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Homogenisation of a class of fourth order equations with application to incompressible elasticity

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Synopsis

Upon formalising an analogy between two-dimensional Stokes flow and two-dimensional isotropic conductivity, we exhibit a class of fourth order equations which behave "isomorphically" like isotropic conductivity from the standpoint of homogenisation and from that of corresponding bounding methods on possible effective behaviours. In particular, Lipton's result on the G-closure problem for mixtures of two incompressible elastic materials is recovered in the two-dimensional case.

Introduction

The underlying motivation of the present work finds its root in the well-known so-called potential approach to Stokes flow. The constitutive incompressibility of two- or three-dimensional Stokes flows allows for the introduction of a potential u – scalar valued in the two-dimensional case and vector valued in the three-dimensional one – such that the velocity field is given as

$$v = \operatorname{Curl} u. \tag{1}$$

The second order Stokes equation which involves the pressure field p and the strain deformation tensor

$$e(v) = \tfrac{1}{2}(\operatorname{grad} v + (\operatorname{grad} v)^t) \tag{2}$$

yields a fourth order equation (in two dimensions) or a fourth order system (in three dimensions) for the potential (cf. Subsection 3.1).

From now onward, our attention is restricted to the two-dimensional case. If ν denotes the viscosity and f the body loadings, the resulting fourth order equation reads as

$$2\frac{\partial^2}{\partial x_1\,\partial x_2}\left(\nu\frac{\partial^2 u}{\partial x_1\,\partial x_2}\right)+\frac{1}{2}\left(\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_1^2}\right)\left(\nu\left(\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_1^2}\right)u\right)=\frac{\partial f_1}{\partial x_2}-\frac{\partial f_2}{\partial x_1}. \tag{3}$$

In equation (3), the viscosity ν can vary from point to point. Upon setting

$$\bar{\partial}_1=\sqrt{2}\frac{\partial^2}{\partial x_1\,\partial x_2},\quad \bar{\partial}_2=\frac{\sqrt{2}}{2}\left(\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_1^2}\right), \tag{4}$$

equation (3) becomes

$$-\bar{\partial}_1(\nu\,\bar{\partial}_1 u)-\bar{\partial}_2(\nu\,\partial_2 u)=\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}. \tag{5}$$

Equation (5) formally resembles an equation of isotropic conduction.

In a seemingly disconnected manner, we propose to investigate the set of all possible two phase mixtures of incompressible elastic materials with prescribed volume fraction of each of the two phases. Renewed interest has been shown of late in the rigorous derivation of bounds on the effective properties of mixtures. Originating with Hashin and Shtrikman's work on bounds for isotropic mixtures of two conducting or elastic materials (cf. [**3**]), several mathematical methods have been successfully developed and have contributed various competing and/or complementary bounds. The reader is invited to refer to Kohn and Milton's excellent survey of various available methods in the case of isotropic conduction [**6**], or to the recent comprehensive study of Milton [**11**].

With the help of the so called variational method developed by Kohn and Milton (cf. [**5**, **6**, **7**, **11**]) Lipton obtained a complete characterisation of the two-dimensional effective properties of a two-phase mixture of incompressible elastic materials at fixed volume fraction [**9**]. One of the goals of the present paper is to recover Lipton's results using a different approach, the "functional method". Specifically, we place ourselves in the context of H-convergence, a notion introduced by Murat and Tartar (cf. e.g. [**12**, **18–20**]). In that context, a method for deriving bounds was firstly described by Tartar in 1977 [**20**] and subsequently applied to the case of mixtures of two isotropic conducting materials [**13**, **22**]. In fact, a complete characterisation of all possible effective tensors was achieved. Later on, further directions were investigated (cf. [**1**] for mixtures of anisotropic conducting materials; [**2**] for the elastic case and references therein). But, in all fairness, mixtures of isotropic conducting materials provide *stricto sensu* the only example of a complete characterisation for mixtures of two materials *at fixed volume fraction* using the "functional method". It is appropriate to emphasise that the other available methods do not fare differently in that respect, with the noteworthy exception of Lipton's characterisation. Note also that Cherkaev and Gibianski have recently used Lipton's result to obtain a complete characterisation of the set of two-dimensional two-phase mixtures of elastic materials with the same bulk modulus at fixed volume fraction (work in progress). In earlier work, Lurie and Cherkaev had obtained a volume fraction independent characterisation of such mixtures [**10**].

In the light of these last remarks, the analogy developed above, between Stokes flow and isotropic conductivity, permits us to hope that the wealth of information at our disposal in the latter setting can be used in the setting of incompressible elasticity (formally equivalent to that of Stokes flow). To formalise such an expectation is the goal of the present study. It should be stressed that we are not in a position to discuss inhomogeneous Stokes flow since we do not take into account the possible transport of the microstructure, or any other "fluid type" effect.

The first section of the paper is very short and uniquely devoted to notation. In the second section, a class of fourth order equations that can be reduced to second order ones is exhibited and the appropriate notions of H-convergence (Subsection 2.1) and compensated compactness (Subsection 2.2) are presented. Following Tartar's method, H-lower semi-continuous functionals are obtained in Subsection 2.3 and the analogue of the bounding theorem of Murat–Tartar (cf. [**22**, Theorem 1]) is obtained (Corollary 2.27). The third section is essentially devoted to two applications of the tools developed in Section 2. Subsection 3.1 rederives Lipton's two-dimensional result, while Subsection 3.2 addresses a N-dimensional problem.

1. Notation

The notation is consistent with that of [**2**, Section 1]. Throughout the paper, Einstein's summation convention is used unless confusion could ensue. Small Greek letters denote vectors in $\mathbb{R}^N$ or $\mathbb{R}^P$ except for α, β (strictly positive real numbers), ε (a small strictly positive parameter), θ (the volume fraction), μ, v (the viscosity), χ (a characteristic function), and φ, ψ (test functions in $\mathscr{C}_0^\infty(\mathbb{R}^N)$). Small Roman letters denote second order tensors on $\mathbb{R}^N$ or $\mathbb{R}^P$, except for x (the position vector in $\mathbb{R}^N$), u, v (vector fields on $\mathbb{R}^N$) or when used as sub/superscripts. Capital Roman letters denote fourth order tensors on $\mathbb{R}^N$ except for N, P (space dimensions), X (a generic vector space) and H (which denotes Sobolev spaces). Capital Greek letters denote third order tensors (elements of $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^P$).

Further, we adopt the following definitions:

(1) $\mathscr{L}_s(X)$ is the set of all symmetric linear mappings on X.

(2) i is the neutral element of $\mathscr{L}_s(\mathbb{R}^P)$ and tr the trace operator, i.e.

$$\operatorname{tr}({}^t p \circ q) = p_{ij} q_{ij}, \tag{1.1}$$

where p_{ij} (respectively q_{ij}) is the representative matrix of p (respectively q) in any given orthonormal basis of $\mathbb{R}^P$.

(3) $\mathscr{M}(\alpha, \beta) = \{A(x) \in L_\infty(\mathbb{R}^N; \mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^N))) \mid \|A\|_{L_\infty} \leqq \beta$ and, for any φ in $\mathscr{C}_0^\infty(\mathbb{R}^N)$,

$$\int_\Omega A_{ijkh}(x) \frac{\partial^2 \varphi}{\partial x_k\, \partial x_h} \frac{\partial^2 \varphi}{\partial x_i\, \partial x_j}\, dx \geqq \alpha \int_\Omega \frac{\partial^2 \varphi}{\partial x_i\, \partial x_j} \frac{\partial^2 \varphi}{\partial x_i\, \partial x_j}\, dx \Big\}. \tag{1.2}$$

Remark 1.1. Inequality (1.2) will be referred to as "functional coercivity". Straightforward application of Lax–Milgram's lemma shows that the Dirichlet problem

$$\frac{\partial^2}{\partial x_i\, \partial x_j}\left(A_{ijkh}(x) \frac{\partial^2 u}{\partial x_k\, \partial x_h}\right) = f \text{ in } \Omega, \tag{1.3}$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where Ω is an arbitrary bounded open domain of $\mathbb{R}^N$ and f an arbitrary element of $H^{-2}(\Omega)$, has a unique solution u in $H_0^2(\Omega)$ wherever A lies in $\mathscr{M}(\alpha, \beta)$.

Contraction of juxtaposed tensorial quantities is assumed unless "$\circ$" is used, i.e. if ξ_i, ζ_i, p_{ij}, Γ_{ijk}, A_{ijkh} are the representatives of ξ, ζ, p, Γ, A, respectively, $\xi\zeta$ is the scalar with values $\xi_i\zeta_i$, $p\xi$ is a vector with coordinates $p_{ij}\xi_j$, $p \circ \Gamma$ is a third order tensor with coefficients $p_{ik}\Gamma_{kjh}$, $\xi \otimes \eta$ is a linear mapping with coefficients $\xi_i\eta_j$, and etc. . . .

