x}$  and  $\Phi_0(\mathbf{x}, \mathbf{u}) \equiv \mathbf{u}$ .) We will sometimes write  $(\Xi_t(\mathbf{x}, \mathbf{u}), \Phi_t(\mathbf{x}, \mathbf{u})) = (\Xi(\mathbf{x}, \mathbf{u}, t), \Phi(\mathbf{x}, \mathbf{u}, t))$  to emphasize the role of the parameter  $t$ , in which case it will be assumed that  $\Xi, \Phi$  are smooth with respect to all their arguments.

An important concept when working with such groups of transformations is the notion of the *infinitesimal generator* of the group which is the differential operator defined by <sup>2</sup>

$$(2.3) \quad \mathbf{v} = \xi^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^\alpha} + \phi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^i},$$

<sup>2</sup>Here and in the rest of this paper we use the convention of summing over repeated indices.

where  $\xi(\mathbf{x}, \mathbf{u}) := \frac{d}{dt} \Xi(\mathbf{x}, \mathbf{u}, t) \Big|_{t=0}$  and  $\phi(\mathbf{x}, \mathbf{u}) := \frac{d}{dt} \Phi(\mathbf{x}, \mathbf{u}, t) \Big|_{t=0}$  (see [10] for further details).

**Remark 2.1.** Hence, in particular, given a real valued function  $\mathbf{F}(\mathbf{x}, \mathbf{u})$  we obtain

$$\frac{d}{dt} \mathbf{F}(\Xi_t(\mathbf{x}, \mathbf{u}), \Phi_t(\mathbf{x}, \mathbf{u})) \Big|_{t=0} = \mathbf{v}(\mathbf{F}(\mathbf{F}(\mathbf{x}, \mathbf{u}))) .$$

Given any particular subdomain  $\Omega' \subset \Omega$ , and any map  $\mathbf{u} : \Omega' \rightarrow \mathbb{R}^n$ , the change of variables given by (2.2) induces a new transformed function  $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})$ ,  $\tilde{\mathbf{u}} : \tilde{\Omega}' \rightarrow \mathbb{R}^n$  where  $\tilde{\Omega}' = (\Xi_t \circ (id \times \mathbf{u}))(\Omega')$ , which is obtained as follows: first note that  $\tilde{\mathbf{x}} = \Xi_t(\mathbf{x}, \mathbf{u}(\mathbf{x})) = (\Xi_t \circ (id \times \mathbf{u}))(\mathbf{x})$ . By our assumptions on the group of transformations it follows, upon application of the Implicit Function Theorem, that this relation can be inverted for small  $t$  to yield  $\mathbf{x} = (\Xi_t \circ (id \times \mathbf{u}))^{-1}(\tilde{\mathbf{x}})$ . The new transformed function  $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})$  depends on  $t$  and is given by

$$(2.4) \quad \tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = [\Phi_t \circ (id \times \mathbf{u})] \circ [\Xi_t \circ (id \times \mathbf{u})]^{-1}(\tilde{\mathbf{x}}) .$$

Through the above construction, the group of transformations (2.2) naturally extends to a corresponding action on the first derivatives  $\nabla_{\mathbf{x}} \mathbf{u}$  (which are mapped to  $\nabla_{\tilde{\mathbf{x}}} \tilde{\mathbf{u}}$ ). It can be shown that this extended action

$$(\mathbf{x}, \mathbf{u}, \nabla_{\mathbf{x}} \mathbf{u}) \rightarrow (\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \nabla_{\tilde{\mathbf{x}}} \tilde{\mathbf{u}})$$

has infinitesimal generator  $\mathbf{v}^{(1)}$  (known as the first prolongation of  $\mathbf{v}$ ) given by

$$(2.5) \quad \mathbf{v}^{(1)} = \xi^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^\alpha} + \phi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^i} + \pi_\alpha^i(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \frac{\partial}{\partial F_\alpha^i}$$

where

$$(2.6) \quad \pi_\alpha^i(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) = \frac{d}{dt} \frac{\partial \tilde{u}^i(\tilde{\mathbf{x}})}{\partial \tilde{x}^\alpha} \Big|_{t=0} = \frac{\partial}{\partial x^\alpha} [\phi^i(\mathbf{x}, \mathbf{u}(\mathbf{x}))] - \frac{\partial u^i}{\partial x^\beta} \frac{\partial}{\partial x^\alpha} [\xi^\beta(\mathbf{x}, \mathbf{u}(\mathbf{x}))]$$

(see [10] page 114).

