When and how do cracks propagate?

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Abstract

Crack propagation in an isotropic 2d brittle material is widely viewed as the interplay between two separate criteria. Griffith's cap on the energy release rate along the crack path decides when the crack propagates, while the Principle of Local Symmetry PLS decides how, that is, in which direction, that crack propagates. The PLS, which essentially predicts mode I propagation, cannot possibly hold in an anisotropic setting. Further it disagrees with its competitor, the principle of maximal energy release, according to which the direction of propagation should coincide with that of maximal energy release. Also, continuity of the time propagation is always implicitly assumed.

In the spirit of the rapidly growing variational theory of fracture, we revisit crack path in the light of an often used tool in physics, *i.e.,* energetic meta-stability of the current state among suitable competing crack states. In so doing, we do not need to appeal to either isotropy, or continuity in time. Here, we illustrate the impact of meta-stability in a 2d setting. In a 2d isotropic setting, it recovers the PLS for smooth crack paths. In the anisotropic case, it gives rise to a new criterion. But, of more immediate concern to the community, it also demonstrates that 2d crack kinking in an isotropic setting is incompatible with continuity in time of the propagation. Consequently, if viewing time continuity as non-negotiable, our work implies that the classical view of crack kinking along a single crack branch is not correct and that a change in crack direction necessarily involves more subtle geometries or evolutions.

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1 Introduction

Fracture is first and foremost an evolution problem and a successful fracture model will be measured through its dual ability to predict the spatial path of the crack, or cracks, as well as the motion in time of those cracks along the predicted path. These are the goals of Griffith's theory in a brittle, elastic setting. In his remarkable paper [18], Griffith specifically focussed on the quasi-static propagation of a connected crack in a 2d, linearly elastic, homogeneous and isotropic medium and he assumed that the crack path was known *a priori*, and consisted in a "nice enough" curve within the domain under investigation. As in all quasi-static evolutions, the time t is merely a loading parameter that describes the loading history and, if $\ell(t)$ denotes the crack length at time t, the goal is now to determine the function $t \mapsto \ell(t)$. Assume that the cracked body is, at all t's, in elastic equilibrium for the values of the loads and of the crack length at time t; this is precisely the quasi-static assumption. Then, Griffith's formulation, henceforth referred to as $\mathsf{Griffith}$, is the following three-pronged criterion :

1. $\mathsf{Irreversibility}$: $\ell(0) = \ell_0, \quad t \mapsto \ell(t)$ is monotonically increasing;
2. $\mathsf{Griffith's\ Criterion}$: $G(t, \ell(t)) \leq \mathsf{G_c}, \quad t \geq 0$;
3. $\mathsf{Energy\ Conservation}$: $\Big(G(t, \ell(t)) - \mathsf{G_c}\Big)\dot{\ell}(t) = 0$, $t \geq 0$.

Above, $G(t, \ell)$ denotes the energy release rate at t associated with the infinitesimal advance of a crack of length ℓ along the pre-defined crack path under the load frozen at time t, *i.e.,*

$$G(t, \ell) := -\partial \mathcal{P}(t, \ell)/\partial \ell,$$

where $\mathcal{P}(t, \ell)$ is the potential energy for the load at t – that is the elastic energy minus the work of the external loads –and the crack of length ℓ, while $\mathsf{G_c}$ is the surface energy density of the material.

Note that, because of irreversibility, the crack length at t depends on the whole history of the loading process, up to time t. Note also that the third item in $\mathsf{Griffith}$ is indeed a statement of energy conservation since it states that the release of potential energy equals the gain of surface energy whenever there is a change in crack length.

Unfortunately, $\mathsf{Griffith}$ grinds to a halt in the absence of a smooth enough solution $t \mapsto \ell(t)$, which is not an uncommon occurrence as demonstrated in e.g. [4, Proposition 2.4]. Among other things, we will prove in this paper that continuity in time of the mapping $t \mapsto \ell(t)$ can never occur along a kinked path, see Property 4. The impact of such a negative result is beyond the scope of this paper and the reader may want to refer to [4] and references therein for a detailed discussion of an enlarged quasi-static formulation where time discontinuous solutions are considered. Of course, energy conservation needs to be re-formulated in such a framework (see [11, 14, 4, 13]). Further, it might be argued that dynamics should provide a more realistic resolution, although, at present, reconciling dynamics with crack path prediction (see below) seems a bit overambitious.

Griffith is also powerless whenever the path is not *a priori* known, even for time-continuous crack growth. For smooth enough connected paths (say continuous and piecewise smooth), path prediction reduces to the determination of the direction of crack growth at the propagation time. Numerous kinking criteria have been proposed in the literature, at least in the isotropic setting. The overall favorite is undoubtedly the Principle of Local Symmetry PLS [17] which postulates that mode I propagation of the crack at t is equivalent to propagation with no kink at that time. Section 2 is devoted to a brief review of and a few comments on the current status of the PLS.

