– Brutal partial damage – a case study for the interaction between evolution and relaxation

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Comes from various works from G. Allaire, me, A. Garroni, C. Larsen, J.-J. Marigo

1/ Brutal damage – a mechanical model – 1

Quasi-static evolution
no kinetic energy

Rate independence
no viscous type behavior

Energy density:
$W(\varepsilon|\chi)$

kinematic variable:
$\epsilon(u) = \varepsilon = 1/2(Du + Du^T)$
$u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$

internal variable:
χ

$+$ loads:

$f(t)$: volume or surface forces

$g(t)$: imposed displacements

Principles:

instantaneous equilibrium:

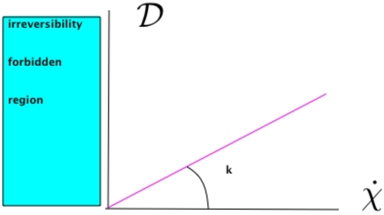
$$-\mathrm{div} D_\varepsilon W(\epsilon(u)(t), \chi(t)) = f(t)$$
$$u(t) = g(t) \text{ on } \partial\Omega$$

positivity of dissipation :

$$-D_\chi W(\epsilon(u)(t), \chi(t)) \in \partial\mathcal{D}(\dot{\chi}(t))$$
$$\mathcal{D} \text{ convex}, \geq 0, \mathcal{D}(0) = 0$$

2/ Brutal damage – a mechanical model – 2

- Assumptions:
  - on $\mathcal{D}$: $\chi \in \{0,1\}$



  - on W: $W(\varepsilon|\chi) := \frac{1}{2}(\chi A_w + (1-\chi)A_s)\varepsilon \cdot \varepsilon$

$$A_w \le A_s$$

isotropy:
$A_w = \lambda_w i \otimes i + 2\mu_w I$
idem for A_s

- $$\begin{cases} -\mathrm{div}((\chi(t)A_w + (1-\chi(t))A_s)\epsilon(u)(t)) = f(t) \\ u(t) = g(t) \text{ on } \partial\Omega \\ 1/2(A_s - A_w)\epsilon(u)(t)\cdot\epsilon(u)(t) \le k \\ \chi(t) \overset{t}{\nearrow};\ \dot{\chi}(t) = 0 \text{ if strict inequality above} \end{cases}$$

3/ Brutal damage and rate independence – 1

Stationarity of:

$$\frac{1}{2}\int_\Omega (\eta A_w + (1-\eta) A_s)\epsilon(v)\cdot\epsilon(v)dx$$
$$-\int_\Omega f(t)\cdot v dx$$
$$+\int_\Omega \mathcal{D}(\eta-\chi(t))dx$$
$$v = g(t) \text{ on } \partial\Omega$$

- $-\,\mathrm{div}(\dots) = f(t)$
  $u(t) = g(t)$ on $\partial\Omega$ $\leftrightarrow$
  $1/2(A_s - A_w)\epsilon(u)(t)\cdot\epsilon(u)(t) \le k$

- $\dot\chi(t) = 0$ if strict ineq. $\leftrightarrow$ Energy Balance:

$$\frac{d\mathcal{E}}{dt} = \int_{\partial\Omega} (\chi(t)A_w + (1-\chi(t))A_s)\epsilon(u)(t)n\cdot\dot g(t)dS - \int_\Omega \dot f(t)\cdot u(t)dx$$

with

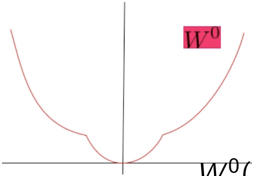
$$\mathcal{E}(t) := 1/2\int_\Omega (\chi(t)A_w + (1-\chi(t))A_s)\epsilon(u)(t)\cdot\epsilon(u)(t)dx$$
$$-\int_\Omega f(t)\cdot u(t)dx + k\int_0^t\int_\Omega \dot\chi(s)dxds$$

4/ Brutal damage and rate independence – 2

- First departure: replace Stationarity by Global Minimality
- Initial time step: u^0, χ^0 minimizes

$$\frac{1}{2}\int_\Omega \{(\chi A_w + (1-\chi)A_s)\epsilon(v)\cdot\epsilon(v) - f^0\cdot v + k\chi\}\, dx$$

$\Downarrow$ get rid of χ



u^0 minimizer of
$\int_\Omega (W^0(\epsilon(u)) - f^0\cdot u)dx$
where

$$W^0(\epsilon) = \min_{\chi\in\{0,1\}}\{\tfrac{1}{2}(\chi A_w\epsilon + (1-\chi)A_s\epsilon) + k\chi\}$$

or still

$$W^0(\varepsilon) = \min\{\tfrac{1}{2}A_s\varepsilon\cdot\varepsilon + k, \tfrac{1}{2}A_w\varepsilon\cdot\varepsilon\}$$

