THE WAVE EQUATION ON A THIN DOMAIN: ENERGY DENSITY AND OBSERVABILITY

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Abstract. We compute the limit energy density of solutions of the linear wave equation in a thin three–dimensional domain, if the wavelength of the Cauchy data is bounded from below by the thickness of the domain. As an application, we obtain a geometric criterion for the uniform observability of solutions of a damped wave equation on such a domain.

Keywords : dimensional reduction, semi–classical, Wigner, and H-measures, wave equation, plate models, control theory, stabilization, propagation of oscillations.

1. Introduction

1.1. **Setting of the problem.** The purpose of this paper is to show how microlocal measures can be used to describe the limit energy density of solutions of the linear wave equation on a thin domain. As an application, we shall derive a geometric necessary and sufficient condition for uniform observability of solutions to the corresponding damped equation.

Specifically, let ω be a smooth domain of $\mathbb{R}^2$, which is either bounded, or all of $\mathbb{R}^2$, and let ε be a small positive real number. We consider the thin three–dimensional domain $\Omega(\varepsilon) := \omega \times (0, \varepsilon)$. Generic points in $\Omega(\varepsilon)$ will be denoted by $x = (y, z)$, where $y \in \omega$ and $z \in (0, \varepsilon)$. Let $\rho = \rho(y)$ and $k = k(y)$ be positive smooth functions on $\overline{\omega}$. We would like to investigate the behaviour, as ε tends to 0, of solutions u^ε to the following wave equation on a finite time interval $[0, T]$:

$$\left\{\begin{array}{rcll} \rho\dfrac{\partial^2 u^\varepsilon}{\partial t^2} &=& \mathrm{div}_x(k\nabla_x u^\varepsilon), & \text{on } \Omega\times(0,T),\\[2mm] u^\varepsilon(t,y,z) &=& 0, & y\in\partial\omega,\\[2mm] \dfrac{\partial u^\varepsilon}{\partial z}(t,y,0) &=& \dfrac{\partial u^\varepsilon}{\partial z}(t,y,\varepsilon)=0, & \\[2mm] u^\varepsilon(t=0,y,z) &=& u_0^\varepsilon(y,z), & \\[2mm] \dfrac{\partial u^\varepsilon}{\partial t}(t=0,y,z) &=& v_0^\varepsilon(y,z). & \end{array}\right. \tag{1.1}$$

Note that, in equation (1.1) above and **onward, the boundary condition on** $\partial\omega \times (0, 1)$ **should be disregarded when** $\omega = \mathbb{R}^2$. Here $v_0^\varepsilon \in L^2(\Omega(\varepsilon))$ and $u_0^\varepsilon \in H^1(\Omega(\varepsilon))$ with the boundary condition $u^\varepsilon_{0\,|\partial\omega\times(0,\varepsilon)} = 0$. Further, the volumic density ρ and the acoustic coefficient k are both assumed to be invariant throughout the thickness of the domain, although they may depend on the horizontal variables y.

We shall assume that the data are normalized so that the total energy satisfies

$$\int_{\Omega(\varepsilon)} (\rho(\partial_t u^\varepsilon)^2 + k|\nabla_x u^\varepsilon|^2)\, dx \leq C. \tag{1.2}$$

Moreover, the frequency of oscillations of the initial data is assumed to be at most in $1/\varepsilon$. This assumption is essential to our analysis (see Remark 2.4 below). Specifically, we suppose that

$$\begin{cases} u_0^\varepsilon \in H^2(\Omega(\varepsilon)), \quad v_0^\varepsilon \in H^1(\Omega(\varepsilon)), \\ \partial_z u_0^\varepsilon(t,y,0) = \partial_z u_0^\varepsilon(t,y,\varepsilon) = 0, \\ v_0^\varepsilon(t,y,z) = 0, \quad y \in \partial\omega, \\ \varepsilon^2 \left(\|\nabla^2 u_0^\varepsilon\|^2_{L^2(\Omega(\varepsilon))} + \|\nabla v_0^\varepsilon\|^2_{L^2(\Omega(\varepsilon))} \right) \leq M\,. \end{cases} \tag{1.3}$$

Remark 1.1. The above assumption is in fact equivalent to an estimate of the Cauchy data in the domain of the infinitesimal generator of the equation. As such it is unaltered by the evolution. In other words, it holds true at all times for $u^\varepsilon(t)$ and $\partial_t u^\varepsilon(t)$. ¶

Our last set of assumptions will simplify the analysis by killing the impact of the lateral boundary $\partial\omega \times (0,1)$. Namely, we assume there exists a neighbourhood A of $\partial\omega$ in $\overline{\omega}$ such that

$$\int_0^T \int_{A\times[0,\varepsilon]} (\rho(\partial_t u^\varepsilon)^2 + k|\nabla_x u^\varepsilon|^2)\, dt\, dx \to 0\,. \tag{1.4}$$

Condition (1.4) can be fulfilled in several situations. First, if the Cauchy data are supported into a fixed compact subset K of ω, then the horizontal finite propagation speed property (see *e.g.* [12]) leads to (1.4) provided the distance from K to $\partial\omega$ is large enough with respect to T. A second class of solutions satisfying (1.4) is related to a stabilization problem we shall discuss below, with an effective damping near the lateral boundary.

Remark 1.2. In view of assumption (1.4), the Dirichlet lateral boundary conditions imposed in (1.1) are not essential to our analysis as will become obvious in the sequel; any kind of lateral boundary conditions that would ensure adequate energy estimates would do. ¶

1.2. **Rescaling.** Let us rescale the whole problem (1.1) in order to work on the fixed domain $\Omega := \Omega(1)$. In other words, we replace $u^\varepsilon(t,y,z)$ by $\sqrt{\varepsilon}u^\varepsilon(t,y,\varepsilon z)$. We obtain the following system (where we kept the same notation u^ε for the unknown, resp. $u_0^\varepsilon, v_0^\varepsilon$ for the initial conditions)

$$
(1.5)\qquad \left\{
\begin{array}{rcll}
\rho \dfrac{\partial^2 u^\varepsilon}{\partial t^2} & = & \mathrm{div}_y(k\nabla_y u^\varepsilon) + \dfrac{1}{\varepsilon^2}\dfrac{\partial}{\partial z}(k\dfrac{\partial u^\varepsilon}{\partial z}), & \text{on } \Omega\times(0,T),\\
u^\varepsilon(t,y,z) & = & 0, & y\in\partial\omega,\\
\dfrac{\partial u^\varepsilon}{\partial z}(t,y,0) & = & \dfrac{\partial u^\varepsilon}{\partial z}(t,y,1) = 0, & \\
u^\varepsilon(t=0,x) & = & u_0^\varepsilon(x), & \\
\dfrac{\partial u^\varepsilon}{\partial t}(t=0,x) & = & v_0^\varepsilon(x). &
\end{array}
\right.
$$