Finally grad and div, respectively, stand for the gradient and divergence operators, ^{t} denotes the adjoint of a tensor and $e(v)$ is the symmetrised gradient mapping, i.e.

$$e(\xi) = \tfrac{1}{2}(\text{grad}\,\xi + (\text{grad}\,\xi)^t).$$

2. H-Convergence of decomposable fourth order tensors

2.1. Preliminaries

Consider an element Δ of $\mathscr{L}(\mathscr{L}_s(\mathbb{R}^N); \mathbb{R}^P)$ and set

$$\hat{\Delta}(\xi) = \Delta(\xi \otimes \xi), \tag{2.1}$$

for every ξ in $\mathbb{R}^N$. Throughout this section the mapping Δ is assumed to satisfy the following hypothesis:

HYPOTHESIS (H). *$\hat{\Delta}(\xi) = 0$ if and only if $\xi = 0$, and $\hat{\Delta}$ maps $\mathbb{R}^N$ onto $\mathbb{R}^P$.*

Remark 2.1. Let Δ_i denote the ith component of Δ $(i = 1, \ldots, p)$. Then Hypothesis (H) immediately implies that at least two eigenvalues of Δ_i (as an element of $\mathscr{L}_s(\mathbb{R}^N)$) have opposite signs.

Remark 2.2. The C^1-character of $\hat{\Delta}$ implies that the measure of $\hat{\Delta}(\mathbb{R}^N)$ is zero if P is strictly greater than N (cf. [**17**, p. 54]). Thus it will be assumed henceforth that $P \leqq N$.

For any given strictly positive real constants α, β, the following definition applies:

DEFINITION 2.3. $\mathscr{D}(\alpha, \beta) = \{B \in \mathscr{M}(\alpha, \beta) \mid$ there exists an element b of $L_\infty(\mathbb{R}^N; \mathscr{L}_s(\mathbb{R}^P))$ such that

$$B(x) = {}^t\Delta \circ b(x) \circ \Delta, \text{ for almost any } x \text{ of } \mathbb{R}^N\}. \tag{2.2}$$

An element of $\mathscr{D}(\alpha, \beta)$ is called decomposable.

Remark 2.4. Throughout the text, we refer to any b that satisfies (2.2) as the *$\mathscr{D}$-associated element of B* with the understanding that it may not be uniquely defined.

Our goal is to investigate the properties of sequences B^ε of elements of $\mathscr{D}(\alpha, \beta)$ from the standpoint of the theory of homogenisation. The following definition and theorem can be applied to an arbitrary sequence B^ε of element of $\mathscr{M}(\alpha, \beta)$:

DEFINITION 2.5. A sequence B^ε of element of $\mathscr{M}(\alpha, \beta)$ H-converges to B^0, element of $\mathscr{M}(\alpha, \beta)$, if and only if for any bounded domain Ω of $\mathbb{R}^N$ and for any element f of $H^{-2}(\Omega)$ the solution $(u^\varepsilon, s^\varepsilon)$, unique in $H_0^2(\Omega) \times L_2(\Omega; \mathscr{L}_s(\mathbb{R}^N))$, of

$$s^\varepsilon = B^\varepsilon \,\text{grad grad}\, u^\varepsilon \text{ in } \Omega, \tag{2.3}$$

$$\text{div div}\, s^\varepsilon = f \text{ in } \Omega, \tag{2.4}$$

converges weakly in $H_0^2(\Omega) \times L_2(\Omega; \mathscr{L}_s(\mathbb{R}^N))$ to the solution (u^0, s^0), unique in that space, of

$$s^0 = B^0 \,\text{grad grad}\, u^0 \text{ in } \Omega, \tag{2.5}$$

$$\text{div div}\, s^0 = f \text{ in } \Omega. \tag{2.6}$$

Remark 2.6. The above definition is a transposition, in the case of an operator of order four, of one of the equivalent definitions of H-limits proposed by Murat and Tartar (cf. e.g. [**12**, **19**, **20**]). It should be emphasised that, in the classical second order elliptic case, the coercivity condition on the conductivity matrix a^ε is a pointwise condition, namely,

$$a^\varepsilon_{ij}(x)\xi_i\xi_j \geqq \alpha\,|\xi|^2, \tag{2.7}$$

for every ξ of $\mathbb{R}^N$ and almost any x of $\mathbb{R}^N$.

Condition (2.7) is equivalent to the "functional" condition

$$\int_{\mathbb{R}^N} a^\varepsilon \operatorname{grad}\varphi \operatorname{grad}\varphi\, dx \geqq \alpha \int_{\mathbb{R}^N} |\operatorname{grad}\varphi|^2\, dx, \tag{2.8}$$

for any φ in $\mathscr{C}_0^\infty(\mathbb{R}^N)$. Indeed, (2.7) trivially implies (2.8) and the converse is easily seen to hold true (cf. e.g. [**16**]).

In the case of a fourth order equation the theory of H-convergence – and in particular Theorem 2.7 below – can be carried through under the "functional" hypothesis (1.2). Hypothesis (1.2) does not imply almost pointwise very strong ellipticity, i.e.

$$B^\varepsilon_{ijkh}(x)e_{ij}e_{kh} \geqq \alpha e_{ij}e_{ij}, \tag{2.9}$$

for all elements e of $\mathscr{L}_s(\mathbb{R}^N)$ and almost every x in $\mathbb{R}^N$. The reader need only refer to Subsection 3.1 where a fourth order version of Stokes equations is presented. Of course, inequality (2.9) is a sufficient condition for (1.2) to be satisfied. Inequality (1.2) also implies strong ellipticity, namely,

$$B^\varepsilon_{ijkh}(x)\xi_i\xi_j\xi_k\xi_j \geqq \alpha\,|\xi|^4, \tag{2.10}$$

for every ξ of $\mathbb{R}^N$ and almost any x of $\mathbb{R}^N$, but the latter condition does not *a priori* guarantee existence or uniqueness of a solution to a Dirichlet problem in $H^2_0(\Omega)$ for any given bounded domain Ω (there is a Fredholm alternative). Similar considerations apply to the system of linearised elasticity where a uniform "functional" coercivity estimate can replace the usually assumed uniform pointwise very strong ellipticity hypothesis (cf. [**2**, **4**]). Once again that estimate implies strong ellipticity (also called rank-1 convexity), i.e.

$$A_{ijkh}(x)\xi_i\eta_j\xi_k\eta_h \geqq \alpha\,|\xi|^2\,|\eta|^2, \tag{2.11}$$

for every ξ, η in $\mathbb{R}^N$ and almost every x in $\mathbb{R}^N$ but inequality (2.11) is known to preclude even $H^1_0(\Omega)$-bounds on solutions to a Dirichlet problem for an arbitrary bounded domain Ω (cf. [**8**]).

The existence of an H-limit is guaranteed through the following theorem:

THEOREM 2.7. *Any sequence of elements of $\mathscr{M}(\alpha, \beta)$ has an H-converging subsequence.*

The proof of this theorem would be a strict reproduction of that given by Tartar in the original case of a second order elliptic equation (cf. e.g. [**12**]). It will not be repeated here.

COROLLARY 2.8. *In the context of Definition* 2.3, *any sequence of elements* B^ε *of* $\mathscr{D}(\alpha, \beta)$ *has an H-converging subsequence (still denoted by* B^ε*) whose limit* B^0 *belongs to* $\mathscr{M}(\alpha, \beta)$.

Remark 2.9. The closedness of the set $\mathscr{D}(\alpha, \beta)$ in the (metrisable) topology associated to H-convergence is unclear. Thus the H-limit of a sequence of decomposable tensors will be assumed to be decomposable whenever necessary.