No minimizers! $\Longrightarrow$ Relaxation through quasiconvexification

u^0 minimizes $\int_\Omega \left\{QW^0(\epsilon(u)) - f^0\cdot u\right\} dx$

where

$$QW^0(\varepsilon) := \min\left\{\int_Y W^0(\varepsilon + \epsilon(\varphi))\, dy : \varphi \in H^1_\#(Y;\mathbb{R}^3)\right\}$$

periodic instead of $\uparrow H^1_0$

5/ Relaxation first time step – 1

- General homogenization formula for two-phase periodic mixtures:

$$A^0\varepsilon\cdot\varepsilon = \int_Y (\chi B + (1-\chi)C)(\varepsilon + \epsilon(w_\varepsilon))\cdot \begin{cases} \varepsilon & \text{or} \\ (\varepsilon & +\,\epsilon(w_\varepsilon)) \end{cases} dy,$$

with w_ε periodic solution of $\operatorname{div} A(y)(\epsilon(w_\varepsilon)+\varepsilon)=0$ with 0-mean.

- Define for any $\theta \in [0,1]$ and any B, C:

$G_\theta(B;C)$ = set of (periodic) homogenized tensors asstd. with

$$\int_Y \chi(y)\, dy = \theta$$

(Also define $G(B;C) = \cup_\theta\{G_\theta(B;C)\}$)

$$\Downarrow$$

$$QW^0(\varepsilon) := \min_{0\le\theta\le1}\left\{\min_{A^0\in\overline{G}_\theta(A_w;A_s)}\left[\tfrac{1}{2}A^0\varepsilon\cdot\varepsilon\right] + k\theta\right\}$$

- How does one compute QW^0?

Problem: $G_\theta(A_w;A_s)$ is unknown as of yet!!!

- Only need to compute $G_\theta(A_w;A_s)\varepsilon\cdot\varepsilon$.

The minimum inside the brackets is attained for a finite rank laminate. Indeed,......

6/ Energy bounds – 1 – Lamination

- Assume $\chi(y) = \chi(y_1)$ with $\int_Y \chi(y)dy = \theta$: then

$$(1-\theta)(A^{lam} - A_w)^{-1} = (A_s - A_w)^{-1} + \theta f_s(e_1), \text{ where}$$
$$f_s(e)\varepsilon := \frac{1}{\mu_w}\varepsilon e_1 \odot e_1 - \frac{\lambda_w+\mu_w}{\mu_w(\lambda_w+2\mu_w)}(\varepsilon e_1 \cdot e_1)e_1 \odot e_1.$$

Proof: Seek $\epsilon(w_\varepsilon) = \varepsilon_w \chi + \varepsilon_s(1-\chi)$, $\varepsilon_w, \varepsilon_s$ constant.

◇ $\varepsilon_w \chi + \varepsilon_s(1-\chi)$ sym. grad. $\Longrightarrow$ $\varepsilon_w - \varepsilon_s = \tau \odot e_1$ for some τ

◇ $A(y)(\varepsilon + \epsilon(w_\varepsilon)) = A_w(\varepsilon+\varepsilon_w)\chi + A_s(\varepsilon+\varepsilon_s)(1-\chi)$ has 0 div.

$$\Longrightarrow [A_w(\varepsilon+\varepsilon_w) - A_s(\varepsilon+\varepsilon_s)]e_1 = 0 \Longrightarrow$$
$$[(A_s - A_w)(\varepsilon+\varepsilon_s)]e_1 = \mu_w\tau + (\lambda_w+\mu_w)(\tau\cdot e_1)e_1$$

◇ Set $h := (A_s - A_w)(\varepsilon+\varepsilon_s) \Rightarrow \tau = \frac{1}{\mu_w}he_1 - \frac{\lambda_w+\mu_w}{\mu_w(\lambda_w+2\mu_w)}(he_1\cdot e_1)e_1$

$$\Longrightarrow \tau \odot e_1 = \frac{1}{\mu_w}he_1 \odot e_1 - \frac{\lambda_w+\mu_w}{\mu_w(\lambda_w+2\mu_w)}(he_1\cdot e_1)e_1 \odot e_1$$

$$◇ \begin{cases} \theta\varepsilon_w + (1-\theta)\varepsilon_s = 0 \Rightarrow \varepsilon_s = -\theta\tau\odot e_1 \overset{def.of\,h}{\Rightarrow} \varepsilon = (A_s - A_w)^{-1}h + \theta\tau\odot e_1 \\ A^{lam}\varepsilon = \theta A_w(\varepsilon+\varepsilon_w) + (1-\theta)A_s(\varepsilon+\varepsilon_s) \end{cases}$$

$$(A^{lam} - A_w)\varepsilon = \theta A_w(\varepsilon+\varepsilon_w) + (1-\theta)A_s(\varepsilon+\varepsilon_s) - A_w(\varepsilon + \theta\varepsilon_w + (1-\theta)\varepsilon_s)$$
$$= (1-\theta)(A_s - A_w)(\varepsilon+\varepsilon_s) = (1-\theta)h$$
$$\Longrightarrow \varepsilon = (1-\theta)(A^{lam} - A_w)^{-1}h \text{ also } = (A_s - A_w)^{-1}h + \theta\tau\odot e_1.$$