The new expression for the energy density is

$$
(1.6)\qquad e^\varepsilon(t,y,z) = \frac{1}{2}\left(\rho(y)(\partial_t u^\varepsilon)^2(t,y,z) + k(y)|\nabla_y u^\varepsilon|^2(t,y,z) + \frac{1}{\varepsilon^2}k(y)(\partial_z u^\varepsilon)^2(t,y,z)\right).
$$

The normalization (1.2) reads as the following energy bound,

$$
(1.7)\qquad \int_\Omega e^\varepsilon(t,x)\,dx \le C.
$$

Our purpose is to describe the weak–$*$ limit of e^ε in the space of measures on $(0,T)\times\Omega$, in terms of energetic informations on the Cauchy data $u_0^\varepsilon, v_0^\varepsilon$. In view of (1.7), $v_0^\varepsilon, \nabla_y u_0^\varepsilon$ and $\frac{1}{\varepsilon}\partial_z u_0^\varepsilon$ are bounded in $L^2(\Omega)$ and u_0^ε is bounded in

$$
H := \Big\{u\in H^1(\Omega); u=0 \text{ on } \partial\omega\times(0,1)\Big\}.
$$

With no loss of generality (see [2]), we may impose the following on the initial conditions $u_0^\varepsilon, v_0^\varepsilon$:

$$
(1.8)\qquad
\begin{array}{l}
u_0^\varepsilon \rightharpoonup 0 \quad \text{in } H,\\
v_0^\varepsilon \rightharpoonup v_0 \quad \text{in } L^2(\Omega), \ \textbf{with } \displaystyle\int_0^1 v_0(y,z)dz = 0,\\
\dfrac{1}{\varepsilon}\dfrac{\partial u_0^\varepsilon}{\partial z} \rightharpoonup w_0 \quad \text{in } L^2(\Omega).
\end{array}
$$

Then, it is easily shown [2] that

$$
(1.9)\qquad u^\varepsilon \rightharpoonup 0, \text{ weak-* in } L^\infty(0,T;H)\cap W^{1,\infty}(0,T;L^2(\Omega)).
$$

Remark 1.3. The convergence in (1.9) is strong if and only if those in (1.8) are strong **and** $v_0 = w_0 = 0$. ¶

1.3. **The limit energy density : statement of the result.** Throughout this subsection, $\omega = \mathbb{R}^2$. Starting from system (1.5), we decompose the solution u^ε (resp. the initial data) along the eigenfunctions of the problem in the z variable, namely

$$
(1.10)\qquad u^\varepsilon(t,y,z) = \sqrt{2}\sum_{n=0}^{\infty} u^{\varepsilon,n}(t,y)\,\cos(n\pi z),
$$

with

$$u^{\varepsilon,n}(t,y) := \sqrt{2}\int_0^1 u^\varepsilon(t,y,z)\,\cos(n\pi z)\,dz\ .$$

The decomposition of the initial data is identical with obvious notation. The function $u^{\varepsilon,n}$ satisfies the following evolution equation,

$$\begin{cases} \varepsilon^2\rho(y)\dfrac{\partial^2 u^{\varepsilon,n}}{\partial t^2} = \varepsilon^2\,\mathrm{div}(k(y)\nabla u^{\varepsilon,n}) - n^2\pi^2 k(y)u^{\varepsilon,n}\ , \\ \\ u^{\varepsilon,n}(t=0,y) = u_0^{\varepsilon,n}(y)\ ,\ \dfrac{\partial u^{\varepsilon,n}}{\partial t}(t=0,y) = v_0^{\varepsilon,n}(y)\ . \end{cases} \tag{1.11}$$

Observe that, by (1.8),

$$u_0^{\varepsilon,n} \overset{H^1}{\rightharpoonup} 0\ ,\ v_0^{\varepsilon,0} \overset{L^2}{\rightharpoonup} 0. \tag{1.12}$$

In order to state our result, we introduce the following microlocal measures related to the data $(u_0^{\varepsilon,n}, v_0^{\varepsilon,n})$. These objects exist up to extraction of a subsequence (see [5], [7], [8], [11], [15]). Set from now onward

$$c(y) := \sqrt{\frac{k}{\rho}(y)}\ .$$

Denote by $N(y,\eta)$ the H–measure associated to the vector valued sequence of functions $(\sqrt{\rho}v_0^{\varepsilon,0}, \sqrt{k}\nabla u_0^{\varepsilon,0})$, which is well defined, thanks to (1.12). This is a 3×3 nonnegative matrix of Radon measures usually defined on $\mathbb{R}^2\times S^1$; here, however, it is more convenient to view those measures as living on the set

$$S_c := \{(y,\eta)\in\mathbb{R}^2\times\mathbb{R}^2 : |\eta|^2 = \frac{1}{c^2(y)}\}.$$

For every $n\geq 1$, denote by $M^n(y,\eta)$ the semiclassical measure associated to the vector valued sequence of functions $(\sqrt{\rho}v_0^{\varepsilon,n}, \sqrt{k}\nabla u_0^{\varepsilon,n}, n\pi\sqrt{k}u_0^{\varepsilon,n}/\varepsilon)$. This is a 4×4 nonnegative matrix of Radon measures defined on $\mathbb{R}^2\times\mathbb{R}^2$.
We set

$$\nu_{0,\pm}(y,\eta) = \frac{1}{2}\Big(N_{00}+N_{11}+N_{22} \mp 2\frac{1}{|\eta|}\,\mathrm{Re}(\eta_1 N_{10}+\eta_2 N_{20})\Big) \tag{1.13}$$

and

(1.14)

$$\begin{aligned}\mu^n_{0,\pm}(y,\eta) \quad = \quad & \frac{1}{2}\Big(M^n_{00}+M^n_{11}+M^n_{22}+M^n_{33} \\ & \mp 2\frac{1}{\sqrt{|\eta|^2+n^2\pi^2}}\,\mathrm{Re}(\eta_1 M^n_{10}+\eta_2 M^n_{20}+in\pi M^n_{30})\Big)\ ,\ n\geq 1.\end{aligned}$$