A possible alternative is the $\mathsf{G}_{\mathsf{max}}$-Criterion which states that the crack will kink along a direction that maximizes the release rate of its potential energy among all kinking angles. It was shown [3] that the two criteria generically yield different kinking angles. There is in truth scant evidence that would support one criterion over another, and even less so in anti-plane shear because the crack is in mode III, so that the notion of mode I, or mode II propagation is rendered meaningless.

We propose in this paper to pair Energy Conservation, the third item in Griffith, with the following least action principle, referred to henceforth as the Stability Criterion: At each time, the total energy, *i.e.,* the sum of potential and surface energies, is evaluated along all possible small variations of the crack from its present state and stability of the latter is declared when the total energy is smallest at that state. Similar (meta)-stability principles are common occurrence in solid mechanics, a striking example being finite elasticity. In all such settings, those principles can never be justified through a mere investigation of mechanical balance which produces at best stationarity principles. But, if meta-stability implies stationarity, the converse is generically false in any non-convex setting. Section 3 details the adopted notion of stability and offers a brief panorama of prior use of that notion in brittle fracture, both by the authors and by other researchers; see notably [10], [26].

We favor in this paper a minimalistic approach to stability because we simply wish to obtain reasonable conditions under which the crack is unstable. To achieve this, we test stability in Section 3 under the only addition of small line segment add-cracks at the crack tip. Indeed, instability under such smooth variations of the crack path should be deemed a reasonable criterion. However, we observe later in the paper that this view is too restrictive to resolve the paradoxical situations that may follow; see also [5].

Please note that stability will not, in and of itself, permit the determination of the optimal direction for crack extension. Indeed, stability does not address evolution, but only the current state. Only through its interaction with Energy Conservation will it potentially be able to predict path.

Let us emphasize once more that, throughout this work, Energy Consevation will be equivalent to the third item in Griffith. This is so because, *throughout, we will assume that the crack propagates continuously in time.* We then propose to derive a few striking results about kinking. Those can be summed up as follows:

- A kinking criterion is obtained in an anisotropic setting (Property 2);
- In an isotropic setting, the PLS strictly overestimates the time at which a pre-crack starts extending (Equation (13));
- In an isotropic setting, a crack cannot kink while propagating continuously in time (Property 4).

Thus, our results essentially settle a longstanding debate between proponents of various kinking criteria in isotropic 2d brittle fracture. The negative answer they provide demonstrate that, modulo the acceptance of metastability, the current vision of crack kinking should be completely overhauled. This stands in sharp contrast to a host of papers that pre-assume the validity of the PLS), e.g. [9, 1], or else derive it – and its anisotropic analogue – from a mere energetic stationarity principle involving both inner and outer variations; see e.g. [25, 19]. The arguments presented below can be recast as mathematical propositions in the framework developed in [5].

2 The prevailing view of kinking

As already mentioned in the introduction, kinking in an isotropic material is often thought of as abiding by the PLS which we now explicitly state, so as to dispel any possible misconception. At a given time t, we denote by α the putative kinking angle at the crack tip, by $\theta^-(t)$ the angle of the tangent to the crack $\Gamma(t)$ at its tip (see Fig. 1), and and by $\mathsf{K}_2(t, \theta^-(t))$ the mode-II stress intensity factor at the tip (so before kinking), then

$$\mathsf{PLS} \qquad \alpha = 0 \iff \mathsf{K}_2(t, \theta^-(t)) = 0.$$

In other words, the crack will not kink if, and only if it is propagating in mode I (pure traction on the crack lips).

Assuming that principle, then the usual argument, somewhat implicit in [9], but truly evidenced through a more accurate expansion of the post-kink stress intensity factors in [3] goes as follows: *Provided* that the crack evolution is continuous in time, then the PLS – or at least the direct implication in the equivalence which we will call $\mathsf{PLS}(\Rightarrow)$ from now on – implies that the limit, as the post-kink add-crack length tends to 0, of the stress intensity factor, *i.e.,* in view of the continuity in time of the crack evolution, $\lim_{t' \searrow t} \mathsf{K}_2(t', \theta(t'))$, denoted henceforth by ${\mathsf{K}_2}^*(t, \theta^-(t) + \alpha)$, where α is the kinking angle, must be 0. In other words, the common view is that

$$\mathsf{PLS}(\Rightarrow) \; + \; \text{Continuity in Time} \Longrightarrow {\mathsf{K}_2}^* = 0.$$

Paradoxically, the most debated implication in the PLS is the reverse implication, *i.e.,* that if propagation occurs with $\mathsf{K}_2(t, \theta^-(t)) = 0$, then $\alpha = 0$; we denote it by $\mathsf{PLS}(\Leftarrow)$. For example, [2] argues that loading and material symmetry imply that no kinking occurs, because, if α were the kinking angle, then $-\alpha$ should be as well, so α must be 0. But, of course, this is only so if uniqueness is assumed!