7/ Energy bounds – 2 – Lamination

• Formula iterates (finite rank laminates):

$$(1-\theta)(A^{lam}-A_w)^{-1} =$$
$$(A_s - A_w)^{-1} + \theta \sum_1^p m_i f_s(e_i), \quad \begin{cases} m_i \geq 0 \\ \sum_1^p m_i = 1. \end{cases}$$

$$\Downarrow$$

• In general any finite rank laminate is given by $(1-\theta)(A^{lam}-A_w)^{-1} = (A_s - A_w)^{-1} + \theta \int_{S^{N-1}} f_s(e) d\nu(e)$, with ν probability measure on the sphere (since extreme points of the set of such measures are Dirac masses).

• Multi-ranks (> 1) laminates are not periodic! A subtle point....

8/ Energy bounds – 3 – Hashin-Shtrikman bounds

• Thm: Any A^0 in $G_\theta(A_w; A_s)$ is such that there exist two finite rank laminates A^- and A^+ with

$$A^- \leq A^0 \leq A^+.$$

Proof: $A^0\varepsilon\cdot\varepsilon =$

$\inf\{\int_Y(1-\chi)(A_s-A_w)(\varepsilon+\epsilon(v))\cdot(\varepsilon+\epsilon(v))dy+\int_Y A_w\text{idem}\, dy : v\in H^1_{per}\}$

$= \inf_v\{\sup_\eta\{\int_Y(1-\chi)(2\eta\cdot(\varepsilon+\epsilon(v))-(A_s-A_w)^{-1}\eta\cdot\eta)dy+\text{idem}\ \}\geq$

same with constant $\eta =$

$$\sup_{\eta cst.}\{A_w\varepsilon\cdot\varepsilon+(1-\theta)[2\eta\cdot\varepsilon-(A_s-A_w)^{-1}\eta\cdot\eta]$$
$$+\inf_v\{\int_Y A_w\epsilon(v)\cdot\epsilon(v)-2\chi\eta\cdot\epsilon(v))dy\}\}$$

• the inf. in v is computed with Fourier series...... $\Longrightarrow$

$$A^0\varepsilon\cdot\varepsilon\geq\sup_{\eta cst.}\Big\{A_w\varepsilon\cdot\varepsilon+(1-\theta)[2\eta\cdot\varepsilon-(A_s-A_w)^{-1}\eta\cdot\eta]$$
$$-\sum_{k\neq 0}|\hat{\chi}_k|^2 f_s(\frac{k}{|k|})\eta\cdot\eta\Big\}$$

$$\sum_{k\neq 0}|\hat{\chi}_k|^2=\int_Y(\chi-\theta)^2dy=\theta(1-\theta)$$

9/ Energy bounds – 4 – Hashin-Shtrikman bounds – part II

Set $\nu := \frac{1}{\theta(1-\theta)} \sum_{k\neq 0} |\hat{\chi}_k|^2 \delta_{\frac{k}{|k|}}$

$$\Downarrow$$

$$A^0\varepsilon\cdot\varepsilon \geq A_w\varepsilon\cdot\varepsilon +$$

$$(1-\theta)\sup_{\eta cst.}\left[2\eta\cdot\varepsilon - \Big(\underbrace{(A_s - A_w)^{-1} + \theta\int_{S^{N-1}} f_s(e)d\nu(e)}\Big)\eta\cdot\eta\right]$$

$= (1-\theta)(A^{lam} - A_w)^{-1}$ for some laminate

Thus: $A^0\varepsilon\cdot\varepsilon \geq A^{lam}\varepsilon\cdot\varepsilon$ □

$$\Downarrow$$

$$\min_{A^0\in\overline{G}_\theta(A_w;A_s)}\left[\tfrac{1}{2}A^0\varepsilon\cdot\varepsilon\right] = A_w\varepsilon\cdot\varepsilon + (1-\theta)\min_\nu\{\sup_\eta ....\}$$

$$\Downarrow$$

$$\min_{A^0\in\overline{G}_\theta(A_w;A_s)}[\frac{1}{2}A^0\varepsilon\cdot\varepsilon] =$$

$$A_w\varepsilon\cdot\varepsilon + (1-\theta)\sup_\eta[2\eta\cdot\varepsilon - (A_s - A_w)^{-1}\eta\cdot\eta - \theta\max_{e\in S^{N-1}} f_s(e)\eta\cdot\eta]$$

$$= A_w\varepsilon\cdot\varepsilon + (1-\theta)\sup_{\eta_1,\eta_2}\left[2(\eta_1\varepsilon_1 + \eta_2\varepsilon_2) - \frac{(\eta_1^2+\eta_2^2)}{4(K_s - K_w)} - \frac{(\eta_1-\eta_2)^2}{4(\mu_s-\mu_w)}\right.$$

$$\left.-\frac{\theta}{\mu_w}\max_{0\leq\nu\leq 1}\left(\eta_1^2\nu + \eta_2^2(1-\nu) - \frac{\lambda_w+\mu_w}{\lambda_w+2\mu_w}(\nu\eta_1 + (1-\nu)\eta_2)^2\right)\right]$$