Following standard terminology (see e.g. Definition 4.10 in [10]) we say that (2.2) is a variational symmetry of (2.1) if for any subdomain  $\Omega' \subset \Omega$  and any map  $\mathbf{u} : \Omega' \rightarrow \mathbb{R}^n$  we have

$$\int_{\Omega'} L(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) dx = \int_{\tilde{\Omega}'} L(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}(\tilde{\mathbf{x}}), \nabla \tilde{\mathbf{u}}(\tilde{\mathbf{x}})) d\tilde{x} .$$

It then follows that (2.2) is a variational symmetry if and only if

$$(2.7) \quad \frac{d}{dt} \int_{\tilde{\Omega}'} L(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \nabla_{\tilde{\mathbf{x}}} \tilde{\mathbf{u}}(\tilde{\mathbf{x}})) \Big|_{t=0} = 0$$

for any map  $\mathbf{u} : \Omega' \rightarrow \mathbb{R}^n$ . It follows from theorem 4.12 in [10] that

$$(2.8) \quad \frac{d}{dt} \int_{\tilde{\Omega}'} L(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \nabla_{\tilde{\mathbf{x}}} \tilde{\mathbf{u}}(\tilde{\mathbf{x}})) \Big|_{t=0} = \int_{\Omega'} \mathbf{v}^{(1)} L(\mathbf{x}, \mathbf{u}, \nabla_{\mathbf{x}} \mathbf{u}(\mathbf{x})) + L(\mathbf{x}, \mathbf{u}, \nabla_{\mathbf{x}} \mathbf{u}(\mathbf{x})) \text{Div } \xi(\mathbf{x}, \mathbf{u}(\mathbf{x})) dx ,$$

where  $\text{Div } \xi(\mathbf{x}, \mathbf{u}(\mathbf{x})) = \frac{\partial}{\partial x^\beta} (\xi^\beta(\mathbf{x}, \mathbf{u}(\mathbf{x})))$ . From (2.7) and the arbitrariness of  $\Omega'$ , it now follows that the integrand on the right hand side must vanish identically and

hence by (2.5) that

$$(2.9) \quad 0 = \left[ \xi^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^\alpha} + \phi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^i} + \pi_\alpha^i(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \frac{\partial}{\partial F_\alpha^i} \right] L(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) + \left[ \frac{\partial}{\partial x^\alpha} \xi^\alpha(\mathbf{x}, \mathbf{u}(\mathbf{x})) \right] L(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) .$$

Using (2.6), we then obtain that

$$(2.10) \quad \phi^i(\mathbf{x}, \mathbf{u}(\mathbf{x})) \frac{\partial L}{\partial u^i} + \frac{\partial}{\partial x^\alpha} (\phi^i(\mathbf{x}, \mathbf{u}(\mathbf{x}))) \frac{\partial L}{\partial F_\alpha^i} + \xi^\alpha \frac{\partial L}{\partial x^\alpha} + \left( \frac{\partial}{\partial x^\alpha} \xi^\beta(\mathbf{x}, \mathbf{u}(\mathbf{x})) \right) M_\alpha^\beta = 0 ,$$

for any map  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$ , where

$$M_\alpha^\beta(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) = \left[ L(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \delta_\alpha^\beta - \frac{\partial u^k}{\partial x^\beta} \frac{\partial L}{\partial F_\alpha^k}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \right]$$

are the components of the  $m \times m$  tensor

$$\mathbf{M}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) = \left[ L(\mathbf{x}, \mathbf{u}, \nabla, \mathbf{u}) \mathbf{I} - (\nabla \mathbf{u})^T \frac{\partial L}{\partial \mathbf{F}}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \right]$$

which is known as the *energy-momentum* tensor.