Remark 2.10. By virtue of Hypothesis (H) and inequality (2.10), any sequence b^ε $\mathscr{D}$-associated with the H-converging sequence B^ε of Corollary 2.8 is easily shown to satisfy, for almost any x of $\mathbb{R}^N$ and every η in $\mathbb{R}^P$,

$$\tilde{\alpha}\eta\eta \leqq b^\varepsilon(x)\eta\eta \leqq \tilde{\beta}\eta\eta, \tag{2.12}$$

where $\tilde{\alpha}$, $\tilde{\beta}$ are strictly positive constants that depend on α, β and Δ only. In particular, we are at liberty, at the possible expense of extracting a further subsequence of B^ε, to assume that, as ε tends to zero,

$$\bar{s}^\varepsilon \underset{\text{def}}{\equiv} b^\varepsilon \Delta \operatorname{grad}(\operatorname{grad} u^\varepsilon) \to \bar{s}^0 \text{ weakly in } L_2(\Omega; \mathbb{R}^P). \tag{2.13}$$

Let Δ_{ijk} $(i = 1, \ldots, P, j, k = 1, \ldots, N)$ denotes the coefficients of Δ and set, for any scalar-valued field u and any $\mathbb{R}^P$-valued field v defined on $\mathbb{R}^N$,

$$\bar{\partial}_i u = \Delta_{ijk} \frac{\partial^2 u}{\partial x_j \, \partial x_k}, \tag{2.14}$$

$$\overline{\operatorname{grad}}\, u = (\bar{\partial}_i u)_{i=1,\ldots,P}, \quad \overline{\operatorname{div}}\, v = \bar{\partial}_j v_j. \tag{2.15}$$

Corollary 2.8 and Remark 2.10 can be suitably rewritten in the light of (2.14), (2.15). We obtain the following corollary:

COROLLARY 2.11. *Assume that Hypothesis* (H) *holds true. Let* B^ε *be an H-converging sequence of elements of* $\mathscr{D}(\alpha, \beta)$ *and* b^ε *be a* $\mathscr{D}$*-associated sequence. Denote by* B^0 *the H-limit of* B^ε. *For any bounded domain* Ω *of* $\mathbb{R}^N$ *and any element f of* $H^{-2}(\Omega)$, *the solution* $(u^\varepsilon, \bar{s}^\varepsilon)$, *unique in* $H_0^2(\Omega) \times L_2(\Omega; \mathbb{R}^P)$, *of*

$$\bar{s}^\varepsilon = b^\varepsilon \, \overline{\operatorname{grad}}\, u^\varepsilon \quad \text{in } \Omega, \tag{2.16}$$

$$\overline{\operatorname{div}}\, \bar{s}^\varepsilon = f \quad \text{in } \Omega, \tag{2.17}$$

converges weakly in $H_0^2(\Omega) \times L_2(\Omega; \mathbb{R}^P)$ *to* $(u^0, \bar{s}^0)$ *with*

$$^t\Delta \bar{s}^0 = B^0 \operatorname{grad}(\operatorname{grad} u^0) \quad \text{in } \Omega, \tag{2.18}$$

$$\overline{\operatorname{div}}\, \bar{s}^0 = f \quad \text{in } \Omega. \tag{2.19}$$

Remark 2.12. In accordance with Remark 2.9, the *a priori* existence of an element b^0 of $L_\infty(\mathbb{R}^N; \mathscr{L}_s(\mathbb{R}^P))$ such that, for almost every x in $\mathbb{R}^N$,

$$B^0(x) = {}^t\Delta \circ b^0(x) \circ \Delta, \tag{2.20}$$

is not obvious. If, however, (2.20) is satisfied, multiplication of (2.18) by $\xi \otimes \xi$, where ξ is an arbitrary element of $\mathbb{R}^N$, yields, in view of Hypothesis (H),

$$\bar{s}^0 = b^0 \, \overline{\operatorname{grad}}\, u^0. \tag{2.21}$$

The reader who is familiar with the classical theory of homogenisation for linear conduction will not fail to acknowledge at this point a striking analogy. It is our purpose in the rest of this section to further develop and formalise such an analogy in the hope that the machinery available in the former setting can be adapted to the latter.

One of the main issues pertaining to homogenisation is to gather as much information as possible on the possible H-limits of a set of H-converging sequences (optimal bounds) with as little knowledge as possible on the structure of the converging sequences. A convenient bounding technique consists in devising functionals that are sequentially lower semi-continuous for the topology associated with H-convergence. This functional method for deriving bounds has its roots in the theory of compensated compactness developed by Murat and Tartar (cf. [**13**, **20**]). The next subsection addresses that theory in our specific context.

Remark 2.13. It should be emphasised that other bounding techniques are available. The variational method devised by Kohn and Milton (cf. e.g. [**6**]) has proved highly successful in numerous problems of homogenisation and was, as already mentioned in the Introduction, the first method to yield bounds in the case of incompressible elasticity (cf. [**9**]).

2.2. Compensated compactness and second order constraints

The theory of compensated compactness is concerned with necessary and sufficient conditions for sequential weak lower semi-continuity of functionals. It usually considers linear constraints on the first order derivatives of the weakly converging sequences, but it can be adapted to the case of linear constraints involving higher order derivatives (see e.g. [**2**]). In this section, a specific set of second order constraints, associated with the mapping Δ and the differential operators $\bar{\partial}_i$ of Subsection 2.1, is investigated.

We set

DEFINITION 2.14. $\hat{\Lambda} = \{(p, q) \in [\mathscr{L}(\mathbb{R}^P)]^2 \mid$ there exists a non-zero ξ of $\mathbb{R}^P$ such that for any element η of $\mathbb{R}^P$ there also exists a real constant α^η satisfying

$$p\eta = \alpha^\eta \xi, \quad (q\eta)\xi = 0\}. \tag{2.22}$$

Then recollection of the definition of Δ at the beginning of Subsection 2.1 yields the following theorem:

THEOREM 2.15. *Assume that Hypothesis* (*H*) *holds true. Let* $\mathscr{B}(p, q)$ *be a quadratic form on* $[\mathscr{L}(\mathbb{R}^P)]^2$. *Then* $\mathscr{B}$ *is positive on* $\hat{\Lambda}$ *if and only if, for any arbitrary bounded domain* Ω *of* $\mathbb{R}^N$ *and two arbitrary sequences* p^ε, q^ε *of elements of* $L_2(\Omega; \mathscr{L}(\mathbb{R}^P))$ *that satisfy relations* (2.23)–(2.26) *as* ε *tends to zero, namely,*

$$p^\varepsilon \rightharpoonup p^0 \text{ weakly in } L_2(\Omega; \mathscr{L}(\mathbb{R}^P)), \tag{2.23}$$

$$q^\varepsilon \rightharpoonup q^0 \text{ weakly in } L_2(\Omega; \mathscr{L}(\mathbb{R}^P)), \tag{2.24}$$

while

$$\bar{\partial}_k p_{ij}^\varepsilon - \bar{\partial}_i p_{kj}^\varepsilon \text{ lies in a compact set of } H_{\text{loc}}^{-2}(\Omega), \tag{2.25}$$

$$\bar{\partial}_l q_{lj}^\varepsilon \text{ lies in a compact set of } H_{\text{loc}}^{-2}(\Omega), \tag{2.26}$$

for all i, j, k in $\{1, \ldots, P\}$, *we have*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \mathscr{B}(p^\varepsilon, q^\varepsilon)\, dx \geqq \int_\Omega \varphi \mathscr{B}(p^0, q^0)\, dx, \tag{2.27}$$

for any φ *in* $\mathscr{C}_0^\infty(\mathbb{R}^N)$.

Remark 2.16. Theorem 2.15 states that, under the Hypothesis (H), one is at liberty to forget the intimate structure of the operators $\bar{\partial}_i$ and to read (2.25), (2.26) as linear constraints on the first order $\bar{\partial}$-derivatives of the fields p^ε, q^ε. The theorem then reduces to a vector-valued version of the "div-curl" (or rather "$\overline{\text{div}}$-$\overline{\text{curl}}$") compensated compactness lemma (cf. [**15**, Subsection 7.1], **21**).

Proof of Theorem 2.15. Recall (2.1) and define $\Lambda = \{(p, q) \in [\mathscr{L}(\mathbb{R}^P)]^2 \mid$ there exists a non-zero element ξ of $\mathbb{R}^N$ such that for all i, j, k in $\{1, \ldots, P\}$

$$\hat{\Delta}_k(\xi)p_{ij} - \hat{\Delta}_i(\xi)p_{kj} = 0, \tag{2.28}$$

$$\hat{\Delta}_l(\xi)q_{lj} = 0\}. \tag{2.29}$$

Then a straightforward adaptation of the classical result of compensated compactness for quadratic forms (cf. [**15**, Theorem 3.2] or [**21**, Theorem 11]) to the case of higher order derivatives would yield the following lemma:

LEMMA 2.17. *In the context of Theorem* 2.15, $\mathscr{B}$ *is positive on* Λ *if and only if, for an arbitrary bounded domain* Ω *of* $\mathbb{R}^N$ *and two arbitrary sequences* p^ε, q^ε *of elements of* $L_2(\Omega; \mathscr{L}(\mathbb{R}^P))$ *that satisfy relations* (2.23)–(2.26) *as* ε *tends to zero, inequality* (2.27) *holds true.*

In view of Lemma 2.17, Theorem 2.15 will be proved if the sets Λ and $\hat{\Lambda}$ are shown to coincide. But the latter statement is an immediate consequence of Hypothesis (H).

The analogy developed in Subsection 2.1 is now complete – at least from the standpoint of the functional method – since Theorem 2.15 enables us to use the very quadratic forms that were successful in deriving bounds (optimal bounds) in the setting of linear conduction. In that respect, three quadratic forms play a decisive role. This is the object of the following lemma:

LEMMA 2.18. *The quadratic forms*

$$\operatorname{tr}({}^t p \circ p) - (\operatorname{tr} p)^2, \tag{2.30}$$

$$(P-1)\operatorname{tr}({}^t q \circ q) - (\operatorname{tr} q)^2, \tag{2.31}$$

are positive on $\hat{\Lambda}$, *whereas the quadratic form*

$$\operatorname{tr}({}^t q \circ p), \tag{2.32}$$

vanishes on $\hat{\Lambda}$.