10/ Relaxation first time step – 2 – 2d case

• We have to compute $\min_{A^0 \in \overline{G}_\theta(A_w;A_s)} [\frac{1}{2} A^0 \varepsilon \cdot \varepsilon]$ using the previous expression.

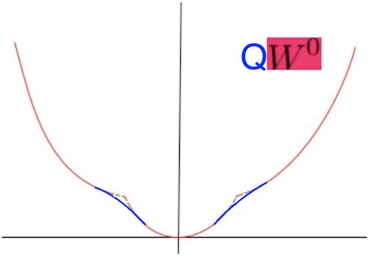
Not a simple task.

• Explicit result is unimportant: There are three regimes

◇ regime 1: $(K_s - K_w)(\theta \mu_s + (1-\theta)\mu_w)|\mathrm{tr}\ \varepsilon| < (\mu_s - \mu_w)(\theta K_s + (1-\theta)K_w)\sqrt{2}\|\varepsilon_d\| \implies$ rank-one layering

◇ regime 2: $\theta(K_s - K_w)|\mathrm{tr}\ \varepsilon| \geq (\mu_w + \theta K_s + (1-\theta)K_w)\sqrt{2}\|\varepsilon_d\| \implies$ rank-one layering

◇ regime 3: the rest $\implies$ rank-2 layering

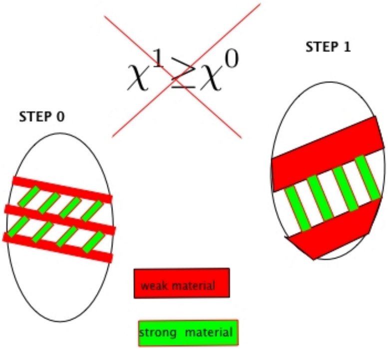


Still have to minimize in θ, so as to obtain $QW^0(\varepsilon)$!

In the end: we get a pair $(\theta^0(\varepsilon), A^0(\varepsilon))$ with $QW^0(\varepsilon) = 1/2 A^0(\varepsilon)\varepsilon \cdot \varepsilon + k\theta_0(\varepsilon)$

11/ Relaxation next time steps t_i^n, $0 = t_0^n \leq ... \leq t_{k(n)}^n = T$

• Say we perform time stepping. At time step 1, it seems that we should impose $\theta_1 \geq \theta_0$. Bad:



At next time step mix weak material with result of previous step, paying at maximum in terms of dissipated energy the vol. frac. of remaining strong material at previous step

at time t_i^n with $\Downarrow$ Θ_{i-1}^n v.f. strong mat. at t_{i-1}^n

$$W(t_i^n, \varepsilon) := \min \left\{ \tfrac{1}{2} A_w \varepsilon \cdot \varepsilon + k\Theta_{i-1}^n, \tfrac{1}{2} A_{i-1}^n \varepsilon \cdot \varepsilon \right\}$$

$$\Downarrow$$

$$QW(t_i^n, \varepsilon) = \min_{0 \leq \theta \leq 1} \left[\min_{A \in \overline{G}_\theta(A_w, A_{i-1}^n)} \{ \tfrac{1}{2} A\varepsilon.\varepsilon \} + k\Theta_{i-1}^n \theta \right]$$

12/ Relaxation next time steps t_i^n, $0 = t_0^n \leq ... \leq t_{k(n)}^n = T$ – 2

- u_i^n minimizer for $I(t_i^n) = \min_v \left\{ \int_\Omega QW(t_i^n, \epsilon(v))dx - \int_\Omega f_i^n \cdot v dx \right\}$
- θ_i^n and A_i^n measurable minimizers for $QW(t_i^n, \epsilon(u_i^n))$
- Set:

$$\text{v.f. strong mat.: } \Theta_i^n := \Theta_{i-1}^n(1 - \theta_i^n), \quad \Theta_{-1}^0 := 1$$

$$\implies \theta_i^n := 1 - \frac{\Theta_i^n}{\Theta_{i-1}^n}$$

$$\Downarrow$$

$$QW(t_i^n, \epsilon(u_i^n)) = \tfrac{1}{2} A_i^n \epsilon(u_i^n) \cdot \epsilon(u_i^n) + k(\Theta_{i-1}^n - \Theta_i^n)$$

$$I(t_i^n) = \int_\Omega \left\{ \tfrac{1}{2} A_i^n \epsilon(u_i^n) \cdot \epsilon(u_i^n)\, dx + k(\Theta_{i-1}^n - \Theta_i^n) \right\} dx - \int_\Omega f_i^n \cdot u_i^n dx$$