**Remark 2.2.** Note that the condition that (2.2) be a variational symmetry is expressible as a pointwise condition in the following way. Since (2.9) holds for all maps  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$ , given any triple  $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times m}$ , we can find a map  $\mathbf{u}_0(\cdot)$  such that  $(\mathbf{x}, \mathbf{u}_0(\mathbf{x}), \nabla \mathbf{u}_0(\mathbf{x})) = (\mathbf{x}, \mathbf{u}, \mathbf{F})$ . It then follows that

$$\left[ \xi^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^\alpha} + \phi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^i} + \pi_\alpha^i(\mathbf{x}, \mathbf{u}, \mathbf{F}) \frac{\partial}{\partial F_\alpha^i} \right] L(\mathbf{x}, \mathbf{u}, \mathbf{F}) + \left[ \frac{\partial}{\partial x^\alpha} \xi^\alpha(\mathbf{x}, \mathbf{u}) + \frac{\partial}{\partial u^i} \xi^\alpha(\mathbf{x}, \mathbf{u}) F_\alpha^i \right] L(\mathbf{x}, \mathbf{u}, \mathbf{F}) = 0$$

pointwise for any choice of  $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times m}$ .

**Remark 2.3.** Assume that  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$  satisfies the Euler-Lagrange equations for (2.1), that is

$$(2.11) \quad \frac{\partial}{\partial x^\alpha} \left( \frac{\partial L}{\partial F_\alpha^i}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \right) = \frac{\partial L}{\partial u^i}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))$$

for  $i = 1, 2, \dots, n$ . Further assume that  $L$  and  $\mathbf{u}$  are smooth enough, then an application of the chain rule yields

$$(2.12) \quad \frac{\partial}{\partial x^\alpha} (M_\alpha^\beta(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))) = \frac{\partial L}{\partial x^\beta}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})),$$

for  $\beta = 1, 2, \dots, m$ .

It now follows from (2.11), (2.12) and (2.10) that if (2.2) is a variational symmetry of (2.1) then  $\mathbf{u}$  satisfies the conservation law

$$(2.13) \quad \frac{\partial}{\partial x^\alpha} \left[ \phi^i(\mathbf{x}, \mathbf{u}) \frac{\partial L}{\partial F_\alpha^i}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) + \xi^\beta(\mathbf{x}, \mathbf{u}) M_\alpha^\beta(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \right] = 0 ,$$

for  $i = 1, 2, \dots, n$ . For later use we note that the weak form of the above conservation law is

$$(2.14) \quad \int_{\Omega} \frac{\partial \theta}{\partial x^{\alpha}}(\mathbf{x}) \left[ \phi^i(\mathbf{x}, \mathbf{u}) \frac{\partial L}{\partial F_{\alpha}^i} + \xi^{\beta}(\mathbf{x}, \mathbf{u}) M_{\alpha}^{\beta}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \right] dx = 0 ,$$

for any  $\theta \in C_0^1(\Omega)$ .

Elaborating on the last remark, our goal in the next section is to show that, if (2.2) is a variational symmetry of  $E$  given by (2.1), and if further  $\mathbf{u}_0$  is a minimizer for  $E(\cdot; \Omega)$  - for any kind of appropriate boundary conditions and under any kind of constraint, provided these are preserved under composition of  $\mathbf{u}_0$  by a small and smooth perturbation of the identity - then the conservation law (2.14) still holds true.

### 3. VARIATIONAL SYMMETRY AND MINIMIZERS

Suppose that  $\mathbf{u}_0$  is a minimiser of the functional  $E(\cdot; \Omega)$  and let  $\theta \in C_0^1(\Omega)$ . Let (2.2) be a variational symmetry of  $E$  and define the one-parameter family of variations by replacing the parameter  $t$  in the definition of the group by the function  $t\theta(\mathbf{x})$ . Thus we set

$$(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = (\tilde{\Xi}_t(\mathbf{x}, \mathbf{u}), \tilde{\Phi}_t(\mathbf{x}, \mathbf{u})) = (\Xi(\mathbf{x}, \mathbf{u}, t\theta(\mathbf{x})), \Phi(\mathbf{x}, \mathbf{u}, t\theta(\mathbf{x}))) .$$