Remark 2.19. The forms (2.30), (2.31) were first introduced by Murat and Tartar in the derivation of optimal conductivity bounds (cf. [**22**, Lemmata 2 and 3]). The form (2.32) is precisely the "energy form".

Proof of Lemma 2.18. This is an immediate consequence of the following result of linear algebra: if r is an element of $\mathscr{L}(\mathbb{R}^P)$ of rank R,

$$R\,\mathrm{tr}\,({}^t r \circ r) \geqq (\mathrm{tr}\, r)^2. \tag{2.33}$$

We have now gathered the ingredients used in the construction of H-lower semi-continuous functionals. The actual construction of such functionals and the resulting bounds are the object of the next subsection.

2.3. *H*-lower semi-continuous functionals

Let z be an arbitrary element of $\mathscr{L}(\mathbb{R}^P; \mathbb{R}^N)$ and define $z\,.\,z$ as the following linear mapping from $\mathbb{R}^P$ into $\mathscr{L}_s(\mathbb{R}^N)$:

$$(z.z\,\eta)_{kl} = \sum_{i=1}^{P} \eta_i z_{ki} z_{li}, \quad k, l = 1, \ldots, N, \tag{2.34}$$

for any η in $\mathbb{R}^P$.

We are now in a position to construct a class of H-lower semi-continuous functionals. Specifically, we obtain the following theorem:

THEOREM 2.20. *Assume that Hypothesis* (H) *holds true. Let* $\mathscr{B}(p, q)$ *be a quadratic form on* $[\mathscr{L}(\mathbb{R}^P)]^2$ *which is positive on* $\hat{\Lambda}$ *and* $\mathscr{J}(p, q)$ *be any linear form on* $[\mathscr{L}(\mathbb{R}^P)]^2$. *Define, for any* b *in* $\mathscr{L}_s(\mathbb{R}^P)$ *and any* B *in* $\mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^N))$ *such that* $\mathrm{Ker}\,\Delta \subset \mathrm{Ker}\,B$,

$$\mathscr{G}(b) = \sup_{p \in \mathscr{L}(\mathbb{R}^P)} \{\mathscr{B}(p, q) + \mathscr{J}(p, q) \mid q = b \circ p\}, \tag{2.35}$$

$$\mathscr{G}_\Delta(B) = \sup_{z \in \mathscr{L}(\mathbb{R}^P; \mathbb{R}^N)} \{\mathscr{B}(p, q) + \mathscr{J}(p, q) \mid p = \Delta \circ z\,.\,z \text{ and } {}^t\Delta \circ q = B \circ z\,.\,z\}. \tag{2.36}$$

Let B^ε *be a H-converging sequence of elements of* $\mathscr{D}(\alpha, \beta)$ *and* b^ε *be a* $\mathscr{D}$*-associated sequence. Denote by* B^0 *the H-limit of* B^ε. *If* $\mathscr{G}(b^\varepsilon)$ *lies in* $L_\infty(\mathbb{R}^N)$ *and is such that, as* ε *tends to zero,*

$$\mathscr{G}(b^\varepsilon) \to \mathscr{G}^0 \text{ weak-* in } L_\infty(\mathbb{R}^N), \tag{2.37}$$

then, for almost every x *in* $\mathbb{R}^N$,

$$\mathscr{G}_\Delta(B^0(x)) \leqq \mathscr{G}^0(x). \tag{2.38}$$

Remark 2.21. It is easily checked, with the help of Hypothesis (H) together with the invertible character of b^ε (cf. (2.12)) that for almost every x of $\mathbb{R}^N$

$$\mathrm{Ker}\, B^\varepsilon(x) = \mathrm{Ker}\,\Delta, \tag{2.39}$$

from which it is deduced that for almost any x of $\mathbb{R}^N$

$$\mathrm{Ker}\, B^0(x) \supset \mathrm{Ker}\,\Delta. \tag{2.40}$$

The proof of (2.40) is to be found at the end of the proof of Theorem 2.20.

Whenever $\mathrm{Ker}\,\Delta \subset \mathrm{Ker}\,B$, the element q in $\mathscr{L}(\mathbb{R}^P)$ that enters the definition of $\mathscr{G}_\Delta(B)$ is immediately seen to be uniquely defined once z is fixed.

Remark 2.22. If B lies in $\mathscr{D}(\alpha, \beta)$ (cf. Remark 2.4) and b is a $\mathscr{D}$-associated element of $\mathscr{L}_s(\mathbb{R}^P)$, then $q = b \circ p$ and

$$\mathscr{G}_\Delta(B) = \mathscr{G}(b). \tag{2.41}$$

In particular,

$$\mathscr{G}_\Delta(B^\varepsilon) = \mathscr{G}(b^\varepsilon). \tag{2.42}$$

Thus if the H-limit B^0 of B^ε lies in $\mathscr{D}(\alpha, \beta)$, inequality (2.38) reads as

$$\mathscr{G}(b^0(x)) \leqq \mathscr{G}^0(x), \text{ for almost every } x \text{ of } \mathbb{R}^N, \tag{2.43}$$

where b^0 is the associated element of $L_\infty(\mathbb{R}^N; \mathscr{L}_s(\mathbb{R}^P))$. Equality (2.42) and/or inequality (2.43) lend a meaning to the statement that $\mathscr{G}$ is a H-lower semi-continuous functional.

Proof of Theorem 2.20. The proof follows closely that of related results in the case of conductivity [**20**, Theorem 8]) or elasticity [**2**, Theorem 3.3]). Let φ be a positive element of $\mathscr{C}_0^\infty(\mathbb{R}^N)$ and denote by Ω^φ a bounded open subset of $\mathbb{R}^N$ which contains its support. Take ψ to be an element of $\mathscr{C}_0^\infty(\Omega^\varphi)$ with value 1 on the support of φ.

For a given p^0 in $\mathbb{R}^P \times \mathbb{R}^P$ with component p^0_{ij}, define

$$^j u^0 = \tfrac{1}{2}(^j\xi^0 x)^2 \psi, \tag{2.44}$$

where $^j\xi^0$ is an element of $\mathbb{R}^N$ whose image by $\hat{\Delta}$ is the vector $(p^0_{ij})_{i=1,\ldots,P}$ of $\mathbb{R}^P$. Then, for i in $\{1, \ldots, P\}$,

$$(\overline{\operatorname{grad}}\, {}^j u^0)_i = (\hat{\Delta}(^j\xi^0))_i = p^0_{ij}, \tag{2.45}$$

on the support of φ.

Define $^j u^\varepsilon$ to be the solution, unique in $H_0^2(\Omega^\varphi)$, of

$$\begin{aligned} {}^j\bar{s}^\varepsilon &= b^\varepsilon\, \overline{\operatorname{grad}}\, {}^j u^\varepsilon, \\ \overline{\operatorname{div}}\, {}^j\bar{s}^\varepsilon &= \operatorname{div} \operatorname{div} (B^0 \operatorname{grad} (\operatorname{grad} {}^j u^0)), \\ {}^j u^\varepsilon = 0, \quad & \frac{\partial^j u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varphi. \end{aligned} \tag{2.46}$$

According to Definition 2.5, $^j u^\varepsilon$ converges weakly in $H_0^2(\Omega^\varphi)$ to $^j u^0$ as ε tends to zero. Further, if we set

$$p^\varepsilon_{ij} = \bar{\partial}_i {}^j u^\varepsilon, \tag{2.47}$$

$$q^\varepsilon_{ij} = {}^j\bar{s}^\varepsilon_i = b^\varepsilon_{ik} p^\varepsilon_{kj} = (b^\varepsilon \circ p^\varepsilon)_{ij}, \tag{2.48}$$

p^ε and q^ε satisfy (2.23)–(2.26) on Ω^φ. Furthermore, as ε tends to zero,

$$p^\varepsilon_{ij} \to \bar{\partial}_i {}^j u^0, \text{ weakly in } L_2(\Omega^\varphi), \tag{2.49}$$

$$q^\varepsilon_{ij} \to {}^j\bar{s}^0_i, \text{ weakly in } L_2(\Omega^\varphi), \tag{2.50}$$

where, according to Corollary 2.11 (relation (2.18)),

$${}^t\Delta {}^j\bar{s}^0 = B^0 \operatorname{grad} (\operatorname{grad} {}^j u^0). \tag{2.51}$$

Almost everywhere on the support of φ,

$$\bar{\partial}_i{}^j u^0(x) = p^0_{ij}, \tag{2.52}$$

$$ {}^j\bar{\xi}^0_i(x) \underset{\text{def}}{\equiv} q^0_{ij}(x), \tag{2.53}$$

with

$$\Delta_{ikl} q^0_{ij}(x) = B^0_{klpq}(x)^j\xi^0_p{}^j\xi^0_q. \tag{2.54}$$

Note that q^0 is not a constant, in contrast to p^0.