Energetic type justifications of $\mathsf{PLS}(\Leftarrow)$ sooner or later lead to a reevaluation of the $\mathsf{G_{max}}$-$\mathsf{Criterion}$ and of its link to the ${\mathsf{K}_2}^* = 0$ criterion. For instance it is noted in [9, p. 164] that, if $\mathsf{PLS}(\Leftarrow)$ holds true, then the energy release rate $G(t, \beta)$, defined in (3) below, seems to admit a local maximum at $\beta = 0$. There, the argument is based on an expansion of the stress intensity factors around $\beta = 0$. However, the proposed expansions in [9] are only given up to the first order in β, so that they cannot warrant such a conclusion because that would require expansions of order 2 in β. By contrast, the later expansions given in [3] do guarantee that $\beta = 0$ is indeed a local maximum for $G(t,\beta)$, although they cannot allow one to conclude that it is a global maximum for $G(t,\beta)$.

In any case, those results cannot constitute proof of the reverse implication $\mathsf{PLS}(\Leftarrow)$, unless the $\mathsf{G_{max}}$-Criterion is viewed as a good kinking criterion, in which case, thanks to the results in [2, 3], we can at least reasonably conjecture (see Remark 1 below) that, when $\mathsf{K_2} = 0$, then 0 is the only true maximum for G. In other words, the statement that a crack that propagates smoothly in time will also propagate smoothly in space if, and only if it is only loaded in mode I can only result from the ab initio adoption of the PLS, and then it is contained in that assumption, or else from the direct implication $\mathsf{PLS}(\Rightarrow)$, together with the $\mathsf{G_{max}}$-Criterion as a kinking criterion.

Assuming the latter, the non trivial results in [3] demonstrate that there is no non-zero angle α for which both $\mathsf{K_2}^*(t,\theta^-(t)+\alpha) = 0$, and α maximizes the energy release rate $G(t,\beta)$ associated with a kinking angle β (see the proof of Property 4 below). Thus, the prevailing view is that the $\mathsf{G_{max}}$-Criterion must be rejected and that only the full PLS should preside. As such, the prevailing view is inoperative when departing from isotropy.

We propose in what follows a completely different approach, which does not rely on that principle, but on a general meta-stability principle which should hold true in all settings and for all quasistatic evolutions. We will then show that the question of how a crack kinks is intimately intertwined with that of when a crack kinks. Temporality has been overlooked in the literature, yet we will see that the PLS overestimates the kinking time.

3 The Stability Criterion

Griffith's Criterion, the second item in Griffith recalled in the introduction, is truly a first order meta-stability condition along the prescribed crack path. Indeed, set

$$\mathcal{E}(t,\ell) := \mathcal{P}(t,\ell) + \mathsf{G_c}\ell$$

to be the total energy of the body at t for a crack length ℓ. Then a natural meta-stability statement of $\mathcal{E}(t,\cdot)$ at $\ell(t)$ is as follows: the total energy at t computed for the actual crack length $\ell(t)$ will be minimal among all total energies computed at t, but for slightly greater crack lengths. Note that, because of irreversibility, we can only compare the actual energy to that associated with crack lengths that are accessible from the actual crack length, *i.e., greater* crack lengths. In other words,

$$\exists \bar{h} > 0, \forall \ell \in (\ell(t), \ell(t) + \bar{h}) \quad \mathcal{E}(t, \ell(t)) \leq \mathcal{E}(t, \ell).$$

Dividing the previous inequality by $\ell - \ell(t)$ and passing to the limit $\ell \searrow \ell(t)$ yields Griffith's criterion as a first order minimality condition for $\mathcal{E}(t, \cdot)$.

We wish to enlarge the set of test cracks, so as to free the crack path. Denote henceforth the actual crack at time t by $\Gamma(t)$, and by Γ^* a virtual crack containing $\Gamma(t)$. Also call $\mathcal{E}(t, \Gamma^*)$ the total energy associated with Γ^* under the actual load at time t. Meta-stability consists in requiring that

$$\mathcal{E}(t, \Gamma(t)) \leq \mathcal{E}(t, \Gamma^*)$$

for an appropriate family of Γ^*'s. If that family is all supersets Γ^* of $\Gamma(t)$, then the meta-stability condition becomes a global stability condition, whereas in all other cases, meta-stability is a local stability condition, *i.e.,* it remains valid for a class of Γ^*'s close in some sense (the length for example) to $\Gamma(t)$.

There is by now a large body of works dedicated to the issue of meta-stability, and this not only within the framework of brittle fracture, but also for cohesive fracture as well as for a slew of rate independent evolutions. As far as brittle fracture is concerned, it can be arguably found in [18], although Nguyen was probably first and foremost in acknowledging and formally importing meta-stability into solid mechanics; see [24] for a review of his prior work.

