A criticism of finite elasto-plasticity

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reporting belatedly on joint work with E. Davoli (Universität Wien)

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1. The trouble with finite plasticity

Many troubles in the 80's .... what is plastic strain, which stress rate is the correct one, which yield criterion, ... ad nauseum and ..... few solutions.

Field might be pronounced "defunct" in the 10's, at least from a mechanician's standpoint: have plasticians retired? .... Well maybe a few left (like, at this seminar???).

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$$P = I \Rightarrow F = E \quad + \quad \text{hyperelastic energy } \mathcal{W} : \mathbb{M}^{3\times 3} \to \bar{\mathbb{R}}$$

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• Advent of plasticity conditioned by threshold on Cauchy stress C:

$$C \in \mathbf{K} \text{ convex (maybe compact)}$$

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Write dissipation as $C \cdot T$

$$C \cdot T \geq 0 \overset{\text{normality rule}}{\Longleftarrow} T \in N_{\mathbf{K}}(C) \text{ (normal cone)}$$

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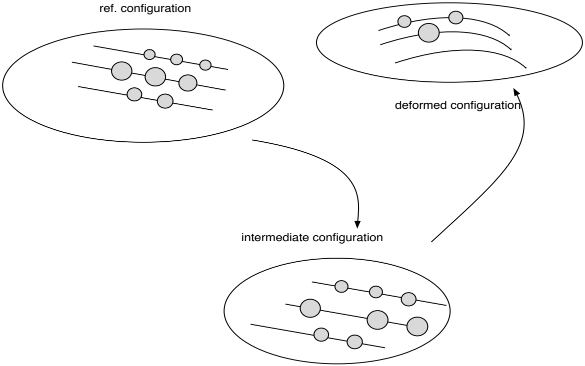
• Isochoric plastic deformation (metals, cryst. solids) $\Rightarrow \det P = 1$

3. The case for $F = EP$ – crystal plasticity

• Crystalline rearrangements due to dislocations lead to new intermediate configuration which is in turn stretched elastically.



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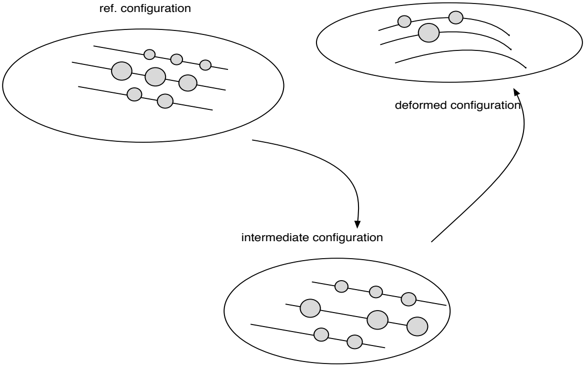

• Intermediate configuration viewed as "'new" reference conf.

• Respects plastic indifference – *was ist das? – not sure*:
$\mathcal{W}(E) = \mathcal{W}(FP^{-1}) = \mathcal{W}(FG(PG)^{-1}),\ \forall G$ with $\det G = 1$.

4. The case for $F = PE$ – polar decomposition

- Decompose P as

$$P = QP'Q^T R \text{ or as } P = \bar{R}\bar{Q}P''\bar{Q}^T$$

$$R, Q, \bar{R}, \bar{Q} \in SO(3)$$

$$P', P'' \text{ diag. } + > 0 \text{ entries}$$

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- With $R' = Q^T R$, $Q' = \bar{R}\bar{Q}$:

$$F = EP \Rightarrow F = EQP'R' = EQ'P''Q^T$$

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- Frame indifference: $\mathcal{W}(\bar{Q}^T E) = \mathcal{W}(R'E) = \mathcal{W}(E)$

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$$F = QP'E',\ P' \text{ diag. }, Q \in SO(3)$$
$$\not\equiv$$
$$\mathcal{W}(EQ) = \mathcal{W}(EQ') = \mathcal{W}(E) \text{ only if } \mathcal{W} \text{ isotropic!}$$

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- Conclusion: Preference for decomposition $F = QPE$ with

$$Q \in SO(3), \ P = \begin{pmatrix} p_1 > 0 & 0 & 0 \\ 0 & p_2 > 0 & 0 \\ 0 & 0 & p_3 > 0 \end{pmatrix}, \ p_1 p_2 p_3 = 1, \ \det E > 0$$

5. A rational(?) model

- Free energy: $\mathcal{W}'(Q,P,F) := \mathcal{W}(P^{-1}Q^TF)$
- Use the thermodynamic machine:

-Compute the first Piola-Kirchhoff stress, then the Cauchy stress via the Piola transform.

$$C = \frac{1}{\det E} QP^{-1} \underbrace{\mathbf{D}\mathcal{W}(E)E^T}_{\text{sym. because of frame indifference}} PQ^T \text{ not nec. symmetric!!}$$

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-Compute back stresses associated with P, Q:

$$B := -\partial\mathcal{W}/\partial P, S := -\partial\mathcal{W}/\partial Q$$

in terms of C.