The weak limit $e^0(t,y,z)$ of the energy density e^ε defined in (1.6) will be expressed in terms of the pushforward of those measures along the following Hamiltonian flows. For every $s\in\mathbb{R}$, we successively define $\Gamma^0_s(y,\eta) = (y(s),\eta(s))$ for $(y,\eta)\in S_c$, and $\Gamma^n_s(y,\eta) = (y(s),\eta(s))$ for $(y,\eta)\in\mathbb{R}^2\times\mathbb{R}^2$, $n\geq 1$, as the solutions to

$$\dot{y} = \frac{c(y)\eta}{\sqrt{|\eta|^2+n^2\pi^2}}\quad ,\quad \dot{\eta} = -\sqrt{|\eta|^2+n^2\pi^2}\,\nabla c(y)\ ,\quad n\geq 0\ , \tag{1.15}$$

with the Cauchy data $y(0)=y, \eta(0)=\eta$. Note that $\Gamma^0_s(y,\eta)\in S_c$ since $(y,\eta)\in S_c$.

Theorem 1.4. *The sequence e^ε converges in $\mathcal{M}_{w*}((0,T)\times\mathbb{R}^2\times(0,1))$ to*

$$e^0(t,y,z) = e^{0,0}(t,y) + \sum_{n=1}^{\infty} e^{0,n}(t,y,z)$$

where

$$\begin{aligned}
e^{0,0}(t,y) &= \frac{1}{2}\int_{c(y)|\eta|=1} \left(\nu_+(t,y,d\eta) + \nu_-(t,y,d\eta)\right), \\
e^{0,n}(t,y,z) &= \frac{1}{2}\int_{\eta\in\mathbb{R}^2} \left(1 + \frac{|\eta|^2}{|\eta|^2+n^2\pi^2}\cos(2n\pi z)\right)(\mu_+^n(t,y,d\eta) + \mu_-^n(t,y,d\eta)),
\end{aligned}$$

and

$$\nu_\pm(t) = \Gamma^0_{\mp t}(\nu_{0,\pm}) \quad ; \quad \mu^n_\pm(t) = \Gamma^n_{\mp t}(\mu^n_{0,\pm}).$$

¶

Remark 1.5. The flows Γ^n rely on geodesic flows of suitable Riemannian metrics. Indeed, introducing the metric

$$g^0 = \frac{dy^2}{c(y)^2}$$

on $\mathbb{R}^2$, it is easy to check that Γ^0 is exactly the arclength parameterized geodesic flow of g^0 restricted to S_c.

In order to interpret Γ^n for $n \geq 1$, first observe that

$$\Gamma^n_s(y,\eta) = n\pi\Gamma_s\left(y, \frac{\eta}{n\pi}\right)$$

where Γ_s is the flow associated with the following system of ordinary differential equations,

$$\dot{y} = \frac{c(y)\eta}{\sqrt{|\eta|^2+1}} \quad , \quad \dot{\eta} = -\sqrt{|\eta|^2+1}\ \nabla c(y)\ .$$

Furthermore, let us introduce on $\mathbb{R}^3$ the metric

$$g = \frac{dy^2+dz^2}{c(y)^2}\ .$$

Then $\Gamma_s(y,\eta)$ can be obtained as the projection on $\mathbb{R}^2\times\mathbb{R}^2$ of the arclength parameterized geodesic curve of g, with initial covector $(y,z,\eta,\zeta=1)$, z being any real number. ¶

1.4. **Application to observation of waves.** The result of the previous subsection is now used, so as to derive an observability result on a bounded thin domain. Let ω be a smooth bounded domain of $\mathbb{R}^2$ and a be a nonnegative continuous function on $\overline{\omega}\times[0,1]$. We consider solutions to the following damped wave equation,

$$\left\{\begin{array}{ll}
\rho(y)\dfrac{\partial^2 u^\varepsilon}{\partial t^2} - \mathrm{div}_x(k(y)\nabla_x u^\varepsilon) + a\left(y,\dfrac{z}{\varepsilon}\right)\dfrac{\partial u^\varepsilon}{\partial t} = 0 & \text{on } \Omega(\varepsilon)\times\mathbb{R}, \\
u^\varepsilon = 0, & y\in\partial\omega, \\
\dfrac{\partial u^\varepsilon}{\partial z} = 0, & z\in\{0,\varepsilon\}, \\
u^\varepsilon(t=0,x) = u_0^\varepsilon(x), & \\
\dfrac{\partial u^\varepsilon}{\partial t}(t=0,x) = v_0^\varepsilon(x). &
\end{array}\right. \tag{1.16}$$

For such an equation, conservation of energy reads, for every positive time t, as

$$E(t,u^\varepsilon)+D(t,u^\varepsilon)=E(0,u^\varepsilon),$$

where

$$E(t,u^\varepsilon)=\frac{1}{2}\int_{\Omega(\varepsilon)}\left(\rho(y)\left(\frac{\partial u^\varepsilon}{\partial t}\right)^2+k(y)|\nabla_x u^\varepsilon|^2\right)dy\,dz$$

is the free energy and

$$D(t,u^\varepsilon)=\int_0^t\int_{\Omega(\varepsilon)}a\left(y,\frac{z}{\varepsilon}\right)\left(\frac{\partial u^\varepsilon}{\partial t}\right)^2 dy\,dz$$

is the loss of energy due to damping.
Further, we assume that the damping is active near the boundary of ω, that is $a(y,z)>0$ in a neighbourhood of $\omega\times(0,1)$, so as to avoid the non trivial technicalities due to the microlocal analysis near the boundary.
A natural question in control theory is to find sharp conditions on the function a, so that the free energy of such solutions decays with a fixed rate as t goes to infinity, namely

$$E(t,u^\varepsilon)\le CE(0,u^\varepsilon)\mathrm{e}^{-\alpha t} \tag{1.17}$$

for $\varepsilon<\varepsilon_0$, with constants $C>0$ and $\alpha>0$ independent on ε.
By a classical argument from semigroup theory, (1.17) is equivalent to the existence of K,T,ε_0 such that the following observability estimate holds

$$E(0,u^\varepsilon)\le K\,D(T,u^\varepsilon), \tag{1.18}$$

for $\varepsilon<\varepsilon_0$.
In the sequel, we address the issue of observability, assuming the following constraint on the initial conditions:

$$\begin{cases} u_0^\varepsilon\in H^2(\Omega(\varepsilon)),\quad v_0^\varepsilon\in H^1(\Omega(\varepsilon)),\\ \partial_z u_0^\varepsilon(t,y,0)=\partial_z u_0^\varepsilon(t,y,\varepsilon)=0,\\ v_0^\varepsilon(t,y,z)=0,\quad y\in\partial\omega,\\ \varepsilon^2\left(\|\nabla^2 u_0^\varepsilon\|^2_{L^2(\Omega(\varepsilon))}+\|\nabla v_0^\varepsilon\|^2_{L^2(\Omega(\varepsilon))}\right)\le M\,E(0,u^\varepsilon).\end{cases} \tag{1.19}$$