Note that by (2.3), the infinitesimal generator corresponding to the above group of transformations is given by

$$\tilde{\mathbf{v}} = \tilde{\xi}^{\alpha}(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^{\alpha}} + \tilde{\phi}^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^i}$$

where

$$(3.1) \quad \tilde{\xi}(\mathbf{x}, \mathbf{u}) = \theta(\mathbf{x})\xi(\mathbf{x}, \mathbf{u}), \quad \tilde{\phi}(\mathbf{x}, \mathbf{u}) = \theta(\mathbf{x})\phi(\mathbf{x}, \mathbf{u})$$

and (by (2.5) and (2.6)) its first prolongation is given by

$$(3.2) \quad \tilde{\mathbf{v}}^{(1)} = \tilde{\xi}^{\alpha}(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^{\alpha}} + \tilde{\phi}^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^i} + \tilde{\pi}_{\alpha}^i(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \frac{\partial}{\partial F_{\alpha}^i} ,$$

where

$$(3.3) \quad \begin{aligned} \tilde{\pi}_{\alpha}^i(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) &= \left. \frac{d}{dt} \frac{\partial \tilde{u}^i(\tilde{\mathbf{x}})}{\partial \tilde{x}^{\alpha}} \right|_{t=0} \\ &= \frac{\partial}{\partial x^{\alpha}} [\theta(\mathbf{x})\phi^i(\mathbf{x}, \mathbf{u}(\mathbf{x}))] - \frac{\partial u^i}{\partial x^{\beta}} \frac{\partial}{\partial x^{\alpha}} [\theta(\mathbf{x})\xi^{\beta}(\mathbf{x}, \mathbf{u}(\mathbf{x}))]. \end{aligned}$$

In analogy with (2.4) we define the corresponding family of variations around  $\mathbf{u}_0$  by

$$\tilde{\mathbf{u}}_t(\tilde{\mathbf{x}}) = [\Phi \circ (id \times \mathbf{u}_0 \times t\theta)] \circ [\Xi \circ (id \times \mathbf{u}_0 \times t\theta)]^{-1}(\tilde{\mathbf{x}}) .$$

Note that, since  $\theta \in C_0^1(\Omega)$ ,  $\tilde{\mathbf{x}} = \mathbf{x}$  if  $\mathbf{x} \in \partial\Omega$  and thus  $\tilde{\Omega} = \Omega$  for  $|t|$  small. Further,  $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = \mathbf{u}(\mathbf{x})$ ,  $\mathbf{x} \in \partial\Omega$ , so that  $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})$  is an admissible test function for  $E(\cdot; \Omega)$ . In other words, since  $\mathbf{u}_0$  is a minimizer for  $E(\cdot; \Omega)$ ,  $E(\tilde{\mathbf{u}}_0; \Omega) \geq E(\mathbf{u}_0; \Omega)$  from which it easily follows that (2.7) holds for  $\tilde{\Omega} = \Omega$  and then, using (2.8), that

$$(3.4) \quad \begin{aligned} 0 &= \left. \frac{d}{dt} \int_{\tilde{\Omega}} L(\tilde{\mathbf{x}}_0, \tilde{\mathbf{u}}_0, \nabla_{\tilde{\mathbf{x}}} \tilde{\mathbf{u}}_0(\tilde{\mathbf{x}})) \right|_{t=0} \\ &= \int_{\Omega} \tilde{\mathbf{v}}^{(1)} L(\mathbf{x}, \mathbf{u}_0, \nabla_{\mathbf{x}} \mathbf{u}_0(\mathbf{x})) + L(\mathbf{x}, \mathbf{u}_0, \nabla_{\mathbf{x}} \mathbf{u}_0(\mathbf{x})) \text{Div } \tilde{\xi}(\mathbf{x}, \mathbf{u}_0(\mathbf{x})) dx . \end{aligned}$$