6. A rational(?) model – part 2

- Clausius-Duhem $B \cdot \dot{P} + S \cdot \dot{Q} \geq 0$ reduces to
  $\det E \; Q^T C Q \cdot (\dot{P} P^{-1} - \dot{Q}^T Q) \geq 0 \Rightarrow \det E \; Q^T C Q \cdot \dot{P} P^{-1} \geq 0$

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- Note that $\det P = 1 \Rightarrow \mathrm{tr}\ \dot{P}P^{-1} = 0$ + frame ind. $\Rightarrow$
  $\mathbf{K} \subset$ trace free mat. + inv. under $SO(3) \Rightarrow$
  Diss. is also $Q^T C_{dev} Q \cdot \det E \ \dot{P}P^{-1} \geq 0$
  $\Downarrow$ (normality)

$$\det E \ \dot{P}P^{-1} \in N_{\mathbf{K}}(C_{dev}) = N_{\mathbf{K}}\left(\frac{1}{\det E}(\mathbf{D}\mathcal{W}(E)E^T)_{dev}\right)$$

or, equivalently,

$$\frac{1}{\det E}(\mathbf{D}\mathcal{W}(E)E^T)_{dev} \in \partial\mathcal{H}(\det E \ \dot{P}P^{-1}) = \partial\mathcal{H}(\dot{P}P^{-1})$$

where $\mathcal{H}(L) := \sup\{L \cdot C' : C' \in \mathbf{K}\}$ supp. fct. of $\mathbf{K}$.

7. Summing up – An elasto-plastic evolution

mass density ρ_0, density of loads f

The transformation field φ satisfies

$$\nabla\varphi = \begin{cases} QPE \\ P \textbf{ diagonal} \text{ with} > 0 \text{ eig.} \\ \det P = 1,\ \det E > 0,\ Q \in SO(3) \end{cases}$$

$$\rho_0 \frac{\partial^2 \varphi}{\partial t^2} - \operatorname{div}\left[QP^{-1}\mathbf{D}\mathcal{W}(E)\right] = f$$

$$\det E\ \dot{P}P^{-1} \in N_{\mathbf{K}}\left(\frac{1}{\det E}(\mathbf{D}\mathcal{W}(E)E^T)_{dev}\right)$$

$$P = I \text{ or } \mathbf{D}\mathcal{W}(E)E^T = E\mathbf{D}\hat{\mathcal{W}}^s(E^T E)E^T \textbf{ commutes with } P.$$

$+$ approp. i.c.'s and b.c.'s

8. That system is "variationalizable" in a quasi-static setting

Formally,

Any $(\varphi(t), E(t), P(t), Q(t))$ with $P(t)$ diag. with >0 entries,
$\det P(t) = 1$; $Q(t) \in SO(3)$;
$\nabla\varphi(t) = Q(t)P(t)E(t)$; e.g. $\varphi(t) = 0$ on $\partial\Omega$; that satisfies

- (Global Minimality) $(E(t), P(t))$ min.

$$\int_\Omega \mathcal{W}(E')dx - \int_\Omega f(t)\cdot\varphi' dx + \int_\Omega \det E(t)\ \mathcal{H}((\log P' - \log P(t)))dx,$$

among all (φ', E', P', Q') with $\nabla\varphi' = Q'P'E', \varphi' \equiv 0$ on $\partial\Omega$
P' diag. with >0 elts., $\det P' = 1$, $Q' \in SO(3)$

- (Energy Conservation)

$$\frac{d}{dt}\left\{\int_\Omega \mathcal{W}(E(t)dx - \int_\Omega f(t)\cdot\varphi(t)dx\right\} + \int_\Omega \det E(t)\ \mathcal{H}(\dot{L}(t))dx = -\int_\Omega \dot{f}(t)\cdot\varphi(t)dx,$$

with $L(t) := \log P(t)$,

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- Note that the commutativity constraint has disappeared!