In brittle fracture the mathematical emphasis was clearly at first on global stability [15],[11], [14], but the recognition that a maybe more realistic criterion would be of a local nature can already be found in [16]. Similar ideas in other rate independent settings were in the air at the time; see e.g. [23], [22]. In [7], a comparison between 1d local minimality and 1d global minimality is undertaken, both in a Griffith and cohesive context. In [10], meta-stability is studied in a Griffith 3d context with the addition of an additional term in the energy that penalizes the mean square distance between the elastic fields. In [21] and [8], meta-stability is used to derive initiation criteria, also both in a Griffith and in a cohesive setting. In [6], meta-stability, together with the strength of elastic singularity at a given point, are combined to decide on whether crack initiation originating at that point will occur with smoothly increasing add-crack length. All of those results are recalled and expanded upon in [4]. In [26], a meta-stability criterion is also proposed in the Griffith setting through the use of test cracks that are simple curves. Further, meta-stability is at the root of many works on cohesive fracture along a specified crack path; see e.g. [12].

Let us be more precise. Consider a 2d structure made of a linearly elastic, maybe anisotropic material and subject to a time-dependent load. A crack is propagating through the structure. The evolution is assumed to be quasi-static: at each time t, the structure is in elastic equilibrium with the load at t, and this for any admissible crack state Γ at t. The potential energy is denoted by $\mathcal{P}(t, \Gamma)$, while, following Griffith, the surface energy for the crack state Γ, denoted by $\mathcal{S}(\Gamma)$, is defined as follows. For a homogeneous, isotropic material, it is proportional to the crack length, *i.e.*, $\mathcal{S}(\Gamma) = \mathsf{G_c}\text{length}(\Gamma)$, where $\mathsf{G_c}$, the fracture toughness, is a material characteristic. In the anisotropic case, toughness is orientation-dependent and, if $\theta(s)$ is the angle of the tangent to Γ with a set direction at the point with arclength s, the surface energy becomes

$$\mathcal{S}(\Gamma) = \int_0^{\text{length}(\Gamma)} \mathsf{G_c}(\theta(s))\ ds. \tag{1}$$

The total energy at t for a crack state Γ is

$$\mathcal{E}(t, \Gamma) = \mathcal{P}(t, \Gamma) + \mathcal{S}(\Gamma).$$

At t, the actual crack state $\Gamma(t)$ has a tip at the interior point x, whereas

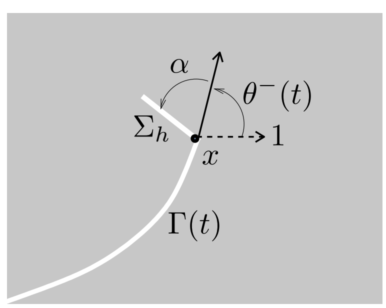

Figure 1: An admissible virtual crack that tests the meta-stability of the actual crack state $\Gamma(t)$ at t

an admissible virtual crack is obtained by adding at the tip x of $\Gamma(t)$ a line segment Σ_h of length h in a direction with angle α with the tangent to $\Gamma(t)$ at x, cf. Fig. 1. Upon computing the various energies associated with those two different crack states, the Stability Criterion demands that the actual total energy be smaller than, or equal to that for the virtual state $\Gamma(t) \cup \Sigma_h$. This should hold for any sufficiently small h and for any kinking direction α. So, for h small enough and for any α,

$$\mathcal{E}(t, \Gamma(t)) \leq \mathcal{E}(t, \Gamma(t) \cup \Sigma_h). \tag{2}$$

As already mentioned, meta-stability entails a decision on the class of admissible test cracks. To achieve this, we have only tested stability under the addition of small line segment add-cracks at the crack tip. In Section 5, we will also consider finite unions of connected small line segments as admissible variations, but will eschew consideration of a larger class of variations [5].

We stress that here, as well as in most of the works cited at the onset of this section, meta-stability is not a kinking criterion, but a general principle that should apply to all quasistatic crack evolutions. Thus, in contrast with the PLS, it is not an ad hoc addition to a pre-existing theory of brittle fracture evolution, but rather a fundational tool that, together with Energy Conservation, will preside over all quasistatic evolutions in all possible settings, independently of any other assumption like smoothness of the crack path, and/or continuity in time of the crack evolution. In all that follows, we restrict the crack path to piecewise smooth connected curves, so as to follow the mechanical tradition, but much more general cracks can be envisioned, for kinking as well as for general crack evolutions; see e.g. [5], but also [4] and references therein.

4 A few hints at how cracks propagate

We proceed to derive a few necessary conditions for propagation in the light of the Stability Criterion. The (virtual) energy release rate $\mathsf{G}(t, \alpha)$ associated with the virtual test crack $\Gamma(t) \cup \Sigma_h$ at t is given by

$$\mathsf{G}(t, \alpha) := \lim_{h \downarrow 0} \frac{1}{h} \left(\mathcal{P}(t, \Gamma(t)) - \mathcal{P}(t, \Gamma(t) \cup \Sigma_h) \right). \tag{3}$$

As such, it depends upon α, t, but also upon $\Gamma(t)$. Similarly, the (virtual) surface energy creation rate is given by

$$\mathsf{G_c}(\alpha + \theta^-(t)) = \lim_{h \downarrow 0} \frac{1}{h} \left(\mathcal{S}(\Gamma(t) \cup \Sigma_h) - \mathcal{S}(\Gamma(t)) \right)$$

where $\theta^-(t)$ is the angle of the tangent to the crack $\Gamma(t)$ with the fixed direction 1 at its tip x.