Next, using (2.10), (3.2), (3.1), (3.3) it follows that (3.4) is equal to

$$\begin{aligned}
 & \int_{\Omega} \left[ \tilde{\xi}^{\alpha}(\mathbf{x}, \mathbf{u}_0) \frac{\partial}{\partial x^{\alpha}} + \tilde{\phi}^i(\mathbf{x}, \mathbf{u}_0) \frac{\partial}{\partial u^i} + \tilde{\pi}_{\alpha}^i(\mathbf{x}, \mathbf{u}_0, \nabla \mathbf{u}_0) \frac{\partial}{\partial F_{\alpha}^i} \right] L \\
 & \quad + L(\mathbf{x}, \mathbf{u}_0, \nabla_{\mathbf{x}} \mathbf{u}_0(\mathbf{x})) \operatorname{Div} \tilde{\xi}(\mathbf{x}, \mathbf{u}_0(\mathbf{x})) \, dx \\
 & = \int_{\Omega} \left[ \theta \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} + \theta \phi^i \frac{\partial}{\partial u^i} \right] L + \\
 & \quad \left[ \frac{\partial}{\partial x^{\alpha}} (\phi^i(\mathbf{x}, \mathbf{u}_0(\mathbf{x})) \theta(\mathbf{x})) - \frac{\partial u_0^i}{\partial x^{\beta}} \frac{\partial}{\partial x^{\alpha}} (\xi^{\beta}(\mathbf{x}, \mathbf{u}_0(\mathbf{x})) \theta(\mathbf{x})) \right] \frac{\partial}{\partial F_{\alpha}^i} L \\
 & \quad + L \operatorname{Div} [\theta \xi] \\
 & = \int_{\Omega} \left[ \mathbf{v}^{(1)} L(\mathbf{x}, \mathbf{u}_0, \nabla \mathbf{u}_0(\mathbf{x})) + L(\mathbf{x}, \mathbf{u}_0, \nabla \mathbf{u}_0(\mathbf{x})) \operatorname{Div} \xi(\mathbf{x}, \mathbf{u}_0(\mathbf{x})) \right] \theta(\mathbf{x}) + \\
 & \quad \left[ \phi^i(\mathbf{x}, \mathbf{u}_0) \frac{\partial L}{\partial F_{\alpha}^i}(\mathbf{x}, \mathbf{u}_0, \nabla \mathbf{u}_0) + \xi^{\beta}(\mathbf{x}, \mathbf{u}_0) M_{\alpha}^{\beta}(\mathbf{x}, \mathbf{u}_0, \nabla \mathbf{u}_0) \right] \frac{\partial \theta}{\partial x^{\alpha}}(\mathbf{x}) \, dx = 0 .
 \end{aligned}$$

Using (2.9), it now follows that the first term in square brackets vanishes identically (since (2.2) is a variational symmetry) and hence  $\mathbf{u}_0$  satisfies (2.14) which is the weak form of the conservation law (2.13).

We have thus recovered (2.14) for any variational symmetry, provided that  $\mathbf{u}_0$  is a minimizer for  $E(\cdot; \Omega)$ . In the next section, we apply this to the specific setting of hyperelasticity and recover known results in a rather straightforward manner.

#### 4. HYPERELASTICITY AND CONSERVATION LAWS

In nonlinear elasticity  $m = n = 2$  or  $3$  and the integral functional

$$(4.1) \quad E(\mathbf{u}) = \int_{\Omega} L(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) \, dx$$

is the energy stored by a deformation  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$  of an elastic body occupying the region  $\Omega$  in its reference configuration. The integrand  $L : \Omega \times M_+^{n \times n} \rightarrow \mathbb{R}$  is known as the stored energy function of the material (where  $M_+^{n \times n}$  denotes the set of real  $n \times n$  matrices with positive determinant.)

Typically, deformations are required to satisfy the local invertibility condition

$$(4.2) \quad \det \nabla \mathbf{u}(\mathbf{x}) > 0 \quad a.e.$$

which is usually incorporated variationally by requiring that

$$(4.3) \quad L(\mathbf{x}, \mathbf{F}) \rightarrow \infty \quad as \quad \det \mathbf{F} \rightarrow 0 .$$

Assumption (4.2) prevents local interpenetration of matter (if  $\mathbf{u}$  is  $C^1$  this result follows directly from the inverse function theorem, see also [5]. If  $\mathbf{u}$  only lies in a Sobolev space then results on the invertibility properties of  $\mathbf{u}$  are contained, for example, in [3], [6], [12].).

If the stored energy function is explicitly independent of  $\mathbf{x}$  so that  $L = L(\mathbf{F})$  then the material is said to be *homogeneous*.

