CRACKS IN FRACTURE MECHANICS : A TIME INDEXED FAMILY OF ENERGY MINIMIZERS

Griffith's theory revisited

G. A. FRANCFORT AND J.-J. MARIGO
LPMTM (UPR-CNRS 9001)
Institut Galilée, Université Paris-Nord
93430 Villetaneuse, France

1. Introductory remarks

Brittle fracture mechanics is classically thought of as operating under various restrictive premises, two of which seem both drastic and unrealistic: no crack will appear unless a crack is already present; cracks propagate along predefined trajectories. Our main goal is to do away with the above, while departing as little as possible from the sanctity of Griffith's criterion. In a nutshell, the following will be achieved:

1. crack initiation, and its subsequent evolution till complete failure of the loaded sample,
2. complete freedom in the mechanical and geometric characteristics of the sample,
3. boundary cracks,
4. unilateral contact if needed.

There is, of course a price to pay, at least for the time being. The proposed formulation does not know well how to handle any kind of loadings, whether body loadings or surface tractions; thus the only admissible boundary conditions are on the displacement field. In the text all such boundary conditions are referred to as "loadings" . The formulation is also, as of yet, unable to model a continuous time evolution of the cracking process, except in the case of linearly increasing boundary displacements; it can however predict the cracking process for any kind of time-discretized boundary displacement.

Without further ado, we briefly present, in the next section, our model; we then implement it in a one-dimensional setting. The interested reader is kindly referred to (Francfort and Marigo, 1997) for a more detailed and thorough exposition of the model.

P. Argoul et al. (eds.),
IUTAM Symposium on Variations of Domains and Free-Boundary Problems in Solid Mechanics, 197–202.

2. The model

In all that follows, Ω denotes a bounded domain of $\mathbf{R}^N$ with $N = 1, 2,$ *or* 3.

Our main ingredient, the potential cracks, can be, quite simply, any compact set; we impose no restrictions of size or shape. Note however that the actual cracks are always of codimension 1, *i.e.* of finite $N-1$-Hausdorff dimension. Following Griffith's fundamental idea, see (Griffith, 1920), the associated surface energy is of the Griffith's type, that is, upon denoting by $k(x)$ the (nonnegative) energy density per unit area required to create a crack at that point,

$$E_s(\Gamma) = \int_\Gamma k(x) dH^{N-1}(x). \tag{1}$$

Note that a dependence of k upon n, the "normal" to the crack Γ could be introduced at the expense of further specifying the geometric structure of Γ.

Our second ingredient is the "loading"; as mentioned before, only displacements can be imposed on part of the boundary of our sample. We denote by U the time-dependent displacement field applied to part of the boundary $\partial_d\Omega$. Assume that Γ denotes the crack at a given time; we have to distinguish between the uncracked part of the boundary, *i.e.*, $\partial_d\Omega\backslash\Gamma$, on which we impose U, and the cracked part of the boundary on which nothing is imposed if unilateral contact is neglected, while unilateral conditions of some type must be imposed otherwise.

Our third ingredient is the actual material. We will assume that the domain Ω is occupied by a linearly elastic material. It should be pointed out that the extension of our model to finite elasticity can be easily performed (the underlying mathematical structure of the problem is actually simpler in the latter setting); see (Fonseca and Francfort, 1995) on a mathematical treatment of fracture in such a setting. The elasticity matrix $A(x)$ satisfies $\alpha\varepsilon\cdot\varepsilon \le A(x)\varepsilon\cdot\varepsilon \le \beta\varepsilon\cdot\varepsilon$, for any symmetric ε.

To each "loading" and potential crack we associate the corresponding elastic energy, $E_d(\Gamma, U)$, defined as

$$E_d(\Gamma, U) = inf\left\{\int_{\Omega\backslash\Gamma}\frac{1}{2}A(x)\varepsilon(u)\cdot\varepsilon(u)dx; \quad u = U \ on\ \partial_d\Omega\backslash\Gamma\right\}. \tag{2}$$

If the crack Γ is smooth enough, as well as $\partial_d\Omega\backslash\Gamma$, then the infimum is classically attained for a displacement field u_Γ, solution to the elastic equilibrium on $\Omega\backslash\Gamma$; if not, then $E_d(\Gamma, U)$ should be understood as a generalized compliance because the associated elastic equilibrium may be ill-posed. In any case it can be conjectured that the actual crack will exhibit more regularity, which will in turn permit a writing of the elastic equilibrium on the uncracked part of the domain. It should be said however that no such regularity has been proved at this point; we do not want to dwell any further on this issue because a precise discussion necessarily involves a detailed explanation of the functional analytic foundation of this formulation, which would distract us from our main goal in the present work, that is, a concise exposition of the proposed model. The interested reader is referred to (Fonseca and Francfort, 1995) and references therein for further inquiries; see also (Ambrosio, 1990) or (De Giorgi *et al.*, 1989) for an exposition of the mathematical results available in the setting of image segmentation as well as linear antiplane elasticity.