Dividing (2) by h and letting h tend to 0 yields the following property:

Property 1. *The crack path $\Gamma(t)$ must be such that, at each t, all virtual kinks produce an energy release rate which remains smaller than the surface energy creation rate, that is*

$$\mathsf{G}(t,\alpha) \le \mathsf{G_c}(\theta^-(t)+\alpha), \quad \forall\alpha. \tag{4}$$

Note that a similar property was derived in [21], as well as in [26].

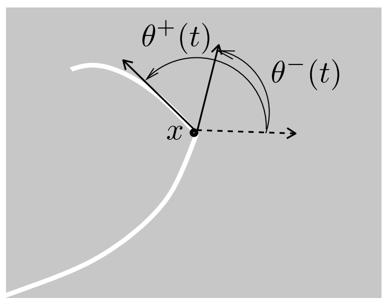

Figure 2: The crack propagates at t from the tip x in the direction $\theta^+(t)$. If $\theta^+(t) \neq \theta^-(t)$, kinking occurs, whereas, if $\theta^+(t) = \theta^-(t)$, there is no kinking.

Assume now that the time t is the actual propagation time, that is the time at which $\Gamma(t)$ is going to extend from x, *continuously as a function of time*, in a direction $\theta^+(t)$ that may coincide, or differ from $\theta^-(t)$, cf. Fig. 2. Energy conservation implies that the (real) energy release rate must be equal to the (real) surface energy creation rate, that is

$$\mathsf{G}(t,\theta^+(t)-\theta^-(t)) = \mathsf{G_c}(\theta^+(t)). \tag{5}$$

The reader is reminded that Energy Conservation (5) becomes meaningless if the crack propagation near x ceases to be smooth in time. Indeed, if the crack tip was to jump from x to y at t, it would become impossible to express the balance of energy in terms of rates and one should then generalize energy balance to a setting that accommodates finite increments in crack length [15, 4].

Recalling (4) and (5), we conclude that

Property 2. *At each continuous propagation time t, the crack orientation $\theta^+(t)$ must be such that*

$$1 = \frac{\mathsf{G}(t, \theta^+(t) - \theta^-(t))}{\mathsf{G_c}(\theta^+(t))} = \max_\alpha \frac{\mathsf{G}(t, \alpha)}{\mathsf{G_c}(\theta^-(t) + \alpha)}.$$

The above result, which applies to any kind of anisotropy, seemingly settles the issue of how and when the crack should propagate. If applied to an isotropic setting for which $\mathsf{G_c}$ is a constant, it does seem to favor the $\mathsf{G_{max}}$-Criterion at the expense of the PLS. But such is not the case, as will be demonstrated below. To pursue the analysis further, we recall Irwin's formula [20] which relates the energy release upon kinking by an angle α at time t to the stress intensity factors, namely, in our notation,

$$\mathsf{G}(t, \alpha) = \mathcal{C} \left\{ (\mathsf{K_1}^*)^2(t, \theta^-(t) + \alpha) + (\mathsf{K_2}^*)^2(t, \theta^-(t) + \alpha) \right\},$$

where $\mathcal{C}$ is a constant that only depends upon the elasticity of the material and upon the type of setting (plane strain, or plane stress) for the specific problem under consideration. The coefficients $\mathsf{K_1}^*(t, \alpha)$ and $\mathsf{K_2}^*(t, \alpha)$ are the coefficients of the singularity at the crack tip after kinking. We also recall [3] that $\mathsf{K_1}^*(t, \theta^-(t) + \alpha)$ and $\mathsf{K_2}^*(t, \theta^-(t) + \alpha)$ are related to their pre-kinking analogues $\mathsf{K_1}(t, \theta^-(t))$ and $\mathsf{K_2}(t, \theta^-(t))$ through

$$\mathsf{K}_i^*(t, \theta^-(t) + \alpha) = \mathsf{F}_{ij}(\alpha) \mathsf{K}_j(t, \theta^-(t)),$$

where the coefficients $\mathsf{F}_{ij}(\alpha)$ are universal constants that only depend upon the kinking angle α; of course $\mathsf{F}_{11}(0) = 1,\ \mathsf{F}_{12}(0) = 0,\ \mathsf{F}_{21}(0) = 0,\ \mathsf{F}_{22}(0) = 1$, so that $\mathsf{K}_i^*(t, \theta^-(t)) = \mathsf{K}_i(t, \theta^-(t))$. An analytic expression for the matrix $F(\alpha)$ is lacking at present, although asymptotic expansions around $\alpha = 0$ have been derived as well as numerical plots for all values of α [3]. In particular, it is known [3] that

$$\begin{aligned} \mathsf{F}'_{11}(0) = 0,\ \mathsf{F}'_{12}(0) = -3/2, \\ \mathsf{F}'_{21}(0) = 1/2,\ \mathsf{F}'_{22}(0) = 0, \end{aligned} \tag{6}$$

and also that

$$\mathsf{F}^2_{12}(\alpha) + \mathsf{F}^2_{22}(\alpha) = 1 + (3/2 - 8/\pi^2)\alpha^2 + O(\alpha^4). \tag{7}$$