Note that (1.19) is a homogeneous version of assumption (1.3). We derive the following

Theorem 1.6. *Assume that $a(y,z)>0$ for every $(y,z)\in\partial\omega\times[0,1]$. Then the following conditions are equivalent.*
i) For every $M>0$, there exists $\varepsilon_0>0,K>0,T>0$ such that every family $(u^\varepsilon)_{\varepsilon<\varepsilon_0}$ of solutions of (1.16) satisfying (1.19)satisfies (1.18);
ii) Every forward geodesic curve of the Riemannian metric $g=\frac{dy^2+dz^2}{c(y)^2}$ on $\omega\times\mathbb{R}$ meets the region $\mathcal{C}:=\{y\in\omega:\exists z\in(0,1),a(y,z)>0\}\times\mathbb{R}$. ¶

Let us mention that, for the usual wave equation, such a result goes back to [13]. For similar results including a discussion of the damping at the boundary, we refer to [1] and [9]. Furthermore, the analogous problem for the wave equation with periodically oscillating coefficients is addressed in [10], where a complete answer is

given in the case of weakly oscillating Cauchy data. Finally, exact controllability by the whole boundary for homogeneous thin plates is studied in [4]. In such a case, the geodesic curves will always run into the boundary, so that the analogue of condition ii) is automatically fulfilled.
The above observability result can be construed as sitting between two and three dimensional geometric control conditions. Indeed, the two-dimensional condition would say that every horizontal forward geodesic curve of g should meet the region $\mathcal{C}$, a weaker condition than ii). Its three dimensional counterpart would impose that every forward geodesic curve of g meets the smaller set $\{(y,z) : a(y,z) > 0\}$, a generically more stringent condition.
The reader will have not failed to notice that we have expressed our result in terms of observability, rather than in terms of stabilization. Indeed, assumption (1.19) is not kept invariant by the action of the semi-group, as pointed out to us by E. Zuazua. Therefore, the usual proof of the equivalence between (1.17) and 1.18) does not seem to fit our setting. At present, we do not know if stabilization holds in the setting of the previous theorem.

2. The limit energy density : proof of the result

2.1. **Quasi-orthogonality.** Let us insert the Fourier decomposition (1.10) into the expression (1.6) for the energy density. We obtain

$$e^{\varepsilon}(t,y,z) = \sum_{n=0}^{\infty} e^{\varepsilon,n}(t,y,z) + \sum_{n\neq p=0}^{\infty} e^{\varepsilon,n,p}(t,y,z) \tag{2.1}$$

where

$$\begin{aligned} e^{\varepsilon,n}(t,y,z) = \rho(y) \Big(&((\partial_t u^{\varepsilon,n}(t,y))^2 + c^2(y)|\nabla_y u^{\varepsilon,n}(t,y)|^2)\cos^2(n\pi z) \\ &+ n^2\pi^2 c^2(y) \frac{(u^{\varepsilon,n})^2}{\varepsilon^2}(t,y)\sin^2(n\pi z)\Big), \end{aligned} \tag{2.2}$$

and

(2.3)

$$\begin{aligned} e^{\varepsilon,n,p}(t,y,z) = \rho(y) \Big(&(\partial_t u^{\varepsilon,n}\, \partial_t u^{\varepsilon,p} + c^2(y)\nabla_y u^{\varepsilon,n}.\nabla u^{\varepsilon,p})\cos(n\pi z)\cos(p\pi z) \\ &+ np\pi^2 c^2(y)\frac{u^{\varepsilon,n}}{\varepsilon}\frac{u^{\varepsilon,p}}{\varepsilon}\sin(n\pi z)\sin(p\pi z)\Big). \end{aligned}$$

In fact, the cross terms will cancel in the limit, as demonstrated in the following

Lemma 2.1. *For fixed $n \neq p$, $e^{\varepsilon,n,p}$ tends to 0 in $\mathcal{M}_{w*}((0,T)\times\mathbb{R}^2\times(0,1))$ as ε tends to 0.* ¶

Proof. We show for example that

$$\partial_t u^{\varepsilon,n}\, \partial_t u^{\varepsilon,p} \rightharpoonup 0,$$

the proof for the other terms being similar.
Keeping track of the energy bound (1.7) through the expansion, we obtain

$$\sum_{n=0}^{\infty} \left(\|\partial_t u^{\varepsilon,n}\|^2 + \|\nabla_y u^{\varepsilon,n}\|^2 + n^2 \|\frac{u^{\varepsilon,n}}{\varepsilon}\|^2 \right) \leq C, \tag{2.4}$$

while keeping track of Remark 1.1 through the rescaling and the expansion, we obtain

$$\sum_{n=0}^{\infty}\big(\|\partial_t^2 u^{\varepsilon,n}\|^2 +\|\partial_t\nabla_y u^{\varepsilon,n}\|^2+\|\nabla_y^2 u^{\varepsilon,n}\|^2 +n^2\|\frac{\partial_t u^{\varepsilon,n}}{\varepsilon}\|^2+n^2\|\frac{\nabla_y u^{\varepsilon,n}}{\varepsilon}\|^2+n^4\|\frac{u^{\varepsilon,n}}{\varepsilon^2}\|^2\Big)\le\frac{M}{\varepsilon^2}. \tag{2.5}$$