Theorem 2.15 is applicable. Since linear forms are weakly continuous, we obtain

$$\int_{\mathbb{R}^N} \varphi[\mathscr{B}(p^0, q^0) + \mathscr{I}(p^0, q^0)]\, dx \leqq \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \varphi[\mathscr{B}(p^\varepsilon, q^\varepsilon) + \mathscr{I}(p^\varepsilon, q^\varepsilon)]\, dx$$

$$\leqq \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \varphi\mathscr{G}(b^\varepsilon)\, dx = \int_{\mathbb{R}^N} \varphi\mathscr{G}^0(x)\, dx, \tag{2.55}$$

where the last equality in (2.55) results from Hypothesis (2.37). Thus, for almost any x of $\mathbb{R}^N$,

$$\mathscr{B}(p^0, q^0(x)) + \mathscr{I}(p^0, q^0(x)) \leqq \mathscr{G}^0(x). \tag{2.56}$$

For any z in $\mathscr{L}(\mathbb{R}^P, \mathbb{R}^N)$, set

$$ {}^j\xi^0_i = z_{ij}\ (j = 1, \ldots, P, i = 1, \ldots, N), \tag{2.57}$$

and take the supremum of the left-hand side of inequality (2.56). This yields inequality (2.38) and completes the proof of Theorem 2.20.

In order to prove (2.40), we introduce for any element p of $\mathscr{L}_s(\mathbb{R}^N)$ the solution u^ε, unique in $H^2_0(\Omega^\varphi)$, of

$$\operatorname{div}\operatorname{div}(B^\varepsilon \operatorname{grad}(\operatorname{grad} u^\varepsilon)) = \operatorname{div}\operatorname{div}(B^0 \operatorname{grad}(\operatorname{grad}(\psi pxx))). \tag{2.58}$$

For any element q of Ker Δ, the following string of inequalities holds true:

$$0 = \lim_{\varepsilon\to 0}\int_{\Omega^\varphi} \varphi B^\varepsilon \operatorname{grad}(\operatorname{grad} u^\varepsilon) q\, dx$$

$$= \int_{\Omega^\varphi} \varphi B^0 \operatorname{grad}(\operatorname{grad}(\psi pxx)) q\, dx$$

$$= \int_{\Omega^\varphi} \varphi B^0 pq\ dx, \tag{2.59}$$

which proves (2.40) since φ is arbitrary.

By virtue of Lemma 2.18, Theorem 2.20 applies to the functionals $\mathscr{G}^l$, $\mathscr{G}^u$, $\mathscr{G}^l_\Delta$, $\mathscr{G}^u_\Delta$ defined as follows:

DEFINITION 2.23. For any b in $\mathscr{L}_s(\mathbb{R}^P)$ and any B in $\mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^N))$,

$$\mathscr{G}(b) = \sup_{p \in \mathscr{L}(\mathbb{R}^P)} \{\bar{\alpha}[\operatorname{tr}({}^t p \circ p) - (\operatorname{tr} p)^2] - \operatorname{tr}({}^t p \circ b \circ p) + 2 \operatorname{tr} p\}, \tag{2.60}$$

$$\mathscr{G}^u(b) = \sup_{p \in \mathscr{L}(\mathbb{R}^P)} \{(P-1)\operatorname{tr}({}^t p \circ b \circ b \circ p) - [\operatorname{tr}(b \circ p)]^2 - \bar{\beta}(P-1) \operatorname{tr}({}^t p \circ b \circ p) + 2\operatorname{tr}(b \circ p)\}, \tag{2.61}$$

$$\mathscr{G}_\Delta(B) = \sup_{z \in \mathscr{L}(\mathbb{R}^P, \mathbb{R}^N)} \{\bar{\alpha}[\operatorname{tr}({}^t p \circ p) - (\operatorname{tr} p)^2] - \operatorname{tr}({}^t q \circ p) + 2 \operatorname{tr} p \mid p = \Delta \circ z \,.\, z, \quad {}^t\Delta \circ q = B \circ z \,.\, z\}, \tag{2.62}$$

$$\mathscr{G}^u_\Delta(B) = \sup_{z \in \mathscr{L}(\mathbb{R}^P, \mathbb{R}^N)} \{(P-1)\operatorname{tr}({}^t q \circ q) - (\operatorname{tr} q)^2 - \bar{\beta}(P-1)\operatorname{tr}({}^t q \circ p) + 2 \operatorname{tr} q \mid p = \Delta \circ z \,.\, z, \quad {}^t\Delta \circ q = B \circ z \,.\, z\}. \tag{2.63}$$

Of course $\bar{\alpha}$ and $\bar{\beta}$ are not arbitrary. Specifically, they are determined with the help of the following lemma whose proof is found in [**22**, Propositions 1 and 2].

LEMMA 2.24. *For any b in $\mathscr{L}_s(\mathbb{R}^P)$ with strictly positive eigenvalues $\lambda_1, \ldots, \lambda_P$, $\mathscr{G}(b)$ and $\mathscr{G}^u(b)$ are finite if and only if $0 \leqq \bar{\alpha} \leqq \lambda_1, \ldots, \lambda_P \leqq \bar{\beta}$, in which case*

$$\mathscr{G}(b) = \cfrac{1}{\bar{\alpha} + \cfrac{1}{\sum_{j=1}^P \cfrac{1}{\lambda_j - \bar{\alpha}}}}, \qquad \mathscr{G}^u(b) = \cfrac{1}{1 + \cfrac{P-1}{\sum_{j=1}^P \cfrac{\lambda_j}{\bar{\beta} - \lambda_j}}}. \tag{2.64}$$

(*Note that if $B = {}^t\Delta \circ b \circ \Delta$, $\mathscr{G}_\Delta(B) = \mathscr{G}(b)$ (respectively, $\mathscr{G}^u_\Delta(B) = \mathscr{G}^u(b)$)*).

Remark 2.25. If in the context of Lemma 2.24 b is isotropic, i.e. $\lambda_1 = \ldots = \lambda_P = \lambda$,

$$\mathscr{G}(b) = \frac{P}{\lambda + (P-1)\bar{\alpha}}, \qquad \mathscr{G}^u(b) = P\frac{\lambda}{\lambda + (P-1)\bar{\beta}}. \tag{2.65}$$

Theorem 2.20 applies to $\mathscr{G}$ and $\mathscr{G}^u$ with $0 \leqq \bar{\alpha} \leqq \tilde{\alpha}$ and $\tilde{\beta} \leqq \bar{\beta}$ (the constants $\tilde{\alpha}$ and $\tilde{\beta}$ were defined in Remark 2.10). The case of interest to us is that of a $\mathscr{D}$-associated sequence b^ε which is of the form

$$b^\varepsilon_{(x)} = (\chi^\varepsilon(x)\tilde{\alpha} + (1 - \chi^\varepsilon(x))\tilde{\beta})i, \tag{2.66}$$

for almost any x of $\mathbb{R}^N$, with

$$\chi^\varepsilon \rightarrow \theta, \text{ weak-}* \text{ in } L_\infty(\mathbb{R}^N) \tag{2.67}$$

as ε tends to zero, and we obtain the following corollary:

COROLLARY 2.26. *Assume that Hypothesis* (H) *holds true and let B^ε be an H-converging sequence of elements of $\mathscr{D}(\alpha, \beta)$ such that a $\mathscr{D}$-associated sequence b^ε is given by* (2.66). *Denote by B^0 the H-limit of B^ε. Then, for almost any x of*

$\mathbb{R}^N$, *and any* $(\bar{\alpha}, \bar{\beta})$ *in* $[0, \tilde{\alpha}] \times [\tilde{\beta}, +\infty)$

$$\mathscr{G}_\Delta(B^0(x)) \leqq P\Big(\frac{\theta(x)}{\bar{\alpha} + (P-1)\bar{\alpha}} + \frac{1-\theta(x)}{\bar{\beta} + (P-1)\bar{\beta}}\Big), \tag{2.68}$$

$$\mathscr{G}^*_\Delta(B^0(x)) \leqq P\Big(\frac{\theta(x)\tilde{\alpha}}{\bar{\alpha} + (P-1)\bar{\beta}} + \frac{(1-\theta(x))\tilde{\beta}}{\bar{\beta} + (P-1)\bar{\beta}}\Big), \tag{2.69}$$

where $\mathscr{G}_\Delta$ *and* $\mathscr{G}^*_\Delta$ *are defined by* (2.62), (2.63).