Consider at first a crack that propagates continuously in both space and time in a homogeneous, isotropic material. At the propagation time t, $\alpha = 0$

and thus, according to Property **P2**, $\alpha = 0$ maximizes $\mathsf{G}(t, \alpha)$ among all α's. Because of Irwin's formula, this means in particular that

$$\mathsf{K}_1(t, \theta^-(t))(\mathsf{K}_1{}^*)'(t, \theta^-(t)) + \mathsf{K}_2(t, \theta^-(t))(\mathsf{K}_2{}^*)'(t, \theta^-(t)) = 0.$$

In view of (6), this is not possible unless $\mathsf{K}_1(t, \theta^-(t))\mathsf{K}_2(t, \theta^-(t)) = 0$, while, if $\mathsf{K}_1(t, \theta^-(t)) = 0$, we should have, by maximality, that $\mathsf{F}_{12}^2(\alpha) + \mathsf{F}_{22}^2(\alpha) \leq 1$, for all α's, which is not the case in view of (7). Thus, we must have $\mathsf{K}_2(t, \theta^-(t)) = 0$. In other words, one implication in the PLS holds in such a setting and we conclude that

Property 3. *Assuming the validity of the* Energy Conservation *and* Stability Criterion*, a crack cannot propagate continuously in space and time in a homogeneous, isotropic material unless it propagates in mode I.*

In this setting, the PLS – or at least one implication in that principle – derives from the Stability Criterion. But note that the Stability Criterion has a much longer reach than the PLS. It could in principle be applied in the anisotropic case and serve to derive an anisotropic equivalent of the PLS, although, in all fairness, this amounts to little more than wishful thinking in the absence of a more appropriate knowledge of the matrix $\mathsf{F}(\alpha)$.

Consider now a crack that propagates (in a homogeneous, isotropic material) continuously in time, while kinking in space at the point x at time t.

Since, after kinking, propagation resumes continuously in both space and time, Property **P3** implies mode I propagation after kinking, and, by passing to the limit in time down to time t, we conclude, thanks to the continuity of the stress intensity factors as a function of the kinked crack length, that $\mathsf{K}_2{}^*(t, \theta^-(t) + [\![\theta]\!]) = 0$. But, in view of **P2**, $[\![\theta]\!]$ must also maximize $\mathsf{G}(t, \alpha)$ among all α's. In other words, the kinking angle must satisfy both the $\mathsf{G}_{\mathsf{max}}$-Criterion and the $\mathsf{K}_2{}^* = 0$ - Criterion. Numerical plots for the $\mathsf{F}_{ij}(\alpha)$'s (see Figs. 3-4) strongly indicate that only $[\![\theta]\!] = 0$ can satisfy both and that the corresponding loading mode must be mode I.

Specifically, the following is easily derived

Property 4. *In a homogeneous, isotropic elastic material, and, provided that*

$$F_{21}(\alpha)F'_{12}(\alpha) \neq F_{22}(\alpha)F'_{11}(\alpha), \forall\alpha \neq 0, \tag{8}$$

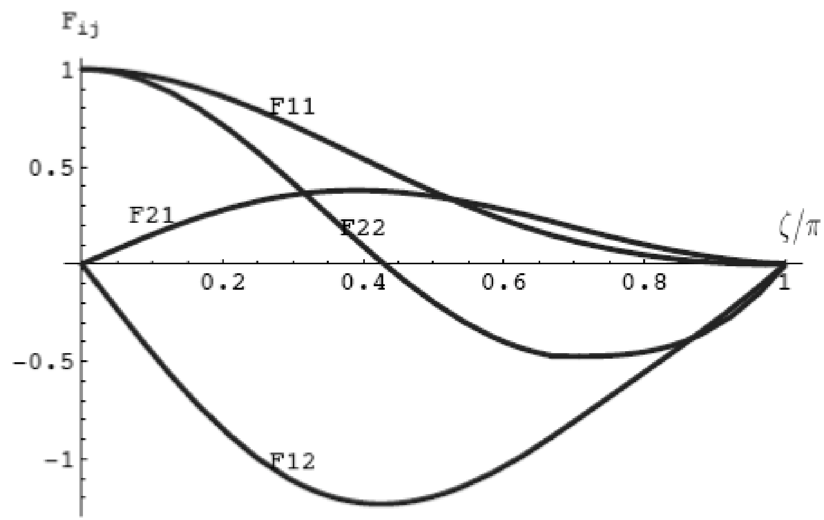

Figure 3: Computed F_{ij}'s; the curves are fitted from the values obtained for discrete angles in [3].

and also that

$$(F_{11}(\alpha))^2 + (F_{21}(\alpha))^2 < 1,\ \alpha \neq 0, \tag{9}$$

then kinking never occurs with a propagation which is continuous in time.