- Note that u_i^n minimizes in particular

$$\frac{1}{2} A_i^n \epsilon(v) \cdot \epsilon(v) dx - \int_\Omega f_i^n \cdot v dx$$

13/ Properties of the discrete time evolution

- Monotonicity: $A_i^n \searrow i \nearrow$
- G-closure: $A_{i+l}^n(x) \in \overline{G}_{(\theta_{i+l}^n + (1-\theta_{i+l}^n)\theta_{i+l-1}^n)(x)}(A_w, A_{i+l-2}^n) = \overline{G}_{1-\left[\frac{\Theta_{i+l}^n}{\Theta_{i+l-2}^n}\right](x)}(A_w, A_{i+l-2}^n)$

$$\Downarrow$$

$$A_j^n(x) \in \overline{G}_{1-\left[\frac{\Theta_j^n}{\Theta_i^n}\right](x)}(A_w, A_i^n),\ j > i$$

- Lower bound total energy:

$\mathcal{T}_i^n := \frac{1}{2}\int_\Omega A_i^n \epsilon(u_i^n).\epsilon(u_i^n)\, dx - \int_\Omega f_i^n \cdot u_i^n dx + k\int_\Omega (1-\Theta_i^n)\, dx = I(t_i^n) + k\int_\Omega (1-\Theta_{i-1}^n)\, dx$

$$\Downarrow$$

$$\mathcal{T}_j^n - \mathcal{T}_i^n + \int_\Omega (f_j^n - f_i^n)\cdot u_j^n dx \geq 0$$

- Continuity Estimate:

$$\|u_j^n - u_i^n\|_{H_0^1} \leq C\left\{\|f_j^n - f_i^n\|_{H^{-1}(\Omega;\mathbb{R}^N)} + \|\Theta_j^n - \Theta_i^n\|_{L^1(\Omega)}^{\frac{1}{2}}\right\}$$

- Upper bound total energy:

$\mathcal{T}_i^n = \int_\Omega \frac{1}{2}A_i^n \epsilon(u_i^n)\cdot\epsilon(u_i^n)\, dx - \int_\Omega f_i^n \cdot u_i^n dx + k\int_\Omega (1-\Theta_i^n)\, dx \leq$

$\mathcal{T}_0 - \sum_{j=1}^{i}\int_\Omega \int_{t_{j-1}^n}^{t_j^n} \dot{f}(\sigma) u_{j-1}^n d\sigma dx$, if e.g. $f \in W^{1,1}(0,T; L^2(\Omega;\mathbb{R}^N))$

14/ Time interpolation

• Define the piecewise constant in time interpolants of all quantities on $[t_i^n, t_{i+1}^n)$:

$$u^n(t),\ A^n(t) \overset{t}{\searrow}, \Theta^n(t) \overset{t}{\searrow}, f^n(t),\ l^n(t),\ \mathcal{T}^n(t)$$

• **H-convergence** (Murat-Tartar):

We say that $A^n \overset{H}{\rightharpoonup} A$ iff, for any $f \in H^{-1}(\Omega;\mathbb{R}^N)$, the solutions of

$$-div\ A^n\epsilon(u^n)) = f, \quad u^n \in H_0^1(\Omega;\mathbb{R}^N)$$

satisfy

$$\begin{cases} u^n \rightharpoonup u, \text{ weakly in } H_0^1(\Omega;\mathbb{R}^N) \\ A^n\epsilon(u^n) \rightharpoonup A\epsilon(u), \text{ weakly in } L^2(\Omega;\mathbb{R}^{N\times N}), \end{cases}$$

where u is the solution of

$$-div\ A\ \epsilon(u) = f, \quad u \in H_0^1(\Omega;\mathbb{R}^N)$$

The following compactness thm. is at the root of H-convergence:

If A^n is uniformly strongly elliptic and bounded, there exists a subsequence, $A^{k(n)}$ and $A \in L^\infty$ with same constants of ellipticity and boundedness such that $A^n \overset{H}{\rightharpoonup} A$.