Corollary 2.27. *If, in the context of Corollary* 2.26, B^0 *is an element of* $\mathscr{D}(\alpha, \beta)$ *and if* $\lambda_1^0, \ldots, \lambda_P^0$ *denote the eigenvalues of a* $\mathscr{D}$*-associated element* b^0, *then, for almost any* x *of* $\mathbb{R}^N$,

$$\mu_-(\theta(x)) \leqq \lambda_i^0(x) \leqq \mu_+(\theta(x)), \tag{2.70}$$

$$\sum_{i=1}^{P} \frac{1}{\lambda_i^0(x) - \tilde{\alpha}} \leqq \frac{1}{\mu_-(\theta(x)) - \tilde{\alpha}} + \frac{P-1}{\mu_+(\theta(x)) - \tilde{\alpha}}, \tag{2.71}$$

$$\sum_{i=1}^{P} \frac{1}{\tilde{\beta} - \lambda_i^0(x)} \leqq \frac{1}{\tilde{\beta} - \mu_-(\theta(x))} + \frac{P-1}{\tilde{\beta} - \mu_+(\theta(x))}, \tag{2.72}$$

where for any θ *in* $[0, 1]$

$$\mu_-(\theta) = \frac{\tilde{\alpha}\tilde{\beta}}{\theta\tilde{\beta} + (1-\theta)\tilde{\alpha}}, \quad \mu_+(\theta) = \theta\tilde{\alpha} + (1-\theta)\tilde{\beta}. \tag{2.73}$$

Proof. Inequalities (2.71), (2.72) of Corollary 2.27 are a mere rewriting of inequalities (2.68), (2.69) when $\bar{\alpha} = \tilde{\alpha}$ and $\bar{\beta} = \tilde{\beta}$. Inequalities (2.70) result from the following classical estimates on B^0 (cf. e.g. [**20**, Theorem 5])

$$B_-(x) \leqq B^0(x) \leqq B_+(x), \quad \text{almost everywhere}, \tag{2.74}$$

where B_+ and $[B_-]^{-1}$, respectively, denote the weak-* limits in $L_\infty(\mathbb{R}^N; \mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^N)))$ of B^ε and $[B^\varepsilon]^{-1}$ as ε tends to zero. The bounds (2.74) (also called Voigt–Reuss bounds) are applied to the sequence B^ε under consideration and the resulting inequalities (2.70) are derived because of Hypothesis (H).

Remark 2.28. Corollary 2.27 is the exact analogue of [**22**, Theorem 1] (or at least of the necessary conditions of that theorem). The machinery developed in this section demonstrates that it can be applied *stricto sensu* to our setting whenever the H-limit of a sequence of decomposable tensors can be shown to remain decomposable.

3. Applications

3.1. A two-dimensional example: incompressible elasticity

In a quasistatic linearised setting, the velocity field v of an incompressible fluid of viscosity ν satisfies Stokes equations, namely,

$$\begin{aligned} -\operatorname{div}(\nu e(v)) + \operatorname{grad} p &= f, \\ \operatorname{div} v &= 0, \end{aligned} \tag{3.1}$$

together with the appropriate set of boundary conditions which will not be specified at this point. The system (3.1) also represents the behaviour of an incompressible isotropic elastic solid with shear modulus $\frac{\nu}{2}$, in which case v should be understood as the displacement field. In the latter setting, a non-isotropic solid may be considered; the system (3.1) then becomes

$$\begin{gathered} -\operatorname{div}(Ae(v)) + \operatorname{grad} p = f, \\ \operatorname{div} v = 0, \end{gathered} \tag{3.2}$$

where A sends trace free symmetric second order tensors into themselves. In a two-dimensional setting, A reads as

$$A = \mu_1 d^1 \otimes d^1 + \mu_2 d^2 \otimes d^2 + \mu_3[d^1 \otimes d^2 + d^2 \otimes d^1], \tag{3.3}$$

where

$$d^1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad d^2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.4}$$

Then system (3.2) can be transformed into a fourth order elliptic equation. Setting

$$v = \operatorname{Curl} u \underset{\text{def}}{\equiv} \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\right), \tag{3.5}$$

and taking the (scalar) rotational of the first equation in (3.2) yields the following scalar equation of order four:

$$\operatorname{div}(\operatorname{div} B(\operatorname{grad}(\operatorname{grad} u))) = -\operatorname{curl} f, \tag{3.6}$$

where

$$\operatorname{curl} f = \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \tag{3.7}$$

and B is the element of $\mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^2))$ whose matrix is

$$B = \frac{1}{2}\begin{bmatrix} \mu_2 & -\mu_2 & -\mu_3\sqrt{2} \\ -\mu_2 & \mu_2 & \mu_3\sqrt{2} \\ -\mu_3\sqrt{2} & \mu_3\sqrt{2} & 2\mu_1 \end{bmatrix} \tag{3.8}$$

in the orthonormal basis f^1, f^2, f^3 of $\mathscr{L}_s(\mathbb{R}^2)$ defined as

$$f^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad f^3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.9}$$

Since

$$e(v) = \begin{bmatrix} \dfrac{\partial^2 u}{\partial x_1\, \partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{\partial^2 u}{\partial x_1^2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{\partial^2 u}{\partial x_1^2}\right) & -\dfrac{\partial^2 u}{\partial x_1\, \partial x_2} \end{bmatrix}, \tag{3.10}$$

it is tempting to introduce

$$\bar{\partial}_1 = \sqrt{2}\frac{\partial^2}{\partial x_1\, \partial x_2}, \quad \bar{\partial}_2 = \frac{\sqrt{2}}{2}\left(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\right), \tag{3.11}$$

where the constants $\sqrt{2}$, $\sqrt{2}/2$ are normalisation constants (cf. (3.14)); relations (3.11) suggest in turn that Δ should be defined as the linear mapping from $\mathscr{L}_s(\mathbb{R}^2)$ into $\mathbb{R}^2$ whose matrix is given by

$$\Delta = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \tag{3.12}$$

in the basis $(f^i)_{i=1,2,3}$ of $\mathscr{L}_s(\mathbb{R}^2)$ and the canonical basis of $\mathbb{R}^2$. Then, setting

$$b \equiv \mu_1 f^1 + \mu_2 f^2 + \mu_3 \sqrt{2} f^3, \tag{3.13}$$

yields

$$B = {}^t\Delta \circ b \circ \Delta. \tag{3.14}$$

Remark 3.1. Since, for any ξ in $\mathbb{R}^2$ with components (ξ_1, ξ_2) in the canonical basis of $\mathbb{R}^2$,

$$\Delta(\xi \otimes \xi) = \left(\sqrt{2}\,\xi_1\xi_2, \frac{\sqrt{2}}{2}(\xi_2^2 - \xi_1^2)\right), \tag{3.15}$$

Δ satisfies Hypothesis (H) of Section 2.

Note that the kernel of Δ is the one-dimensional subspace of $\mathscr{L}_s(\mathbb{R}^2)$ generated by i (the identity matrix).

Remark 3.2. b defined in (3.13) has the following matrix representation with respect to the canonical basis of $\mathbb{R}^2$:

$$b = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}. \tag{3.16}$$

Remark 3.3. Whenever A is an element of $L_\infty(\mathbb{R}^2; \mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^2)))$ and u is an element of $\mathscr{C}_0^\infty(\mathbb{R}^N)$, it is easily checked that

$$Ae(\operatorname{Curl} u)e(\operatorname{Curl} u) = B \operatorname{grad}(\operatorname{grad} u) \operatorname{grad}(\operatorname{grad} u) \tag{3.17}$$

almost everywhere in $\mathbb{R}^2$.

Remark 3.4. In the context of Remarks 2.6 and 3.3, existence and uniqueness of v in $H_0^1(\Omega)$ satisfying (3.2) on a given bounded domain Ω of $\mathbb{R}^2$, together with Dirichlet boundary conditions on the boundary of Ω, is ensured through Korn's inequality if A is "functionally" coercive, i.e. if for any divergence free element φ of $\mathscr{C}_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} A(x)e(\varphi)(x)e(\varphi)(x)\,dx \geqq 2\alpha \int_{\mathbb{R}^2} e(\varphi)(x)e(\varphi)(x)\,dx. \tag{3.18}$$

In (3.18), α is a strictly positive real number. Then, in view of (3.17),

$$\int_\Omega B(x) \operatorname{grad}(\operatorname{grad} u)(x) \operatorname{grad}(\operatorname{grad} u)(x)\,dx \geqq 2\alpha \int_\Omega e(v)(x)e(v)(x)\,dx. \tag{3.19}$$

But

$$\int_\Omega e(v)(x)e(v)(x)\,dx = \int_\Omega \left[2\left(\frac{\partial^2 u}{\partial x_1\,\partial x_2}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1^2}\right)^2\right] dx, \tag{3.20}$$

and through integration by parts of the term involving $\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}$, we obtain

$$\int_\Omega e(v)(x)e(v)(x)\,dx = \frac{1}{2}\int_\Omega \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x_1\,\partial x_2}\right)^2\right] dx, \tag{3.21}$$

which implies that B satisfies the "functional" coercivity hypothesis (1.2) (although B clearly does not satisfy very strong ellipticity (cf. Remark 2.6)). Note that the converse also holds true.