*Frame indifference* of the stored energy requires that

$$L(QF) = L(F)$$

for all  $F \in M_+^{n \times n}$  and all  $Q \in SO(n)$  (the  $n \times n$  special orthogonal matrices: i.e. orthogonal matrices with determinant +1). If in addition the material is *isotropic* then

$$L(FQ) = L(F)$$

for all  $F \in M_+^{n \times n}$  and all  $Q \in SO(n)$ . (See [5] for further background on nonlinear elasticity.) In the context of nonlinear elasticity, the structure and properties mentioned above give rise to the following variational symmetries: homogeneity of the material yields

$$(4.4) \quad (\mathbf{x}, \mathbf{u}) \rightarrow (\mathbf{x} + t\mathbf{c}, \mathbf{u}),$$

translational invariance yields

$$(4.5) \quad (\mathbf{x}, \mathbf{u}) \rightarrow (\mathbf{x}, \mathbf{u} + t\mathbf{c}),$$

rotational invariance of the energy yields

$$(\mathbf{x}, \mathbf{u}) \rightarrow (\mathbf{x}, \mathbf{Q}(t)\mathbf{u}),$$

and isotropy of the material yields

$$(\mathbf{x}, \mathbf{u}) \rightarrow (\mathbf{Q}(t)\mathbf{x}, \mathbf{u}),$$

where  $\mathbf{c} \in \mathbb{R}^n$  is a constant vector and  $\mathbf{Q} : (-\delta, \delta) \rightarrow SO(n)$ ,  $\delta > 0$ , satisfies  $\mathbf{Q}(0) = I$ . Each of the above variational symmetries immediately gives rise to corresponding conservation laws via the formula (2.13) ( see [10] pages 281-283).

Of notable interest is the conservation law

$$(4.6) \quad \frac{\partial}{\partial x^\alpha} \left[ L(\nabla \mathbf{u}_0(\mathbf{x})) \delta_\alpha^\beta - \frac{\partial u^k}{\partial x^\beta} \frac{\partial L}{\partial F_\alpha^k}(\nabla \mathbf{u}_0(\mathbf{x})) \right] = 0, \quad \beta = 1, 2, \dots, n,$$

which we obtain here from the use of the family of symmetries given by (4.4) and upon application of (2.13). That conservation law was derived by J.M. BALL in [1] using ‘‘inner’’ variations of the form  $\mathbf{u}_0(\mathbf{x} + t\phi(\mathbf{x}))$  (with  $\phi \in C_0^1(\Omega)$  and  $t \in \mathbb{R}$  to palliate the difficulty encountered in using usual variations of the form  $\mathbf{u}_0(\mathbf{x}) + t\phi(\mathbf{x})$  because these may violate (4.2) for arbitrarily small  $t \neq 0$  even though  $\mathbf{u}_0$  satisfies (4.2).

Ball also considered ‘‘outer’’ variations of the form  $\mathbf{u}_0(\mathbf{x}) + t\phi(\mathbf{u}_0(\mathbf{x}))$ ,  $\phi \in C_0^1(\mathbf{u}_0(\Omega))$  from which he derived the weak form of the Cauchy equilibrium equations satisfied by  $\mathbf{u}_0$ , namely,

$$(4.7) \quad (\text{Div}_{\mathbf{y}} \mathbf{T})_i = \frac{\partial}{\partial y^\alpha} [T_{i\alpha}] = 0$$

in  $\mathbf{u}_0(\Omega)$ , where

$$\mathbf{T}(\mathbf{y}) = \frac{\partial L}{\partial \mathbf{F}}(\nabla \mathbf{u}_0(\mathbf{x})) [\nabla \mathbf{u}_0(\mathbf{x})]^T (\det \nabla \mathbf{u}_0(\mathbf{x}))^{-1}$$

is called the Cauchy stress tensor and  $\mathbf{x} = \mathbf{u}_0^{-1}(\mathbf{y})$ . In the context of this paper (and the weak form of the conservation laws (2.14)) this same result obtained in [1] follows from the use of the family of symmetries (4.5) together with test functions  $\theta(\mathbf{x}) = \tilde{\theta}(\mathbf{u}_0(\mathbf{x}))$  where  $\tilde{\theta} \in C_0^1(\mathbf{u}_0(\Omega))$ . (See [1] and [4] for further details.)