15/ Time interpolation – 2

$$\text{Assume } \Delta_n = t_i^n - t_{i-1}^n \searrow 0$$

- Then, in particular $\exists\{k(n)\}_n$ such that

$A^{k(n)}(t) \overset{H}{\rightharpoonup} A(t), \quad \Theta^{k(n)}(t) \overset{L^\infty}{\rightharpoonup} \Theta(t), \quad A(t,x) \in \overline{G}_{1-\Theta(t,x)}(A_w, A_s)$
Proof: metrizable char. H-conv. + G-closure prop. +Thm. Mainik-Mielke:

> *Let $(\mathcal{Y}, d)$ be a compact metric space and let $Y_n : [0,T] \to \mathcal{Y}$ be a sequence with equibounded total variation $\mathrm{Var}_d(Y_n, [0,T])$ with respect to the distance d. Then, there exists a subsequence $\{k(n)\}$ of $\{n\}$ and a function $Y : [0,T] \to \mathcal{Y}$ such that*
>
> $$d(Y_{k(n)}(t), Y(t)) \overset{n}{\to} 0, \ \forall \ t \in [0,T] .$$

- The associated $u^{k(n)}$ satisfies $u^{k(n)}(t) \overset{H_0^1}{\rightharpoonup} u(t)$ with

$$u(t) \text{ minimizes } \tfrac{1}{2}\int_\Omega A(t)\epsilon(v)\cdot\epsilon(v) - \int_\Omega f(t)\cdot v dx$$

$\overset{cont.est.}{\Longrightarrow} \|u(t)\|_{H_0^1} \leq C$, provided e.g. $f \in W^{1,1}(0,T; L^2(\Omega;\mathbb{R}^N))$

16/ Minimality in the limit

- Take $\left\{ \begin{array}{l} \theta \in L^\infty(\Omega;[0,1]) \\ A(x) \in \overline{G}_{\theta(x)}(A_w, A(t,x)), \text{ a.e. in } \Omega \end{array} \right.$ arbitrary $\Longrightarrow$

$\exists \chi_p$ char. fct. with $\left\{ \begin{array}{l} \chi_p \overset{L^\infty}{\rightharpoonup} \theta \\ \chi_p A_w + (1-\chi_p)A(t) \overset{H}{\rightharpoonup} A, \ p \nearrow \infty. \end{array} \right.$

$\Downarrow$ locality

$$\chi_p A_w + (1-\chi_p)A^n(t) \in G_{[\theta^n(t)(1-\chi_p)+\chi_p]}(A_w, A^n(t-\Delta_n)) \overset{H}{\rightharpoonup} \chi_p A_w + (1-\chi_p)A(t)$$

forgetting the x-dependence $\uparrow$ with $\theta^n(t) := \frac{\Theta^n(t-\Delta_n)-\Theta^n(t)}{\Theta^n(t-\Delta_n)}$

- Then: $\int_\Omega \frac{1}{2} A^n(t)\epsilon(u^n(t)) \cdot \epsilon(u^n(t))\, dx - \int_\Omega f^n(t) \cdot u^n(t) dx +$

$k \int_\Omega (\Theta^n(t-\Delta_n) - \Theta^n(t)) dx \le \int_\Omega QW^n(t, \epsilon(v_p^n))\, dx - \int_\Omega f^n(t) \cdot v_p^n dx \le$

$\int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A^n(t))\epsilon(v_p^n) \cdot \epsilon(v_p^n)\, dx - \int_\Omega f^n(t) \cdot v_p^n dx$

$+k \int_\Omega \Theta^n(t-\Delta_n)(\theta^n(t)(1-\chi_p) + \chi_p)\, dx$

$= \int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A^n(t))\epsilon(v_p^n) \cdot \epsilon(v_p^n)\, dx - \int_\Omega f^n(t) \cdot v_p^n dx$

$+k \int_\Omega [\, \Theta^n(t-\Delta_n) - \Theta^n(t))(1-\chi_p) + \Theta^n(t-\Delta_n)\chi_p]\, dx.$

17/ Minimality in the limit – 2

- Choose v_p^n minimizer of

$$\int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A^n(t))\epsilon(v)\cdot\epsilon(v)dx - \int_\Omega f(t)\cdot v dx$$

and assume $\Theta^{n_t}(t-\Delta_{n_t}) \overset{L^\infty}{\rightharpoonup} \Psi$

$$\Downarrow \text{ pass to limit in previous ineq. in } n$$

$$\int_\Omega \frac{1}{2}A(t)\epsilon(u(t))\cdot\epsilon(u(t)dx - \int_\Omega f(t)\cdot u(t)dx + k\int_\Omega(\Psi-\Theta(t))dx \leq$$

$$\int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A(t))\epsilon(v_p)\cdot\epsilon(v_p)\, dx - \int_\Omega f(t)\cdot v_p dx$$

$$+ k\int_\Omega [(\Psi-\Theta(t))(1-\chi_p) + \Psi\chi_p]\ dx$$

with v_p minimizer of

$$\int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A(t))\epsilon(v)\cdot\epsilon(v)dx - \int_\Omega f(t)\cdot v dx$$

$$\Downarrow \text{ pass to limit in previous ineq. in } p$$

$$\text{idem} \leq \int_\Omega \frac{1}{2}A\epsilon(\bar{v})\cdot\epsilon(\bar{v})\, dx - \int_\Omega f(t)\cdot\bar{v}dx + k\int_\Omega[(\Psi-\Theta(t))(1-\theta)+\Psi\theta]$$