In the above equations, all norms are taken in $L^2(\omega)$.
Take the time derivative of equation (1.11), multiply it by $\varphi(t,y)\partial_t u^{\varepsilon,p}(t,y)$, where φ is a smooth compactly supported function, and integrate in all variables. Then subtract from the resulting expression that obtained by interchanging the roles of n and p. We obtain

$$(p^2-n^2)\pi^2\int\varphi k\partial_t u^{\varepsilon,n}\,\partial_t u^{\varepsilon,p}\,dt\,dy=\varepsilon\int\Big\{\rho\partial_t\varphi(\varepsilon\partial_t^2u^{\varepsilon,p}\partial_t u^{\varepsilon,n}-\varepsilon\partial_t^2u^{\varepsilon,n}\partial_t u^{\varepsilon,p}) -k\nabla_y\varphi.(\partial_t u^{\varepsilon,n}\varepsilon\nabla_y\partial_t u^{\varepsilon,p}-\partial_t u^{\varepsilon,p}\varepsilon\nabla_y\partial_t u^{\varepsilon,n})\Big\}dt\,dy.$$

In view of estimates (2.4), (2.5), the right hand-side of the above identity tends to 0 with ε. □
As a consequence of the lemma above, we know how to compute the limit energy density associated to a finite number of modes in terms of the individual energy densities

$$e^{0,n}:=\lim_{\varepsilon}e^{\varepsilon,n}. \tag{2.6}$$

We will see below that those quantities are well defined. The general case of an infinite number of modes is the object of the following

Lemma 2.2. *With the definition (2.6),*

$$e^0=\sum_{n=0}^{\infty}e^{0,n}.$$

¶

Proof. Set

$$u_N^\varepsilon(t,y,z):=\sqrt{2}\sum_{n=0}^{N}u^{\varepsilon,n}(t,y)\cos(n\pi z),$$

and denote by e_N^ε the associated energy density. The energy $E(t,u^\varepsilon-u_N^\varepsilon)$ associated to the tail $u^\varepsilon-u_N^\varepsilon$ is given by

$$E(t,u^\varepsilon-u_N^\varepsilon)=\frac12\sum_{n=N+1}^{\infty}\int\left(\rho(v_0^{\varepsilon,n})^2+k(y)(|\nabla_y u_0^{\varepsilon,n}|^2+n^2\pi^2\frac{(u_0^{\varepsilon,n})^2}{\varepsilon^2})\right)dy.$$

Recalling (2.5), we conclude that

$$E(t,u^\varepsilon-u_N^\varepsilon)\le\frac{C}{N^2}.$$

Then, using (1.7), together with Cauchy-Schwarz' inequality,

$$\int|e^\varepsilon-e_N^\varepsilon|dydz\le\frac{C}{N}.$$

We then pass to the limit, first in ε, then letting $N\nearrow\infty$. ¶

2.2. **The energy of the ground mode.** This short subsection is devoted to the computation of $e^{0,0}$. In the case $n = 0$, the scale ε drops out (1.11), so that $u^{\varepsilon,0}$ satisfies a plain two-dimensional wave equation, namely

$$\begin{cases} \rho(y)\dfrac{\partial^2 u^{\varepsilon,0}}{\partial t^2} = \operatorname{div}(k(y)\nabla u^{\varepsilon,0}), \\ u^{\varepsilon,0}(t=0,y) = u_0^{\varepsilon,0}(y), \\ \dfrac{\partial u^{\varepsilon,0}}{\partial t}(t=0,y) = v_0^{\varepsilon,0}(y)\ . \end{cases}$$

The limit energy density $e^{0,0}$ was computed in [3] and in [6]; the corresponding part of Theorem 1.4 is a mere rewriting of the results of those papers.

2.3. **The energy of the excited modes.** When $n > 0$, the scale ε is explicitly present in (1.11), endowing it with a semi-classical structure. The analysis of the weak limit $e^{0,n}$ of the corresponding energy density $e^{\varepsilon,n}$ is conveniently addressed in the framework developed in [7].
To this effect, we transform (1.11) into a 4×4-system by introducing

$$\mathcal{U}^{\varepsilon,n} := \left(\sqrt{\rho}\partial_t u^{\varepsilon,n}, \sqrt{k}\partial_{y_1} u^{\varepsilon,n}, \sqrt{k}\partial_{y_2} u^{\varepsilon,n}, n\pi\sqrt{k}\frac{u^{\varepsilon,n}}{\varepsilon}\right).$$

Then $\mathcal{U}^{\varepsilon,n}$ satisfies

$$\text{(2.7)} \qquad \begin{cases} \varepsilon\dfrac{\partial \mathcal{U}^{\varepsilon,n}}{\partial t} + (P^{\varepsilon,n})^W(y,\varepsilon D_y)\mathcal{U}^{\varepsilon,n} = 0, & \text{on } \mathbb{R}^2 \times (0,T), \\ \mathcal{U}^{\varepsilon,n}(0) = \mathcal{U}_0^{\varepsilon,n}, & \text{on } \mathbb{R}^2, \end{cases}$$

where $(P^{\varepsilon,n})^W(y,\varepsilon D_y)$ is the Weyl quantization of $P^{\varepsilon,n}(y,\eta)$ defined as

$$\text{(2.8)} \qquad P^{\varepsilon,n}(y,\eta) := P^{0,n}(y,\eta) + \varepsilon Q^0(y),$$

with

$$\text{(2.9)} \qquad P^{0,n}(y,\eta) = -c(y)\begin{bmatrix} 0 & i\eta_1 & i\eta_2 & -n\pi \\ i\eta_1 & 0 & 0 & 0 \\ i\eta_2 & 0 & 0 & 0 \\ n\pi & 0 & 0 & 0 \end{bmatrix},$$

and $Q^0(y)$ is skew-symmetric (and independent of η).

Remark 2.3. For the notion of Weyl quantization, the reader is referred to [7]. In the above computation, we only use the following elementary formula for the Weyl quantization of a symbol of the form $ib(y).\eta$, namely

$$(ib(y).\eta)^W = b(y).\nabla_y + \frac{\varepsilon}{2}\operatorname{div} b.$$

Note that the specific expression for $\mathcal{U}^{\varepsilon,n}$ in terms of $u^{\varepsilon,n}$ implies the following constraint on $\mathcal{U}^{\varepsilon,n}$:

$$\text{(2.10)} \qquad (A^{\varepsilon,n})^W(y,\varepsilon D_y)\mathcal{U}^{\varepsilon,n} = 0, \ \text{ on } \mathbb{R}^2 \times [0,T],$$

where

$$\text{(2.11)} \qquad (A^{\varepsilon,n})(y,\eta) := A^{0,n}(y,\eta) + \varepsilon B(y),$$

with

$$A^{0,n}(y,\eta) = \begin{bmatrix} 0 & i\eta_2 & -i\eta_1 & 0 \\ 0 & n\pi & 0 & -i\eta_1 \\ 0 & 0 & n\pi & -i\eta_2 \end{bmatrix}, \tag{2.12}$$

the same holding true for $\mathcal{U}_0^{\varepsilon,n}$.