Our fourth ingredient is the total energy, which we define to be

$$E(\Gamma, U) = E_s(\Gamma) + E_d(\Gamma, U). \tag{3}$$

It now remains to postulate an evolution law for the cracking process. This is done firstly for a sequence of set times $0 = t_0 < t_1 < \cdots < t_i < \cdots$; upon denoting by U_i and Γ_i the corresponding "loadings" and resulting cracks, we propose the following principle:

Assume that Γ_0, ...,Γ_{i-1},*are known, then the crack* Γ_i *at time* t_i *will be such that*

$$\Gamma_{i-1} \subset \Gamma_i, \tag{4}$$

$$E(\Gamma_i, U_i) \leq E(\Gamma, U_i) \quad \forall \Gamma \supset \Gamma_{i-1}. \tag{5}$$

The first inclusion translates the irreversible character of the fracturing process while the second expresses a principle of least energy: the preferred state, among all admissible cracks, *i.e.* , among all cracks that include the actual crack at the previous time step, is that which produces the smallest total energy.

As it stands, the proposed evolution law allows for any kind of "loading"; we emphasize however our current inability at producing a time-continuous version of this evolution law. This is why the continuous analogue is only proposed for increasing "loadings", that is, for loadings U of the form

$$U(t) = \begin{cases} tU_0 & , \quad t \geq 0, \\ 0 & , \quad t < 0. \end{cases} \tag{6}$$

The time $t \in (-\infty, +\infty)$ should be understood as a loading parameter, rather than an actual time, as is always the case when investigating quasistatic evolutions; in this respect, the only relevant feature of time is its monotonically increasing character and, as such, any increasing function of t would do. The associated evolution law for the cracking process becomes

- **No initial crack** (any initial condition would do) :

$$\Gamma^-(0) = \emptyset; \tag{7}$$

- **Irreversibility of the process** :

$$\Gamma(t) \textit{ increases with } t; \tag{8}$$

- **First least energy principle** :

$$E(\Gamma(t), tU_0) \leq E(\Gamma, tU_0), \qquad \forall \Gamma \supset \Gamma^-(t); \tag{9}$$

- **Second least energy principle** :

$$E(\Gamma(t), tU_0) \leq E(\Gamma(s), tU_0), \qquad \forall s, \quad 0 \leq s \leq t, \tag{10}$$

where $\Gamma^-(t) = \overline{\bigcup_{s<t} \Gamma(s)}$.

If the three first relations are an exact transcription of the time-discretized model in a continuous framework, the fourth relation is new. It can be formally deduced from the discrete version for increasing "loadings" (see (Francfort and Marigo, 1997)); it is not contained in the third relation but does provide an additional restriction which proves essential in preventing too large a set of possible solutions (cf. section 2 below).

Note also that the evolution law guarantees that $\Gamma(0) = \emptyset$, at least when $k(x) > 0, H^{N-1} - a.e.$ on Ω. Indeed, since, at time 0, the "loading" is 0, the associated compliance is also 0, so that $E(\Gamma(0), 0) = E_s(\Gamma(0))$ and (9) yields $E_s(\Gamma(0)) \leq E_s(\emptyset)$. Since the reverse inequality is a direct consequence of irreversibility, that implies that $\Gamma(0) \subset \emptyset$ up to a set of null H^{N-1}-Hausdorff measure.

We refer the reader to (Francfort and Marigo, 1997) for a detailed presentation of the properties of the proposed evolution model. The following features are particularly noteworthy and should be compared to Griffith's theory results, see (Liebowitz, 1968) :

1. *A crack will appear at some finite time, provided that some elastic energy is stored in the sound loaded body,* i.e. *provided that* $E_d(\emptyset, U_0) \neq 0$;
2. *Cracking will stop only when there will be no more elastic energy inside the body;*
3. *If the crack follows a smooth enough space-trajectory and if it further advances (with time) along that trajectory in a regular manner,* i.e. , *if the H^{N-1}-Hausdorff measure of the crack is an absolutely continuous function of t, then the crack evolution obeys the classical Griffith's law;*
4. *But even if the space-trajectory is smooth, the crack evolution may differ from that predicted by Griffith's law if it propagates with (a) jump(s) of* **finite** *length;*

5. *If the "initial" displacement field corresponding to elastic equilibrium in the absence of any cracks (or more generally in the presence of the initial crack) exhibits singular points, then a crack — of finite or infinitesimal length — will form at such points, either right away or at finite time, according to the strength of the singularity of the displacement field at such a point;*
6. *In the absence of singularities on the "initial" displacement field, then, either the sample will remain crack free — whether $E_d(\emptyset, U_0) = 0$ —, or a crack of* **finite** *length will appear at a finite non-zero time.*

3. The one-dimensional case

Consider a one-dimensional bar $\Omega = (0, L)$ of cross-sectional area S made of a heterogeneous material with Young's modulus $E(x)$. Assume that the end displacements are respectively $u(0) = 0$ and $u(L) = \varepsilon t$ where $\varepsilon = \pm 1$.