with $\bar{v}$ minimizer of $\int_\Omega \frac{1}{2}A\epsilon(v)\cdot\epsilon(v)dx - \int_\Omega f(t)\cdot v dx$

$$\Downarrow\ \forall v$$

$$\frac{1}{2}\int_\Omega A(t)\epsilon(u(t))\cdot\epsilon(u(t)dx - \int_\Omega f(t)\cdot u(t)dx \leq \frac{1}{2}\int_\Omega A\epsilon(v)\cdot\epsilon(v)dx - \int_\Omega f(t)\cdot v dx + k\int_\Omega \Theta(t)\theta dx$$

18/ Minimality in the limit – 3

• v.f. of weak mat. is $1 - \Theta(t)$ for a solution, and $(1-\theta)(1-\Theta(t)) + \theta = 1 - \Theta(t) + \Theta(t)\theta$ for a competitor $\Longrightarrow$ previous condition is equivalent to:

$$\frac{1}{2}\int_\Omega A(t)\epsilon(u(t))\cdot\epsilon(u(t)dx - \int_\Omega f(t)\cdot u(t)dx + k\int_\Omega(1-\Theta(t))dx \le \frac{1}{2}\int_\Omega A\epsilon(v)\cdot\epsilon(v)dx - \int_\Omega f(t)\cdot vdx + k\int_\Omega(1-\Theta)dx$$

Θ v.f. of strong material for $A(x) \in \overline{G}(A_w, A(t,x))$

19/ Energy balance

- Upper bound on total energy $\Rightarrow$

$$\begin{aligned}\mathcal{T}(t) &:= \int_\Omega \frac{1}{2}A(t)\epsilon(u(t))\cdot\epsilon(u(t))dx - \int_\Omega f(t)\cdot u(t)dx + k\int_\Omega(1-\Theta(t))dx \\ &\leq \mathcal{T}_0 - \int_0^t\int_\Omega \dot{f}(\sigma)\cdot u(\sigma)dxd\sigma\end{aligned}$$

- Lower bound on the total energy $\Longrightarrow$

$$\mathcal{T}(t') - \mathcal{T}(t) \geq -\int_\Omega (f(t') - f(t))\cdot u(t')dx,\ t' > t$$

$$+ \text{ continuity estimate}$$

$$\Downarrow$$

$$\begin{aligned}\mathcal{T}(t) &:= \int_\Omega \frac{1}{2}A(t)\epsilon(u(t))\cdot\epsilon(u(t))dx - \int_\Omega f(t)\cdot u(t)dx + k\int_\Omega(1-\Theta(t))dx \\ &\geq \mathcal{T}_0 - \int_0^t\int_\Omega \dot{f}(\sigma)\cdot u(\sigma)dxd\sigma\end{aligned}$$

20/ A relaxed evolution

- We have established the following

Thm.: For $f \in W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^N))$ there exist $u(t) \in H^1_0(\Omega;\mathbb{R}^N)$, $\Theta(t) \in L^\infty(\Omega)$, $A(x,t) \in \overline{G}_{1-\Theta(x,t)}(A_w, A_s)$, such that

- Initial time: $(u(0), A(0), (1-\Theta(0))$ minimizes $\int_\Omega \frac{1}{2}A\epsilon(v)\cdot\epsilon(v)dx - \int_\Omega f(0)\cdot v dx + k\int_\Omega (1-\Theta)\, dx$;
- Monotonicity: $A(t)$ and $\Theta(t)$ are decreasing functions of t, as well as $\overline{\Theta}(t) := \int_\Omega \Theta(t)\, dx$;
- Continuity: u is continuous with values in H^1_0, except at the (at most countable) discontinuity points of $\overline{\Theta}$;
- One-sided minimality: $(u(t), A(t), \Theta(t))$ minimizes $\int_\Omega \frac{1}{2}A'\epsilon(v)\cdot\epsilon(v)dx - \int_\Omega f(t)\cdot v dx + k\int_\Omega (1-\Theta)dx$, among all $(v, \Theta \le \Theta(t), A'(x,t) \in \overline{G}(A_w, A(x,t))$;
- Energy balance: $\mathcal{T}(t) := \int_\Omega \frac{1}{2}A(t)\epsilon(u(t))\cdot\epsilon(u(t))\, dx - \int_\Omega f(t)\cdot u(t)dx + k\int_\Omega(1-\Theta(t))dx$ satisfies
$$\mathcal{T}(t) = \mathcal{T}(0) - \int_0^t \int_\Omega \dot{f}(\sigma)\cdot u(\sigma)dxd\sigma$$

21/ Optimality of the evolution

In which sense is this relaxed evolution close to that of a putative classical evolution?