**Remark 4.1.** As a further example, we now suppose that  $L(\mathbf{F})$  is homogeneous of degree  $p$  so that  $L(\lambda\mathbf{F}) = \lambda^p L(\mathbf{F})$  for all  $\mathbf{F} \in M_+^{n \times n}$  for any  $\lambda > 0$ , then

$$(\mathbf{x}, \mathbf{u}) \rightarrow ((1+t)\mathbf{x}, (1+t)^{\frac{p-n}{p}}\mathbf{u}) = (\tilde{\mathbf{x}}, \tilde{\mathbf{u}})$$

is a further variational symmetry of (4.1).

Application of (2.13) immediately yields the following conservation law:

$$\frac{\partial}{\partial x^\alpha} \left[ M_\alpha^\beta (\nabla \mathbf{u}_0(\mathbf{x})) x^\beta + \frac{p-n}{p} \frac{\partial L}{\partial F_\alpha^k} (\nabla \mathbf{u}_0(\mathbf{x})) \mathbf{u}_0^k(\mathbf{x}) \right] = 0, \quad \beta = 1, 2, \dots, n,$$

(see [10] Example 4.32). Note however that the requirement of  $p$ -homogeneity of  $L$  is incompatible with condition (4.3) which must therefore be dropped if such a setting is adopted. The next example illustrates such a situation and demonstrates that minimisers may satisfy the weak form of some conservation laws but not others.

**Example 4.1.** Consider minimising

$$(4.8) \quad E(\mathbf{u}) = \int_A |\nabla \mathbf{u}|^2 dx$$

on  $W^{1,2}(A)$  subject to the boundary condition

$$(4.9) \quad \mathbf{u}|_{\partial A} = \text{identity}$$

The Euler-Lagrange equations for (4.8), (4.9) are formally given by

$$(4.10) \quad \begin{cases} \Delta \mathbf{u} = 0 \\ \mathbf{u}|_{\partial A} = \text{identity} \end{cases}$$

and by the strict convexity of (4.8) the unique global minimiser is given by the homogeneous map

$$\mathbf{u}^h \equiv \mathbf{x}.$$

There are clearly no other weak solutions for this Dirichlet problem in  $W^{1,2}(A)$ . However, the results of [11] show that for each  $n \in \mathbf{N}$  there exists a minimiser  $\mathbf{u}^N$  of  $E$  on

$$\mathcal{A}_N = \{ \mathbf{u} \in W^{1,2}(A) : \det \nabla \mathbf{u} \geq 0 \text{ a.e.}, \mathbf{u} = \mathbf{x} \text{ on } \partial A, \mathbf{u} \text{ satisfies } (H_N) \},$$

where  $(H_N)$  is a homotopy condition that the map  $\mathbf{u}$  twists the annulus through  $2\pi N$  (see [11] Definition 2.8).<sup>3</sup>

The arguments in our current paper show that each  $\mathbf{u}^N$  must satisfy the weak form of the corresponding energy-momentum equations (4.6), namely

$$\text{Div } M = \text{Div} [|\nabla \mathbf{u}|^2 I - \nabla \mathbf{u}^T \nabla \mathbf{u}] = 0$$

which are given in component form by

$$\frac{\partial}{\partial x^\alpha} M_\alpha^\beta = \frac{\partial}{\partial x^\alpha} \left[ |\nabla \mathbf{u}|^2 \delta_\alpha^\beta - \frac{\partial u^k}{\partial x^\beta} \frac{\partial u^k}{\partial x^\alpha} \right] = 0.$$

<sup>3</sup>More precisely, using polar coordinates, write

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(r, \theta), \quad r \in [a, b], \quad \theta \in [0, 2\pi)$$

then for a.e.  $\theta \in [0, 2\pi)$

$$\gamma_\theta(r) = \frac{\mathbf{u}(r, \theta)}{|\mathbf{u}(r, \theta)|}, \quad r \in [a, b]$$

is a closed continuous curve in the plane with a well defined winding number around the origin, denoted  $\text{wind}(\gamma_\theta)$ . The map  $\mathbf{u}$  is said to satisfy  $(H_N)$  provided that  $\text{wind}(\gamma_\theta) = N$  for a.e.  $\theta \in [0, 2\pi)$ .