9. Comparison with classical model $F = EP$

• In the classical model, it seems impossible to diagonalize the plastic strain P except for isotropic materials

• In the classical model, it seems impossible to accommodate a constraint on the Cauchy stress that does not depend on the elastic strain gradient: $C_D \in \partial\mathcal{H}(EP^{-1}E^T \det E)$

• In the classical model, it seems impossible to derive a formally equivalent variational evolution without an ad-hoc modification of the dissipation (Mielke 2003):

$$\mathcal{D}(P, P') := \inf \left\{ \int_0^1 \mathcal{H}(P^{-1}(s)\dot{P}(s))ds : P \text{ smooth on } [0,1]; P(0) = P, P(1) = P' \right\}$$

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$$\Downarrow$$

Dissipation gap

surely better than allowing a "mineshaft gap" says

as-of-today-President Trump  , *oops General Turgidson*  *I mean*

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Existence theorem à la Mielke – see next slide

11. Existence of a regularized variational evolution II

- Elastic energy $\mathcal{W}$:
  1. classical: polyconv., $C^1(\mathbb{M}^{3\times3}_+)$, $\mathcal{W} \equiv \infty$ on $\mathbb{M}^{3\times3} \setminus \mathbb{M}^{3\times3}_+$,
     $\mathcal{W}(Id) = 0$, $\mathcal{W}(RF) = \mathcal{W}(F)$, $R \in SO(3)$, $\mathcal{W}(F) \geq c_1 \mathrm{dist}^p(F; SO(3))$, $\mathbf{p > 3}$;
  2. less classical: $|\mathbf{D}\mathcal{W}(F)F^T| \leq c_2(\mathcal{W}(F)+1)$, and $\exists\omega$, mod. cont. s.t., for all $N \in \mathbb{M}^{3\times3}_+$,
     $|\mathbf{D}\mathcal{W}(F)F^T - \mathbf{D}\mathcal{W}(NF)(NF)^T| \leq \omega(\|N - Id\|)(\mathcal{W}(F)+1)$ (example: **Ogden mat.**)

- Hardening functional $\mathcal{W}_{hard}$:

constrained model of kin. hardening: continuous, $\mathcal{W}_{hard}(P) := \widetilde{\mathcal{W}}_{hard}(P)$,
$P \in \mathbf{V} \subset \mathbb{R}^3_1 := (\mathbb{R}^*_+)^3 \cap \{P : \det P = 1\}$ (compact with $\mathbb{1}_3$ as an interior point), $+\infty$ else.

- Dissipation functional $\mathcal{H}$: convex, positively one-hom. with $r|F| \leq \mathcal{H}(F) \leq R|F|$.

- Boundary conditions $\mathbf{g}$ on $\Gamma \subset \partial\Omega$: smooth enough, spec.

$\mathbf{g} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$: $\mathbf{g}(t,\cdot)$ global diffeom. with $\mathbf{g} \in C^2([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$, $\|\nabla\mathbf{g}\|_{L^\infty} \leq C$, $\|\nabla\dot{\mathbf{g}}\|_{L^\infty} \leq C$,

$$\|\nabla\ddot{\mathbf{g}}\|_{L^\infty} \leq C, \quad \|(\nabla\mathbf{g})^{-1}\|_{L^\infty} \leq C.$$

- Deformations φ: can be decomposed as $\varphi(t,x) = \mathbf{g}(t, y(t,x))$, with

$$y \in \mathscr{Y} := \{y \in W^{1,p}(\Omega;\mathbb{R}^3) : y\lfloor_\Gamma = id\} \text{ (mult. decomp.)}.$$

- Energy functional : for any $(y,Q,P) \in \mathscr{A} := \mathscr{Y} \times W^{1,p}(\Omega; SO(3)) \times W^{1,p}(\Omega;\mathbb{R}^3_1)$,

$$\begin{cases} \mathcal{E}(t,y,Q,P) := \int_\Omega \mathcal{W}(P^{-1}Q^T\nabla\mathbf{g}(t,y)\nabla y)\,dx \\ \mathcal{F}(t,y,Q,P) := \mathcal{E}(t,y,Q,P) + \int_\Omega \mathcal{W}_{hard}(P)dx + \int_\Omega |\nabla P|^p dx + \int_\Omega |\nabla Q|^p dx \end{cases}$$

- Variational evolution: $t \in [0,T] \mapsto (y(t),Q(t),P(t)) \in \mathscr{A}$, $L(t) := \log P(t)$ is a variational evolution if

(Glob.Min.) $\mathcal{F}(t,y(t),Q(t),P(t)) \leq \mathcal{F}(t,y',Q',P') + \int_\Omega \mathcal{H}(L' - L(t))dx$, $\forall (y',Q',P') \in \mathscr{A}$, $L' := \log P'$;

(En. Cons.) $\mathcal{F}(t,y(t),Q(t),P(t)) + \mathrm{Diss}_{\mathcal{H}}(0,t;L) = \mathcal{F}(0,y^0,Q^0,P^0) +$
$\int_0^t\int_\Omega \mathbf{D}\mathcal{W}(P^{-1}(s)Q^T(s)\nabla\mathbf{g}(s,y(s))\nabla y(s)) \cdot P^{-1}(s)Q^T(s)\nabla\dot{\mathbf{g}}(s,y(s))\nabla y(s)dxds$,

where $\mathrm{Diss}_{\mathcal{H}}(t_1,t_2;L) := \sup_{\{s_i\}} \left\{ \sum_{i=1}^N \int_\Omega \mathcal{H}(L(s_i) - L(s_{i-1}))\,dx \right\}$.