- Recovery – we are not too low: $\exists \chi^n(t) \overset{t}{\nearrow}$ s.t. for the solution $v^n(t)$ of pb. with $\chi^n(t)$,

$$\begin{cases} \chi^n(t) \overset{L^\infty}{\rightharpoonup} 1 - \Theta(t) \\ \chi^n(t) A_w + (1 - \chi^n(t)) A_s \overset{H}{\rightharpoonup} A(t). \end{cases}$$

Indeed, by metrizability, can find $\chi^n_k \overset{t}{\nearrow}$ with
$\chi^n_k(t) \overset{L^\infty}{\rightharpoonup} 1 - \Theta^n(t),\ \chi^n_k(t) A_w + (1 - \chi^n_k(t)) A_s \overset{H}{\rightharpoonup} A^n(t).$
Then by a diagonalization argument, we can construct $\chi^{n_k(n)}$ that satisfies the statement.

$$\Downarrow$$

$$\text{Sort of } \Gamma - \limsup \text{ statement}$$

- Are we too high? Probably. Indeed,....

22/ A different relaxation – 1

- For a sequence of sets D_n, define $\mathcal{G}_{1-\Theta'}(\{D_n\}, A_w, A_s) := \{$ set of H-lims. of $\chi_{D'_n} A_w + (1-\chi_{D'_n})A_s : D'_n \supset D_n;\ \chi_{D'_n} \overset{L^\infty}{\rightharpoonup} 1 - \Theta'\}$

$$\Downarrow$$

Thm.: $\exists D_n(t) \overset{t}{\nearrow} \begin{cases} \chi_{D_n(t)} \overset{L^\infty}{\rightharpoonup} 1 - \Theta(t) \\ \chi_{D_n(t)} A_w + (1-\chi_{D_n(t)})A_s \overset{H}{\rightharpoonup} A(t) \end{cases}$ and one-sided minimality holds $\forall (v, \Theta'(\leq \Theta), A' \in \mathcal{G}_{1-\Theta'}(\{D_n\}, A_w, A_s))$

- Improved minimality: Take $A'(x) \in \overline{G}_\theta(A_w, A(x,t))$; there exists E_h s.t. $A'_h := \chi_{E_h} A_w + (1 - \chi_{E_h})A(t) \overset{H}{\rightharpoonup} A'$ with $\chi_{E_h} \overset{L^\infty}{\rightharpoonup} \theta$.

◇ loc.H-conv. $\Longrightarrow$ $A'_h \in \mathcal{G}_{1-\Theta(t)+\Theta(t)\chi_{E_h}}(\{D_n\}, A_w, A_s) \overset{\text{improved minim.}}{\Longrightarrow}$

$$\int_\Omega \frac{1}{2} A(t)\epsilon(u(t)) \cdot \epsilon(u(t))dx - \int_\Omega f(t) \cdot u(t)dx \leq$$

$$\min_v \{\int_\Omega \frac{1}{2} A'_h \epsilon(v) \cdot \epsilon(v)dx - \int_\Omega f(t) \cdot vdx + k \int_{E_h} (1 - \Theta)dx\}$$

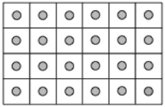
$$\Downarrow h \to 0$$

previous minimality

- But prev. min. $\Longrightarrow$ $A(t,x) \in \overline{G}_{1-\frac{\Theta(x,t)}{\Theta(x,s)}}(A_x, A(s,x))$; not this one!

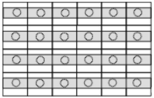
23/ A different relaxation – 2

- However, if $\{D_n\}$ is such that $\chi_{D_n} A_w + (1 - \chi_{D_n}) A_s \xrightarrow{H} A$, then
$B \in \mathcal{G}_{1-\Theta}(\{D_n\}, A_w, A_s)$ for some $\Theta \not\Longrightarrow B(x) \in \overline{G_\theta}(A_w, A)$ for some θ
- Indeed, for a scalar 2d pb.:



$\Longrightarrow$ isotropic material β'

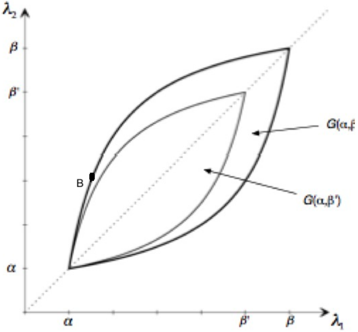
periodic set D_n



$\Longrightarrow$ material on boundary of G-closure B

periodic set $E_n \supset D_n$, a lamination

But:



- Proof of existence of relaxed evol. very similar to that of previous theorem: replace relaxation by sequences of near minimizers,.....

24/ Final remarks

- Can be shown that any evolution where global minimality is replaced by a decent notion of local minimality $\Longrightarrow$ local minimizers are also global minimizers.
- No possibility of total brutal damage, i.e., $A_w \equiv 0$.
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