If the bar is crack-free, its "surface" energy is 0, while its bulk energy is

$$E(\emptyset, \varepsilon t) = \frac{1}{2}\bar{E}S\frac{t^2}{L}, \qquad \text{with} \quad \frac{1}{\bar{E}} = \frac{1}{L}\int_0^L \frac{1}{E(x)}dx. \tag{11}$$

For any crack Γ with finite surface energy, *i.e.* any countable set $\mathcal{T}$ of points in $[0, L]$ (we implicitly assume that k is continuous on $[0, L]$ and that $k(x) > \alpha, x \in \bar{\Omega}$, for some $\alpha > 0$),

$$E(\Gamma, \varepsilon t) = E_s(\Gamma) = S \sum_{x \in \mathcal{T}} k(x). \tag{12}$$

Introduce the critical time

$$t_r = \sqrt{\frac{2k_{min}L}{\bar{E}}}, \qquad \text{with} \quad k_{min} = \min_{x\in[0,L]} k(x). \tag{13}$$

Then, the bar remains crack-free for all t's with $t < t_r$. Indeed note that $\Gamma(0) = \emptyset$; if there exists $t < t_r$ such that $\Gamma(t) \neq \emptyset$, then, since (12) and (10) give $E(\Gamma(t), \varepsilon t) = E_s(\Gamma(t)) \leq E(\emptyset, \varepsilon t)$, we should have

$$k_{min}S \leq \sum_{x\in\Gamma(t)} k(x)S = E_s(\Gamma(t)) \leq \frac{1}{2}\bar{E}S\frac{t^2}{L} < \frac{1}{2}\bar{E}S\frac{t_r^2}{L} = k_{min}S, \tag{14}$$

which is impossible.

Further, the bar is cracked as soon as $t > t_r$. Indeed, if such is not the case, by (9) we should have $E(\Gamma, \varepsilon t) \geq E(\emptyset, \varepsilon t), \forall\Gamma$'s, *i.e.*,

$$k(x)S \geq \frac{1}{2}\bar{E}S\frac{t^2}{L} > \frac{1}{2}\bar{E}S\frac{t_r^2}{L} = k_{min}S, \quad \forall x \in \bar{\Omega}, \tag{15}$$

which is impossible.

Thirdly, since (10) gives $E(\Gamma(t_r), \varepsilon t_r) \leq E(\emptyset, \varepsilon t_r) = k_{min} S$, while (9) and (12) imply

$$E(\Gamma(t_r), \varepsilon t) \geq E(\Gamma(t), \varepsilon t) = E_s(\Gamma(t)) \geq k_{min} S, \quad \forall t > t_r, \tag{16}$$

by passing to the limit when $t \to t_r$, we obtain

$$E(\Gamma(t_r), \varepsilon t_r) = \lim_{t \to t_r} E_s(\Gamma(t)) = k_{min} S. \tag{17}$$

But by using once more (10), we have, when $t_r < s < t$:

$$k_{min} S \leq E_s(\Gamma(t)) = E(\Gamma(t), \varepsilon t) \leq E(\Gamma(s), \varepsilon t) = E_s(\Gamma(s)). \tag{18}$$

By passing to the limit when $s \to t_r$ and using (17), we get $E_s(\Gamma(t)) = k_{min} S, \forall t > t_r$, which is only possible if $\Gamma(t)$ is reduced to a single point x such that $k(x) = k_{min}$. Then, for any $t > t_r$, $\Gamma(t) = \{x\}$. We have thus proved that *the bar remains uncracked until time t_r, at which it suddenly breaks at a single point $\{x\}$ where k reaches its minimum.*

Remark that, in the setting of unilateral contact, it is easily verified that the bar will not break under compression but that it will break at time t_r under traction. The contrast with Griffith's theory of fracture is striking : our formulation predicts "crack" initiation where Griffith's cannot.

As a final note, the reader's attention is drawn to a peculiar byproduct of the formulation. The end displacement, at the time of breakage, grows like the square root of the length of the bar so that the average force per unit length at that time decreases like the square root of the length of the bar. This effect, which is equally present in the classical theory of fracture mechanics, is due to the interplay between surface and bulk energies which in turn produces a characteristic length. The physical plausibility of such a scaling might be deemed debatable by the physically inclined reader.

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