Existence Theorem: Let $(y^0,Q^0,P^0) \in \mathscr{A}$ be a stable initial condition, that is that it satisfies

$$\mathcal{F}(0,y_0,Q_0,P_0) \leq \mathcal{F}(0,y',Q',P') + \int_\Omega \mathcal{H}(L' - L_0)dx$$

for every $(y',Q',P') \in \mathscr{A}$ with $L_0 := \log P_0$. Then, there exists a variational quasi-static evolution $t \mapsto (y(t),Q(t),P(t))$ such that $y(0) = y_0$, $Q(0) = Q_0$, $P(0) = P_0$ and $L(0) = L_0$.

12. Rigid Plasticity as a first indictement

- Assume

$$\mathcal{W}(E) = \begin{cases} 0, & E \in SO(N) \\ \infty, & \text{else.} \end{cases} \Rightarrow \mathcal{W}(E) = 0 \text{ iff } F = QPR \Rightarrow$$

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- Define $\hat{\mathcal{W}}(F) := \min\left\{ \int_\Omega \mathcal{W}(E) + \mathcal{H}(\log P);\ F = QPE \text{ for } \ldots.\ \right\}$

$\Downarrow$ Von Mises plasticity

$$\hat{\mathcal{W}}(F) := h(\det F\ ) + c\sqrt{|\log \lambda_1|^2 + |\log \lambda_2|^2 + |\log \lambda_3|^2},$$

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- Lower semi-cont. envelope of $\varphi \to \int_\Omega \hat{\mathcal{W}}(\nabla\varphi)dx$ (in smooth enough space like $W^{1,p}$)?

If you believe it is local, that is of the form $\varphi \to \int_\Omega Q\hat{\mathcal{W}}(\nabla\varphi)dx$, then $Q\hat{\mathcal{W}}$ is quasi-convex, then rank-one convex and we find that

$Q\hat{\mathcal{W}} \equiv 0$ if $\det F\ = 1$!

- Not good and not specific to our model. Log. growth + mult. decomp. are the culprits!!

13. 1d as a second indictement

The deformation φ is such that $\varphi' = ep = pe$.

- The free energy $\mathcal{W}$ is taken to be such that

$$\mathcal{W}(1) = 0, \quad \begin{cases} W \geq 0 \text{ strictly convex on its domain} \\ \mathcal{W}(e) = \frac{1}{2}A(e-1)^2, e \geq 1, A > 0. \end{cases}$$

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• There is a unique spatially homogeneous solution to the variational formulation: $\varphi_{hom}(t), p_{hom}(t)$.

• Define the energy coming out of the variational formulation:

$$\mathcal{F}(t, \hat{\varphi}, \hat{p}) := \int_{(0,L)} \mathcal{W}(\hat{e})dx + \int_{(0,L)} c\left|\log \hat{p} - \log p(t)\right| dx, \ \hat{e} := \frac{\hat{\varphi}'}{\hat{p}}.$$

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The deformation φ is such that $\varphi' = ep = pe$.

- The free energy $\mathcal{W}$ is taken to be such that

$$\mathcal{W}(1) = 0, \quad \begin{cases} W \geq 0 \text{ strictly convex on its domain} \\ \mathcal{W}(e) = \frac{1}{2}A(e-1)^2, e \geq 1, A > 0. \end{cases}$$

- The boundary conditions are a stretch: $\varphi(0) = 0, \ \varphi(L) = L + td$.
- There is a unique spatially homogeneous solution to the variational formulation: $\varphi_{hom}(t), p_{hom}(t)$.
- Define the energy coming out of the variational formulation:

$$\mathcal{F}(t, \hat{\varphi}, \hat{p}) := \int_{(0,L)} \mathcal{W}(\hat{e}) dx + \int_{(0,L)} c \left| \log \hat{p} - \log p(t) \right| dx, \ \hat{e} := \frac{\hat{\varphi}'}{\hat{p}}.$$

- The spatially homogeneous solution is unstable, that is that there exists a smooth admissible variation (w, μ) of $(\varphi_{hom}(t), p_{hom}(t))$ such that

$$D^2\mathcal{F}(\varphi_{hom}(t), p_{hom}(t))[(w, \mu), (w, \mu)] < 0.$$

14. Conclusion

Is it that finite
plasticity with no hardening
makes no sense?...

or...

Is it that the multiplicative
decomposition makes
no sense?...

or....

Am I missing something?[1]

[1] .... *most likely so*