In order to apply Theorem 6.1 in [7], we need to analyze the spectrum of $P^{0,n}(y,\eta)$.
The eigenvalues of $P^{0,n}(y,\eta)$ are

$$0,\ \lambda_n^{\pm}(y,\eta) := \pm ic(y)(|\eta|^2 + n^2\pi^2)^{\frac{1}{2}}.$$

The eigenspace associated to 0 is a plane generated by the vectors

$$e_{1,n}^0(\eta) := \begin{pmatrix} 0 \\ n\pi \\ 0 \\ i\eta_1 \end{pmatrix}, \quad e_{2,n}^0(\eta) := \begin{pmatrix} 0 \\ 0 \\ n\pi \\ i\eta_2 \end{pmatrix}.$$

That associated to $\lambda_n^{\pm}(y,\eta)$ is a line generated by

$$e_n^{\pm}(\eta) := \begin{pmatrix} \mp(|\eta|^2 + n^2\pi^2)^{\frac{1}{2}} \\ \eta_1 \\ \eta_2 \\ -in\pi \end{pmatrix}.$$

We conclude in particular that the multiplicity of the eigenvalues of $P^{0,n}(y,\eta)$ is constant.
We denote by $\Pi_n^0(y,\eta)$, resp. $\Pi_n^{\pm}(y,\eta)$, the orthogonal projections of $\mathbb{C}^4$ onto the eigenspaces associated to the eigenvalues 0, resp. $\lambda_n^{\pm}(y,\eta)$, and by $\mathfrak{M}^n(t,y,\eta)$ the matrix-valued semi-classical measure associated to (a subsequence of) $\mathcal{U}^{\varepsilon,n}$.
Localization applied to (2.10) immediately implies that

$$\begin{cases} A^{0,n}(y,\eta)\,\mathfrak{M}^n\,(t,y,\eta) = 0 \\ \\ A^{0,n}(y,\eta)M^n(y,\eta) = 0. \end{cases} \tag{2.13}$$

Recall that $M^n(y,\eta)$ is the matrix-valued semi-classical measure associated to the sequence

$$(\sqrt{\rho}v_0^{\varepsilon,n}, \sqrt{k}\nabla u_0^{\varepsilon,n}, n\pi\sqrt{k}u_0^{\varepsilon,n}/\varepsilon).$$

But $e_{j,n}^0(\eta)(j=1,2) \perp \mathrm{Ker}A^{0,n}(y,\eta)$ for the inner product on $\mathbb{C}^4$, therefore

$$\Pi_n^0(y,\eta)\,\mathfrak{M}^n\,(t,y,\eta)\Pi_n^0(y,\eta) \equiv 0.$$

Consequently, thanks to the analysis leading to Theorem 6.1 in [7],

$$\mathfrak{M}^n\,(t,y,\eta) = \Pi_n^+(y,\eta)\,\mathfrak{M}^n\,(t,y,\eta)\Pi_n^+(y,\eta) + \Pi_n^-(y,\eta)\,\mathfrak{M}^n\,(t,y,\eta)\Pi_n^-(y,\eta). \tag{2.14}$$

Since $\Pi_n^{\pm}(y,\eta)$ are rank-one projectors, (2.14) is equivalent to

$$\mathfrak{M}^n\,(t,y,\eta) = \mu_+^n(t,y,\eta)\Pi_n^+(y,\eta) + \mu_-^n(t,y,\eta)\Pi_n^-(y,\eta), \tag{2.15}$$

where $\mu_{\pm}^n(t,y,\eta) := \mathrm{tr}\left(\Pi_n^{\pm}(y,\eta)\,\mathfrak{M}^n\,(t,y,\eta)\Pi_n^{\pm}(y,\eta)\right)$.

According to Theorem 6.1(i) in [7], $\mu_\pm^n(t,y,\eta)$ is solution to the following transport equation:

$$\begin{cases} \dfrac{\partial \mu_\pm^n}{\partial t} + \{\lambda_\pm^n, \mu_\pm^n\} = 0, & \text{on } \mathbb{R}_t \times \mathbb{R}_y^2 \times \mathbb{R}_\eta^2, \\ \mu_\pm^n(t=0) = \mu_{0,\pm}^n, & \text{on } \mathbb{R}_y^2 \times \mathbb{R}_\eta^2, \end{cases} \tag{2.16}$$

where $\mu_{0,\pm}^n(y,\eta) := \mathrm{tr}\left(\Pi_n^\pm(y,\eta) M^n(y,\eta) \Pi_n^\pm(y,\eta)\right)$.
Since the eigenspaces associated to $\lambda_\pm^n(y,\eta)$ are one-dimensional, we obtain

$$\mu_{0,\pm}^n(y,\eta) = \frac{M^n(y,\eta) e_n^\pm(\eta) . e_n^\pm(\eta)}{|e_n^\pm(\eta)|^2}.$$

Using the explicit expressions for $e_n^\pm(\eta)$ and appealing to the second relation in (2.13), a straightforward, but tedious computation leads to the formula (1.14) for $\mu_{0,\pm}^n$.
We now invoke (2.5), together with Proposition 1.7 in [7], to compute the energy density $e^{0,n}$ defined in (2.6). We obtain

$$e^{0,n}(t,y,z) = \int_{\eta\in\mathbb{R}^2} \left(\left[\sum_{j=0,1,2} \mathfrak{M}_{jj}(t,y,d\eta) \right] \cos^2(n\pi z) + \mathfrak{M}_{33}(t,y,d\eta) \sin^2(n\pi z) \right).$$

Because of the explicit expressions for $e_n^\pm(\eta)$, the diagonal coefficients of the matrix $\mathfrak{M}^n$ are immediately computed through (2.15) and we finally obtain

$$e^{0,n}(t,y,z) = \frac{1}{2} \int_{\eta\in\mathbb{R}^2} \left(1 + \frac{|\eta|^2}{|\eta|^2 + n^2\pi^2} \cos(2n\pi z) \right) (\mu_+^n(t,y,d\eta) + \mu_-^n(t,y,d\eta)).$$