Remark 1. Condition (9) ensures that, if $\mathsf{K}_2 = 0$, then $\alpha = 0$ is indeed the angle that maximizes G, which ensures that Property 2 is satisfied. Thus, see Section 2, if $\mathsf{K}_2 = 0$, then necessarily $\alpha = 0$, *i.e.,* the reverse implication $\mathsf{PLS}(\Leftarrow)$ holds true.

As already stated, only numerical evidence presently corroborates the validity of (8), (9), for want of explicit analytical expressions for the matrix $\mathsf{F}(\alpha)$, cf. Fig. 4.

The argument that led to the negative Property **P4** is quite different from the classical argument put forth by the proponents of the PLS as described in Section 2. For us, the only principle is the Stability Criterion and we obtain that

$$\text{Stability} + \text{Continuity in time} \Rightarrow {\mathsf{K}_2}^* = 0 + \mathsf{G}_{\mathsf{max}}\text{-Criterion}\ .$$

The continuity in time of the propagation plays an essential role: the crack tip has to pass at some time through every nearby point y of the

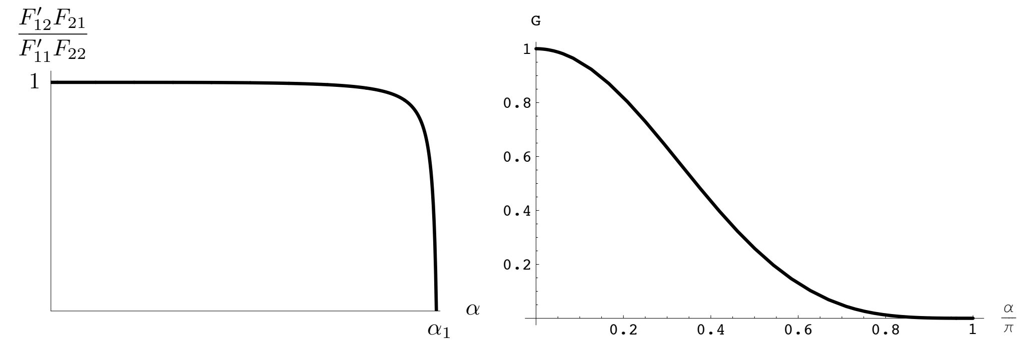

Figure 4: *Left:* Numerically, (8) can only be met at $\alpha = 0$. The value $\alpha_1 \approx .425\pi$ is that for which $F'_{12} = 0$; it corresponds to the minimum of F_{12} in Fig. 3. Between α_1 and $\alpha_2 \approx .430\pi$, the value for which $F_{22} = 0$, the conjecture holds true because the signs of $F_{21}F'_{12}$ and of $F_{22}F'_{11}$ are opposite, so that they cannot cancel out at some ζ, unless they are both 0, which is not the case. Between α_2 and π, it is easily checked directly that both criteria are not met at the same angle through a comparison of the values of G at $\pm\alpha$; we conjecture that, in that interval as well, the sufficient condition (8) is met. Note that, was the conjecture to be false, analyticity would imply that it can only be so at a finite number of universal angles for which the ratio K_1/K_2 would be given, independently of the values of the Lamé coefficients, or of the values and type of loads; see [5]. –
Right: Numerically, $\mathsf{G}(\alpha)$ is maximal at $\alpha = 0$ when $\mathsf{K}_2 = 0$, *i.e.,* (9) is satisfied. Both computations are based on numerical values given in [2, 3].

kinking point x on the kinked trajectory and equilibrium has to be satisfied at that time (or those times if it lingers at a given point). Hence the validity of the limit process $y \to x$. This is precisely what permits one to assert the validity of the condition $\mathsf{K_2}^* = 0$ at x. Otherwise, we would only have that condition at some point y, maybe close to x, where the crack jumps to at time t, and $\mathsf{K_2}^*(t, 0)$ would only be 0 at y, which yields no information whatsoever on $[\![\theta]\!]$ at x.

The knowledge that

$$\mathsf{K_2}^* = 0 + \mathsf{G_{max}}\text{-Criterion} \ \Rightarrow \text{No kinking},$$

leads to the rather arbitrary rejection of the $\mathsf{G_{max}}$-Criterion . However, adoption of the Stability Criterion leads either to a rejection of the continuity in time of the evolution during kinking, or else to the rejection of the Stability

Criterion. The reader is certainly at liberty to reject the latter, but, in doing so, she will have to reflect upon the arbitrariness of accepting at faith value identical meta-stability criteria in many other fields of physics and mechanics. Note also that the criterion also leads to the universally acknowledged mode I propagation in the case of a spatially smooth crack path as shown above.