Our analysis has shown that to any two-dimensional anisotropic incompressible elastic material – with bounded measurable elasticity coefficients satisfying "functional" coercivity – can be associated an element B of $\mathscr{D}(\alpha, \beta)$, where 2α is the ellipticity constant of the material (cf. Remark 3.4), β is the essential-sup norm of the elastic coefficients and $\mathscr{D}(\alpha, \beta)$ is given through Definition 2.3 (with Δ defined by (3.12)). The following simple characterisation of $\mathscr{D}(\alpha, \beta)$ will be used in the sequel:

Lemma 3.5. *$\mathscr{D}(\alpha, \beta) = \{B \in \mathscr{M}(\alpha, \beta) \mid$ the kernel of B contains the subspace generated by i almost everywhere$\}$.*

Proof. An element B of $\mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^2))$ is of the form (3.14) if and only if it is of the form (3.8). But checking that the elements of $\mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^2))$ of the form (3.8) are precisely those whose kernel contains the identity is a trivial task.

The analysis of H-converging sequences of elements of $\mathscr{D}(\alpha, \beta)$ in the spirit of Section 2 is then facilitated by the following corollary:

Corollary 3.6. *The H-limit B^0 of a H-converging sequence B^ε of elements of $\mathscr{D}(\alpha, \beta)$ also belongs to $\mathscr{D}(\alpha, \beta)$.*

Proof. By virtue of (2.40) in Remark 2.21

$$\operatorname{Ker} B^0(x) \supset \operatorname{Ker} \Delta \tag{3.22}$$

for almost any x of $\mathbb{R}^2$, which in view of Lemma 3.5 and Remark 3.1 yields the result.

We now investigate two-dimensional mixtures of two isotropic incompressible elastic materials in prescribed local volume fraction $\theta(x)$. We consider sequences of elements A^ε of $L_\infty(\mathbb{R}^2; \mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^2)))$ of the form

$$A^\varepsilon(x) = (\chi^\varepsilon(x)\alpha + (1-\chi^\varepsilon(x))\beta)I \quad \alpha \leqq \beta, \tag{3.23}$$

for almost any x of $\mathbb{R}^2$, where χ^ε is the characteristic function of the first material, I the identity matrix on $\mathscr{L}_s(\mathbb{R}^2)$ and

$$\chi^\varepsilon \to \theta, \text{ weak-* in } L_\infty(\mathbb{R}^2), \tag{3.24}$$

as ε tends to zero. According to the analysis developed above, a sequence B^ε of elements of $\mathscr{D}(\alpha/2, \beta)$ can be associated to the sequence A^ε. Specifically, by virtue of (3.3), (3.13), (3.16), the $\mathscr{D}$-associated b^ε's are, for almost every x of $\mathbb{R}^2$,

$$b^\varepsilon(x) = (\chi^\varepsilon(x)\alpha + (1-\chi^\varepsilon(x))\beta)i, \tag{3.25}$$

and Corollary 2.27 applies. We obtain the following theorem:

THEOREM 3.7. *Let A^ε be a sequence of elements of $L_\infty(\mathbb{R}^2; \mathscr{L}_s(\mathscr{L}_s(\mathbb{R}^2)))$ of the form* (3.23), (3.24) *such that the associated sequence B^ε of elements of $\mathscr{D}(\alpha, \beta)$ H-converge to B^0 and set, for almost any x of $\mathbb{R}^2$,*

$$B^0(x) = {}^t\Delta \circ b^0(x) \circ \Delta, \tag{3.26}$$

where $b^0(x)$ lies in $\mathscr{L}_s(\mathbb{R}^2)$.

Then the eigenvalues $\lambda_1^0(x)$, $\lambda_2^0(x)$ of $b^0(x)$ satisfy

$$\frac{1}{\lambda_1^0(x) - \alpha} + \frac{1}{\lambda_2^0(x) - \alpha} \leqq \frac{1}{\mu_-(\theta(x)) - \alpha} + \frac{1}{\mu_+(\theta(x)) - \alpha}, \tag{3.27}$$

$$\frac{1}{\beta - \lambda_1^0(x)} + \frac{1}{\beta - \lambda_2^0(x)} \leqq \frac{1}{\beta - \mu_-(\theta(x))} + \frac{1}{\beta - \mu_+(\theta(x))} \tag{3.28}$$

almost everywhere, with

$$\mu_-(\theta) = \frac{\alpha\beta}{\theta\beta + (1-\theta)\alpha}, \quad \mu_+(\theta) = \theta\alpha + (1-\theta)\beta. \tag{3.29}$$

Remark 3.8. Because an elastic tensor acting on trace free elements of $\mathscr{L}_s(\mathbb{R}^N)$ can be trivially associated to any element of $\mathscr{D}(\alpha, \beta)$ (cf. (3.3)) and since attention could be further restricted to H-converging sequences of tensors of the form A^ε (cf. [**2**] for a detailed treatment of H-convergence in the setting of linearized elasticity), Theorem 3.7 could be restated solely in terms of the eigenvalues of the possible H-limits A^0 of A^ε (these eigenvalues are precisely λ_1^0 and λ_2^0).

Proof of Theorem 3.7. Theorem 3.7 is a mere rewriting of Corollary 2.27 in the context of Subsection 3.1, but for the fact that inequality (2.70) has been deleted. In two dimensions however, (2.71) and (2.72) are well known to imply (2.70) (cf. [**22**]).

Remark 3.9. As mentioned before, this result was firstly derived by Lipton in [**9**, theorem 2.0]. In that paper, bounds are actually derived in an N-dimensional setting ($N \geqq 2$) and optimality is proved for $N = 2$. The machinery developed in Section 2 would be inadequate to handle dimensions higher than two because there is no scalar equation of order four which corresponds to the Stokes system even for constant viscosities. Lipton's bounds could, however, be recovered in the case $N \geqq 3$ by adapting the H-convergence technique described in [**2**] to the case of incompressible elasticity and by using the following quadratic forms:

$$(N-1)\operatorname{tr}({}^tP \circ P) - (\operatorname{tr} P)^2, \tag{3.30}$$

$$N\left(\frac{N-1}{2}\right)\operatorname{tr}({}^tQ \circ Q) - (\operatorname{tr} Q)^2, \tag{3.31}$$

$$\operatorname{tr}({}^tQ \circ P), \tag{3.32}$$

where P, Q belong to $\mathscr{L}(\mathscr{L}_s(\mathbb{R}^N))$. This result will not be dwelt upon any further, since our main purpose in this paper is to examine the second-fourth order analogy exhibited in Section 2.

3.2. A N-dimensional example

In this subsection, a N-dimensional example is briefly developed. The investigated fourth order equation is a N-dimensional extension of the equation (3.6) derived in the previous subsection, but we are not aware of a physical interpretation in the case of a dimension greater than two.

Consider the mapping Δ from $\mathscr{L}_s(\mathbb{R}^N)$ into $\mathbb{R}^N$ defined as

$$\Delta f^{11} = -\Delta f^{22} = \ldots = -\Delta f^{NN} = \delta^1; \tag{3.33}$$

$$\Delta f^{1j} = \bar{\beta}\delta^j, j = 2, \ldots, N; \quad \Delta f^{ij} = 0, \quad i \neq j, \quad i, j \geqq 2, \tag{3.34}$$

where f^{ij} ($i = 1, \ldots, N, j = 1, \ldots, N$) is the orthonormal basis of $\mathscr{L}_s(\mathbb{R}^N)$ given by

$$f^{ij} = \frac{1}{\sqrt{2}}(\delta^i \otimes \delta^j + \delta^j \otimes \delta^i), \quad i \neq j; \quad f^{ii} = \delta^i \otimes \delta^i \text{ (no summation)}, \tag{3.35}$$

and $\delta^1, \ldots, \delta^N$ denote the canonical orthonormal basis of $\mathbb{R}^N$. In (3.34), $\bar{\beta}$ is a strictly positive real constant.

Then for any ξ in $\mathbb{R}^N$ with components $(\xi_1, \ldots, \xi_N)$

$$\hat{\Delta}(\xi) = \left(\xi_1\xi_1 - \sum_{j=2}^{N} \xi_j\xi_j\right)\delta^1 + \bar{\beta}\sqrt{2}\,\xi_1 \sum_{j=2}^{N} \xi_j\delta_j. \tag{3.36}$$

The mapping Δ is easily checked to satisfy Hypothesis (H).