The proof of the theorem is complete once we observe that the solutions to (2.16) are precisely

$$\mu_\pm^n(t) = \Gamma_{\mp t}^n(\mu_{0,\pm}^n).$$

¶

Remark 2.4. *In the above proof – that which led to the expression for* $e^{0,n}$ *–, as well as in establishing the orthogonality of the modes in Subsection 2.1, the assumption (1.3) is essential because it ensures that no energy is lost in high frequency modes, by which we mean* $|\eta| >> \frac{1}{\varepsilon}$ *or* $n >> 1$. ¶

3. Observation of waves

In this last section, we prove Theorem 1.6. Set

$$\overline{a}(y) = \int_0^1 a(y,z)\, dz\ .$$

Proceeding by contradiction, we are led to proving the equivalence between the following two statements :

- a) There exists a sequence $\{u^\varepsilon\}$ of solutions of (1.16) that satisfies (1.19) with $E(0,u^\varepsilon) = 1$ – that is precisely (1.3)– and, for any $T > 0$, $D(T,u^\varepsilon) \to 0$.
- b) There exists a geodesic curve $t \mapsto (y(t), z(t))$ of the metric g on $\omega \times \mathbb{R}$ such that $\overline{a}(y(t)) = 0$ for every $t \geq 0$, or equivalently, which lies in the complement of $\mathcal{C} = \{y \in \omega : \exists z \in (0,1), a(y,z) > 0\} \times \mathbb{R}$.

As in the previous sections, let us carry our problem onto the fixed domain $\Omega = \omega \times (0,1)$ by stretching the vertical variable. Replacing $u^\varepsilon(t,y,z)$ by $\sqrt{\varepsilon}u^\varepsilon(t,y,\varepsilon z)$, statement a) is equivalent to the existence of a family of solutions of

(3.1)
$$\begin{cases} \rho(y)\dfrac{\partial^2 u^\varepsilon}{\partial t^2} - \mathrm{div}_y(k(y)\nabla_y u^\varepsilon) - \dfrac{1}{\varepsilon^2}k(y)\dfrac{\partial^2 u^\varepsilon}{\partial z^2} + a(y,z)\dfrac{\partial u^\varepsilon}{\partial t} = 0, & \text{on } \Omega\times\mathbb{R}, \\ u^\varepsilon = 0, & y \in \partial\omega, \\ \dfrac{\partial u^\varepsilon}{\partial z} = 0, & z \in \{0,1\}, \\ u^\varepsilon(t=0,x) = u_0^\varepsilon(x), & \\ \dfrac{\partial u^\varepsilon}{\partial t}(t=0,x) = v_0^\varepsilon(x)\,, & \end{cases}$$

with the following properties,

$$\frac{1}{2}\int_\Omega \rho(y)|v_0^\varepsilon|^2 + k(y)[|\nabla_y u_0^\varepsilon|^2 + \frac{1}{\varepsilon^2}|\partial_z u_0^\varepsilon|^2]\,dy\,dz = 1, \tag{3.2}$$

$$\forall T>0,\ \int_0^T\int_\Omega a(y,z)|\partial_t u^\varepsilon|^2\,dy\,dz\,dt \to 0, \tag{3.3}$$

together with condition (1.3) written in the rescaled variables.

We transform statement a) into an equivalent statement involving this time a family of solutions to the undamped wave equation on all of $\mathbb{R}^2\times(0,1)$. To this effect, we first observe that $u^\varepsilon \rightharpoonup 0$. Indeed, in view of (3.2), it is easily shown (as in *e.g.* [2]) that $u^\varepsilon(t,y,z) \rightharpoonup u(t,y)$, where u is characterized as the solution of

$$\begin{cases} \rho(y)\dfrac{\partial^2 u}{\partial t^2} - \mathrm{div}_y(k(y)\nabla_y u) = 0, & \text{on } \omega\times\mathbb{R}, \\ u = 0, & y\in\partial\omega, \\ u(t=0,y) = u_0(y), & \\ \dfrac{\partial u}{\partial t}(t=0,y) = u_1(y)\,, & \end{cases} \tag{3.4}$$

and where

$$\int_0^1 u^\varepsilon(0,y,z)\,dz \rightharpoonup u_0,$$

$$\int_0^1 \frac{\partial u^\varepsilon}{\partial t}(0,y,z)\,dz \rightharpoonup u_1,$$

up to the possible extraction of a subsequence.
Moreover, by lower semi-continuity, $\overline{a}(y)(\partial_t u)^2(t,y) = 0$ for every $t\geq 0$. Using classical uniqueness theorems for second order partial differential equations (see *e.g.* Robbiano [14]), we infer that $u=0$.

As a second step, we invoke equipartition of the energy (obtained by multiplying the equation by u^ε) combined with (3.3) and the fact that $a(y,z)>0$ for every y in a neighbourhood of $\partial\omega$; this implies that, for some neighbourhood A of $\partial\omega$, and for any $T>0$,

(3.5)
$$\int_0^T \int_{A\times]0,1[} \left(\rho(y)\left(\frac{\partial u^\varepsilon}{\partial t}\right)^2 + k(y)|\nabla_y u^\varepsilon|^2 + \frac{k(y)}{\varepsilon^2}\left(\frac{\partial u^\varepsilon}{\partial z}\right)^2\right) dt\, dz\, dy \longrightarrow 0\ .$$

We then introduce a cut-off function $\chi(y) \in C_0^\infty(\omega)$ with $\chi(y) \equiv 1$ near the complement of $\overline{A}$. The function $v^\varepsilon := \chi u^\varepsilon$ satisfies the undamped wave equation on $\mathbb{R}^2 \times (0,1)$ with a non-zero right hand-side $f^\varepsilon(t,y,z)$, which goes to 0 strongly in $L^2((0,T)\times\mathbb{R}^2\times(0,1))$, in view of (3.5) and (3.3). Further, v^ε satisfies (3.3), together with the rescaled version of (1.3). We claim that the energy $E(0,v^\varepsilon)$ of v^ε at $t=0$ tends to 1. Indeed, writing successively the energy conservation law for u^ε and v^ε, we get

$$TE(0,u^\varepsilon) = \int_0^T E(t,u^\varepsilon)\,dt + o(1) = \int_0^T E(t,v^\varepsilon)\,dt + o(1) = TE(0,v^\varepsilon) + o(1),$$

and the claim is a consequence of (3.2).