Note that  $\mathbf{u}^N$  cannot satisfy the weak form of the Euler-Lagrange equations (4.10) because of the uniqueness result alluded to above.

It is interesting to note that the integrand in (4.8) is homogeneous of degree two and so by Remark 4.1 ( $p = n = 2$ ) each  $\mathbf{u}^N$  also satisfies the weak form of the conservation law

$$\frac{\partial}{\partial x^\alpha} [M_\alpha^\beta x^\beta] = 0 .$$

We anticipate that in this example each  $\mathbf{u}^N$  will be degenerate in the sense that  $\det \nabla \mathbf{u}^N = 0$  on a set of non zero measure. In this case the corresponding Cauchy stress tensor will be undefined and so the weak form of the Cauchy equations (4.7) will not hold.

## 5. CONCLUDING REMARKS

For sufficiently regular deformations  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$  it is possible to change variables to rewrite the energy functional (4.1) as an integral over the deformed configuration  $\mathbf{u}(\Omega)$  :

$$E(\mathbf{u}) = \int_{\Omega} L(\nabla \mathbf{u}(\mathbf{x})) dx = \int_{\mathbf{u}(\Omega)} \hat{L}(\nabla_{\mathbf{u}} \mathbf{x}(\mathbf{u})) du ,$$

where  $\hat{L}(\mathbf{F}) = \det \mathbf{F} L(\mathbf{F}^{-1})$  and  $\mathbf{x} : \mathbf{u}(\Omega) \rightarrow \mathbb{R}^n$  is the inverse of  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$  (see e.g. [2]). In the context of this paper, it is interesting to note that the energy momentum tensor for  $\hat{L}$  is equal to the Cauchy stress tensor for  $L$  and vice versa, i.e.,

$$\frac{\partial \hat{L}}{\partial \mathbf{F}} \frac{\mathbf{F}^T}{\det \mathbf{F}} = \left[ L(\mathbf{F}^{-1}) \mathbf{I} - (\mathbf{F}^{-1})^T \frac{\partial L}{\partial \mathbf{F}}(\mathbf{F}^{-1}) \right] = \mathbf{M}(\mathbf{F}^{-1})$$

and

$$\hat{\mathbf{M}}(\mathbf{F}) = \left[ \hat{L}(\mathbf{F}) - \mathbf{F}^T \frac{\partial \hat{L}}{\partial \mathbf{F}}(\mathbf{F}) \right] = \frac{\partial L}{\partial \mathbf{F}}(\mathbf{F}^{-1}) \frac{(\mathbf{F}^{-1})^T}{\det(\mathbf{F}^{-1})} .$$

Hence these two tensors are in a precise sense dual to one another.

Finally, we note that in this paper we have only considered variational point symmetries of the integral functional (2.1) and the corresponding conservation laws (2.13). However, the general version of Noether's theorem applies to more general symmetries known as divergence symmetries for which the infinitesimal invariance criterion (2.10) is replaced by the requirement that

$$\begin{aligned} \phi^i(\mathbf{x}, \mathbf{u}(\mathbf{x})) \frac{\partial L}{\partial u^i} + \frac{\partial}{\partial x^\alpha} (\phi^i(\mathbf{x}, \mathbf{u}(\mathbf{x}))) \frac{\partial L}{\partial F_\alpha^i} + \xi^\alpha \frac{\partial L}{\partial x^\alpha} + \left( \frac{\partial}{\partial x^\alpha} \xi^\beta(\mathbf{x}, \mathbf{u}(\mathbf{x})) \right) M_\alpha^\beta \\ = \text{Div } \mathbf{B} , \end{aligned}$$

where  $\mathbf{B}$  is a vector function of  $\mathbf{x}, \mathbf{u}$  and the derivatives of  $\mathbf{u}$  (see [10] page 283). The only modification that arises in our initial derivation of the conservation law (2.13) is the inclusion of a term involving  $\mathbf{B}$ . The modification to our derivation of the weak form of this (more general) conservation law as a necessary condition for a minimiser is straight forward and we leave this as an exercise for the interested reader.

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