5 Revisiting the when

The negative Property **P4** renders obsolete the $\mathsf{K_2}^* = 0$ versus $\mathsf{G_{max}}$-Criterion conflict and points to a rethinking of kinking in global terms, that is without resorting to the local notion of energy release rates, which become meaningless if a crack jumps at a given time. If resolutely opposed to the notion of a crack jump, one has to abandon the simple picture of a crack trajectory as a piecewise smooth curve in 2d.

Consider a homogeneous, isotropic structure with an initial crack Γ under proportional loading (the load increases proportionally to time while its spatial variation remains fixed). In that case, the potential energy depends quadratically upon t. Also assume that Γ is not in pure mode I to start with. Call t_i the time at which the crack starts extending, and apply the Stability Criterion with test cracks of the form $\Gamma \cup \Sigma_h$ where Σ_h is an add-crack of length h added at the tip x of Γ. If Σ_h is, as before, a line-segment, then **P1** implies that

$$t_i^2 \leq \frac{\mathsf{G_c}}{\max_\alpha \mathsf{G}(1,\alpha)}. \tag{10}$$

The proponents of the PLS usually infer, with the additional help of Energy Conservation, the propagation time T_i given by

$$T_i^2 = \frac{\mathsf{G_c}}{\mathsf{G}(1,[\![\theta]\!])}, \tag{11}$$

where $[\![\theta]\!]$ is the kinking angle predicted by the $\mathsf{K_2}^*$=0 - Criterion , *i.e.,* such that $\mathsf{K_2}^*(1, \theta^-(t) + [\![\theta]\!]) = 0$. But, since we have assumed that the initial crack is not in pure mode I, so that, because of the incompatibility between the $\mathsf{G_{max}}$-Criterion and the $\mathsf{K_2}^*$=0 - Criterion, $\mathsf{G}(1,[\![\theta]\!]) < \max_\alpha \mathsf{G}(1,\alpha)$, then

$$T_i > t_i.$$

Thus, T_i given by (11) is too large and the crack should have already propagated at the kinking time!

Is the propagation time then equal to t_i? The answer depends on the type of "kink" one allows. For line-segment only add-cracks, the answer is positive, and then we are left with a jump as only possible outcome of a kink, as demonstrated above. More complex geometries of Σ_h produce a different result. Assume for example that Σ_h is a union of two line segments of respective length ηh and $(1-\eta)h$, and respective orientation α_1 and α_2; see Fig. 5. Consider a unit load, and define the energy release rate for this

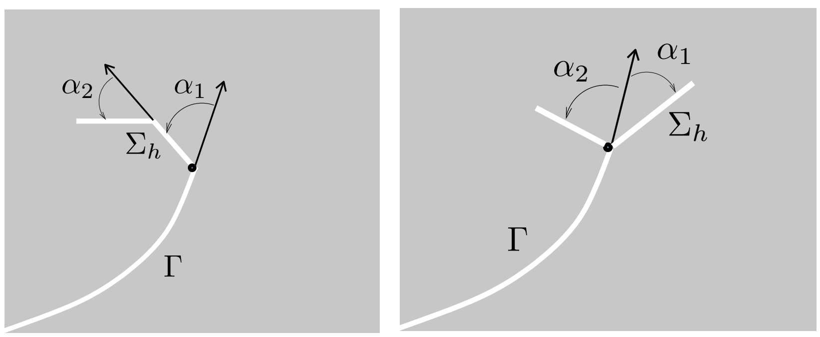

Figure 5: Testing stability with a kinked or a branched add-crack

type of add-crack – a kinked or a branched add-crack – to be

$$\mathcal{G}(\eta,\alpha_1,\alpha_2) := \lim_{h\downarrow 0} \frac{1}{h}\left(\mathcal{P}(1,\Gamma)-\mathcal{P}(1,\Gamma\cup\Sigma_h)\right). \tag{12}$$

The Stability Criterion implies a new bound for t_i, namely,

$$t_i^2 \le \frac{\mathsf{G_c}}{\max\limits_{(\eta,\alpha_1,\alpha_2)} \mathcal{G}(\eta,\alpha_1,\alpha_2)}, \tag{13}$$

which is lower than, or equal to that in (10) since taking $\eta=1$ would have us recover (10). It can be shown [5] that, if considering a kinked add-crack, it is actually strictly lower than the bound given by (10). The proof uses once more the incompatibility between the $\mathsf{G_{max}}$-Criterion and $\mathsf{K_2}^*=0$ criteria.

This also demonstrates [5] that the energy release rate due to a line-segment add-crack cannot equal $\mathsf{G_c}$ at the time of kinking.

All of this contributes to our opinion that the actual propagation time is strictly less than t_i. Of course, the perspicacious reader will object that the kinked add-crack is not realistic because the geometric shape of Σ_h changes with h. Maybe so, but this has the arguable merit to demonstrate that the questions of when and how a crack propagates should be simultaneously investigated and that Energy Conservation is not sufficient for such a task. We portend that the Stability Criterion is a good candidate for filling in the conceptual gap. In any case, kinking remains a mystery.

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