Subtracting from v^ε the solution to the undamped wave equation on $\mathbb{R}^2\times(0,1))$ with 0 initial data and f^ε as right hand-side, we obtain a solution w^ε to the homogeneous wave equation, which satisfies $E(0,w^\varepsilon)\to 1$, (3.3), together with the rescaled version of (1.3). Dividing w^ε by the square root of $E(0,w^\varepsilon)$, we finally get **a solution $\tilde{w}^\varepsilon$ to the homogeneous wave equation, which satisfies (3.2), (3.3), together with the rescaled version of (1.3)**. The other implication would result from similar constructions.

Let us first prove that statement a) implies statement b). In view of the above construction, w^ε satisfies assumptions of Theorem 1.4, and we conclude that its limit energy density is given by the density produced in that theorem, that is

$$e^0(t,y,z) = e^{0,0}(t,y) + \sum_{n=1}^{\infty} e^{0,n}(t,y,z),$$

where

$$e^{0,0}(t,y) = \frac{1}{2}\int_{c(y)|\eta|=1} \left(\nu_+(t,y,d\eta) + \nu_-(t,y,d\eta)\right),$$

$$e^{0,n}(t,y,z) = \frac{1}{2}\int_{\eta\in\mathbb{R}^2} \left(1 + \frac{|\eta|^2}{|\eta|^2+n^2\pi^2}\cos(2n\pi z)\right)\left(\mu_+^n(t,y,d\eta) + \mu_-^n(t,y,d\eta)\right),$$

and

$$\nu_\pm(t) = \Gamma^0_{\mp t}(\nu_{0,\pm}) \quad ; \quad \mu^n_\pm(t) = \Gamma^n_{\mp t}(\mu^n_{0,\pm}).$$

In the expressions above,

$$\nu_{0,\pm},\ \mu^n_{0,\pm}$$

are given through (1.13), resp. (1.14), in terms of the microlocal defect measure N, resp. the semi-classical measure M^n, associated to

$$\left(\sqrt{\rho}\frac{\partial w^{\varepsilon,0}}{\partial t}, \sqrt{k}\nabla w^{\varepsilon,0}\right)(0,y),\ \left(\sqrt{\rho}\frac{\partial w^{\varepsilon,n}}{\partial t}, \sqrt{k}\nabla w^{\varepsilon,n}, n\pi\sqrt{k}\frac{w^{\varepsilon,n}}{\varepsilon}\right)(0,y),$$

respectively.

In particular, the projection on the (t,y) space of the support of $\nu_\pm$ is a union of curves $t\mapsto(t,y(t))$, where $t\mapsto y(t)$ is a geodesic curve for the metric $dy^2/c(y)^2$ on ω. More generally, for every $n\geq 1$, the projection on the (t,y) space of the support of

$\mu^n_\pm$ is a union of curves $t \mapsto (t, y(t))$, where, for some appropriate function $t \mapsto z(t)$, the curve $t \mapsto (y(t), z(t))$ is geodesic for the metric $g = \dfrac{dy^2 + dz^2}{c(y)^2}$ on $\omega \times \mathbb{R}$ (the case $n = 0$ corresponds to $z = cst$).
Moreover, appealing to the equipartition of energy, the measure limits of $\rho(y)(\dfrac{\partial w^\varepsilon}{\partial t})^2$ and $k(y)(|\nabla_y w^\varepsilon|^2 + \dfrac{1}{\varepsilon^2}(\dfrac{\partial w^\varepsilon}{\partial z})^2)$ are identical, so that, recalling the above expression for e^0, property (3.3) reads as

$$\int_0^T \int_\omega \int_{S^2} \overline{a}(y)\, d\nu_\pm(t, y, \eta) = 0$$

$$\int_0^T \int_\Omega \int_{\mathbb{R}^2} a(y,z)(1 + \frac{|\eta|^2}{|\eta|^2 + n^2\pi^2} \cos(2n\pi z))\, d\mu^n_\pm(t, y, \eta)\, dz = 0,\ n \geq 1,$$

for all $T > 0$.
As a consequence, the supports of $\nu_\pm$ and $\mu^n_\pm$ do not meet the region above the set $\{\overline{a}(y) > 0\}$. Since $E(0, w^\varepsilon) \to 1$, at least one of the measures $\nu_\pm$, $\mu^n_\pm$ does not vanish and since the y-support of all these measures is a union of y-projections of geodesics for the metric g, we conclude statement b).

Conversely, assume that b) holds. Suppose for instance that the geodesic in b) does not live in $\{z = cst.\}$. Let (y^0, η^0) be such that the concerned geodesic is precisely $\frac{1}{\pi}\Gamma^1_s(y^0, \eta^0) := (y(s), \eta(s))$ (see Remark 1.5). We consider

$$u^\varepsilon_0(y, z) := \sqrt{\varepsilon} U \left(\frac{y - y^0}{\sqrt{\varepsilon}}\right) \exp \left(i\, \frac{y.\eta^0}{\varepsilon}\right) \cos(\pi z),$$

and

$$v^\varepsilon_0(y, z) := i \frac{\kappa}{\sqrt{\varepsilon}} U \left(\frac{y - y^0}{\sqrt{\varepsilon}}\right) \exp \left(i\, \frac{y.\eta^0}{\varepsilon}\right) \cos(\pi z),$$

with a suitable choice of $U \in C^\infty_0(\mathbb{R}^2)$ and of $\kappa \in \mathbb{R}$, so that

$$\mu^{1,0}_+ = 0\ ,\ \mu^{1,0}_- = 2\delta(y - y^0)\, \delta(\eta - \eta^0).$$

The corresponding solution u^ε to equation (1.5) satisfies (3.2), together with the rescaled version of (1.3). Moreover, by Theorem 1.4, the limit energy density is given by

$$e^0(t, y, z) = \delta(y - y(t))(1 + \beta \cos(2\pi z))$$

for some $\beta \in \mathbb{R}$. Since $a(y(t), z) = 0$ for every z, we conclude that u^ε also satisfies property (3.3). In the light of the equivalence established at the onset of this proof, this leads to statement a). ¶

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