Let b be an arbitrary element of $\mathscr{L}_s(\mathbb{R}^N)$

$$b = \sum_{i=1}^{N} b_{ii} f^{ii} + \sum_{i<j}^{N} b_{ij}\sqrt{2} f^{ij}. \tag{3.37}$$

Then

$$\begin{aligned} B = {}^t\Delta \circ b \circ \Delta = {} & b_{11}\left(f^{11} - \sum_{i>1}^{N} f^{ii}\right) \otimes \left(f^{11} - \sum_{i>1}^{N} f^{ii}\right) \\ & + \bar{\beta} \sum_{j>1}^{N} b_{1j}\left[\left(f^{11} - \sum_{i>1}^{N} f^{ii}\right) \otimes f^{1j} + f^{1j} \otimes \left(f^{11} - \sum_{i>1}^{N} f^{ii}\right)\right] \\ & + \bar{\beta}^2 \sum_{j,k=2}^{N} b_{kj} f^{1k} \otimes f^{1j}. \end{aligned} \tag{3.38}$$

Remark 3.10. Assume that b is an element of $L_\infty(\mathbb{R}^N; \mathscr{L}_s(\mathbb{R}^N))$ that satisfies strong ellipticity, i.e. that there exists a strictly positive real constant $\tilde{\alpha}$ such that

$$b(x)\xi\xi \geqq \tilde{\alpha}\xi\xi, \tag{3.39}$$

almost everywhere on $\mathbb{R}^N$ and for all ξ's in $\mathbb{R}^N$. If u is an arbitrary element of $\mathscr{C}_0^\infty(\mathbb{R}^N)$,

$$\overline{\text{grad}}\, u = \begin{bmatrix} \dfrac{\partial^2 u}{\partial x_1^2} - \displaystyle\sum_{j=2}^{N} \dfrac{\partial^2 u}{\partial x_j^2} \\ \bar{\beta}\sqrt{2}\dfrac{\partial^2 u}{\partial x_1\,\partial x_2} \\ \vdots \\ \bar{\beta}\sqrt{2}\dfrac{\partial^2 u}{\partial x_1\,\partial x_N} \end{bmatrix} \begin{matrix} \\ \uparrow \\ N-1 \\ \downarrow \end{matrix}, \tag{3.40}$$

where $\overline{\text{grad}}$ is defined in (2.15) (with Δ given by (3.33), (3.34)); hence $B(x) = {}^t\Delta \circ b(x) \circ \Delta$ satisfies

$$\int_{\mathbb{R}^N} B(x)\,\text{grad}\,(\text{grad}\,u)\,\text{grad}\,(\text{grad}\,u)\,dx = \int_{\mathbb{R}^N} b(x)\,\overline{\text{grad}}\,u\,\overline{\text{grad}}\,u\,dx, \quad (3.41)$$

and, by virtue of (3.39),

$$\int_{\mathbb{R}^N} B(x)\,\text{grad}\,(\text{grad}\,u)\,\text{grad}\,(\text{grad}\,u)\,dx$$

$$\geqq \bar{\alpha}\int_{\mathbb{R}^N}\left[\sum_{j=1}^{N}\left(\frac{\partial^2 u}{\partial x_j^2}\right)^2 + 2\sum_{\substack{j,k=2\\ j\neq k}}^{N}\frac{\partial^2 u}{\partial x_j^2}\frac{\partial^2 u}{\partial x_k^2} + 2\bar{\beta}^2\sum_{i=2}^{N}\left(\frac{\partial^2 u}{\partial x_1\,\partial x_j}\right)^2 - 2\frac{\partial^2 u}{\partial x_1^2}\sum_{j=2}^{N}\frac{\partial^2 u}{\partial x_j^2}\right]dx. \quad (3.42)$$

Integration by parts of the second and fourth term of the right-hand side of inequality (3.42) yields

$$\int_{\mathbb{R}^N} B(x)\,\text{grad}\,(\text{grad}\,u)\,\text{grad}\,(\text{grad}\,u)\,dx$$

$$\geqq \bar{\alpha}\inf\,[1,\, 2(\bar{\beta}^2 - 1)]\int_{\mathbb{R}^N}\text{grad}\,(\text{grad}\,u)\,\text{grad}\,(\text{grad}\,u)\,dx. \quad (3.43)$$

Thus B satisfies (1.2) (with appropriately defined α and β) as soon as

$$\bar{\beta} > 1, \quad (3.44)$$

and the converse also holds true (cf. Remark 2.10).

In view of Remark 3.10 to any N-dimensional second rank tensor b – with bounded measurable coefficients satisfying strong ellipticity (i.e. (3.39)) – there can be associated an element B of $\mathcal{D}(\alpha, \beta)$, where α and β have to be appropriately chosen; $\mathcal{D}(\alpha, \beta)$ is given through Definition 2.3 (with Δ defined by (3.33), (3.34)). Once again a useful characterisation of $\mathcal{D}(\alpha, \beta)$ is proposed through the following lemma:

LEMMA 3.11. *$\mathcal{D}(\alpha, \beta) = \{B \in \mathcal{M}(\alpha, \beta) \mid Bf^{11} = -Bf^{ii},\ Bf^{ij} = 0$, for all i and j in $(2, \ldots, N)$ with $i \neq j$, almost everywhere$\}$.*

The proof of this lemma is a straightforward algebraic computation. Thus, the exact analogue of the argument that led to Corollary 3.6 would yield the following corollary:

COROLLARY 3.12. *The H-limit B^0 of a H-converging sequence B^ε of element of $\mathcal{D}(\alpha, \beta)$ also belongs to $\mathcal{D}(\alpha, \beta)$.*

We investigate N-dimensional mixtures of two elements B^1 and B^2 of the form

$$B^1 = \bar{\alpha}\,{}^t\Delta \circ \Delta, \quad B^2 = \bar{\beta}\,{}^t\Delta \circ \Delta, \quad (3.45)$$

i.e.

$$B^1(\text{respectively } B^2) = \bar{\alpha}(\text{respectively } \bar{\beta})\left(f^{11} - \sum_{i>1}^{N} f^{ii}\right) \otimes \left(f^{11} - \sum_{i>1}^{N} f^{ii}\right) + \bar{\beta}^2\sum_{j=2}^{N} f^{1j} \otimes f^{1j}. \quad (3.46)$$

If we set

$$B^{\varepsilon}(x) = \chi^{\varepsilon}(x)B^1 + (1 - \chi^{\varepsilon}(x))B^2 \tag{3.47}$$

for almost any x of $\mathbb{R}^N$, where χ^{ε} is a characteristic function and

$$\chi^{\varepsilon} \rightarrow \theta, \text{ weak-* in } L_{\infty}(\mathbb{R}^N) \tag{3.48}$$

as ε tends to zero we obtain the following theorem:

THEOREM 3.13. *Let B^{ε} be a sequence of elements of $\mathscr{D}(\alpha, \beta)$ of the form* (3.46), (3.47), (3.48) *that H-converges to B^0 and set, for almost any x of $\mathbb{R}^N$,*

$$B^0(x) = {}^t\Delta \circ b(x) \circ \Delta,$$

where $b^0(x)$ lies in $\mathscr{L}_s(\mathbb{R}^N)$. Then the eigenvalues $\lambda_1^0(x), \ldots, \lambda_N^0(x)$ of $b^0(x)$ satisfy (2.70)–(2.73) *of Corollary* 2.27.

Theorem 3.13 is a mere rewriting of Corollary 2.27 in the context of Subsection 3.2.

Remark 3.14. In the case where $N = 3$, the matrix ${}^t\Delta \circ \Delta$ reads as

$$\begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\beta}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\beta}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.49}$$

in the basis $(f^{11}, \ldots, f^{23})$ of $\mathscr{L}_s(\mathbb{R}^3)$. We have in essence derived bounds for a class of anisotropic fourth order tensors on $\mathbb{R}^3$. The optimality of these bounds should be discussed but is beyond our purpose here.

3.3. A concluding remark in the two-dimensional setting

It would be tempting to apply the decomposition technique described in Section 2 to other two-dimensional systems in which a divergence free tensor may be exhibited. Indeed, such a system will give rise to a fourth order scalar equation with a decomposable coefficients tensor in the spirit of Subsection 3.1.

It is easily shown that, if $N = 2$, then

$$P = 2. \tag{3.50}$$

The associated Δ maps $\mathscr{L}_s(\mathbb{R}^2)$ into $\mathbb{R}^2$. We set

$$\Delta = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3/\sqrt{2} \\ \beta_1 & \beta_2 & \beta_3/\sqrt{2} \end{bmatrix} \tag{3.51}$$

in the basis $(f^i)_{i=1,2,3}$ of $\mathscr{L}_s(\mathbb{R}^2)$ (cf. (3.9)) and the canonical basis of $\mathbb{R}^2$. Then Δ is verified to satisfy (H) if and only if the vector

$$\zeta = \begin{bmatrix} \alpha_2\beta_3 - \alpha_3\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \tag{3.52}$$

lies strictly inside the double cone of vertex 0 of all points ξ in $\mathbb{R}^3$ whose components satisfy

$$\xi_3^2 \leqq \xi_1\xi_2, \tag{3.53}$$

i.e. if and only if

$$(\alpha_1\beta_2 - \alpha_2\beta_1)^2 < (\alpha_2\beta_3 - \alpha_3\beta_2)(\alpha_3\beta_1 - \alpha_1\beta_3) \tag{3.54}$$

We do not know of any two-dimensional systems *other than the Stokes system* for which even the dimensionality condition (3.50) is satisfied.

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