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AN INTRODUCTION TO H–MEASURES AND THEIR APPLICATIONS

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ABSTRACT. These notes attempt a simple introduction to H–measures (microlocal defect measures), a tool designed independently by P. GÉRARD and by L. TARTAR to compute weak limits of quadratic products of oscillating fields. The canvas around which the concepts are presented is that of the linear wave equation with smooth coefficients and rapidly oscillating initial data. The weak limit of the energy density is computed and a compactness result of the L^6–norm of the field (in $\mathbb{R}^3$) at each time is established.

KEYWORDS. Symbolic calculus, microlocal analysis, oscillations, compactness, wave equation.

1. INTRODUCTION

In these lectures, I attempt a self–contained treatment of H–measures, which are also called microlocal defect measures. The concept was introduced under the former name by L. TARTAR, and under the latter by P. GÉRARD at the beginning of the 1990's. The key references are [13], [4].

In a nutshell, the basic mathematical issue is that of the computation of the weak limit of quadratic products of weakly converging fields. Older tools are available.

In an elliptic setting, compensated compactness often proves successful (see e.g. [14, 9, 10]). The "div–curl" lemma provides a prototypical example. That lemma states that, if $u^\varepsilon \in L^p(\Omega;\mathbb{R}^N),\ v^\varepsilon \in L^{p'}(\Omega;\mathbb{R}^N)$ $(1 < p < \infty,\ 1/p + 1/p' = 1)$ are such that

$$u^\varepsilon \rightharpoonup u, \quad \text{weakly in } L^p(\Omega;\mathbb{R}^N)$$
$$v^\varepsilon \rightharpoonup v, \quad \text{weakly in } L^{p'}(\Omega;\mathbb{R}^N)$$

while curl u^ε and div v^ε lie in a compact set of $W^{-1,p}_{loc}(\Omega;\mathbb{R}^N)$ and $W^{-1,p'}_{loc}(\Omega;\mathbb{R}^N)$ respectively, then,

$$u^\varepsilon.v^\varepsilon \rightharpoonup u.v, \text{ weak–* in } \mathcal{D}'(\mathbb{R}^N).$$

Although very handy in the study of elliptic equations in divergence form, compensated compactness does not allow a general computation for any quadratic product, precisely because it requires specific compensations between the various derivatives of the weakly converging quantities.

YOUNG measures, on the other hand, compute the weak limit of any non–linear function of the weakly converging fields (see e.g. [14, 1]): if

$$u^\varepsilon \rightharpoonup u, \text{ weakly in } L^p(\Omega),\ 1 < p < \infty,$$

there exists a subsequence, still indexed by ε and, for a.e. $x \in \Omega$, a probability measure ν_x on $\mathbb{R}$, such that, for any $f \in \mathcal{C}^0(\Omega)$ with $|f(t)| \le cst.(1 + |t|^q),\ q < p$,

$$f(u^\varepsilon(x)) \rightharpoonup \int_{\mathbb{R}} f(\lambda)\, d\nu_x(\lambda), \text{ weakly in } L^r(\Omega),\ rq < p.$$

The measure ν_x has adequate weak measurability properties in x, so as to lend a meaning to the above limit. Unfortunately, the determination of the YOUNG measure is in general impossible, unless the investigated problem possesses a variational structure (see e.g. [11]). In essence, YOUNG measures fail to inherit any kind of differential structure from the equations satisfied by the field u^ε, with the notable exception of certain classes of conservation laws (see e.g. [14], [2]).

H–measures may be seen as a middle ground between compensated compactness and YOUNG measures. In contrast to the former, no compensation is necessary in order to pass to the limit in quadratic products, while, in contrast to the latter, the differential structure of the investigated problem results in localization and transport properties for the H–measures.

The notes will articulate around the non-homogeneous wave equation in $\mathbb{R}^N$ (with mostly $N = 3$) with oscillating initial conditions. From a mathematical standpoint, such a setting demonstrates the power of the method and eventually leads to a beautiful result of P. GÉRARD on the generic compactness in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, at each time t, of the solution $u^\varepsilon(x,t)$ to the classical wave equation with oscillating initial data. From a mechanical standpoint, the computation of the limit of the energy density associated to the solution $u^\varepsilon(x,t)$ is a key to the investigation of energy transfer from high frequencies into heat, a prerequisite to the determination of a sound thermomechanical model.

Consider, for any $\rho, k \in \mathcal{C}^\infty(\mathbb{R}^N)$ with

$$0 < \alpha \le \rho(x), k(x) \le \beta < \infty \tag{1.1}$$

the following wave equation:

$$\begin{aligned} \rho\frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(k \operatorname{grad} u^\varepsilon) = 0 \\ u^\varepsilon(0) = u_0^\varepsilon \\ \frac{\partial u^\varepsilon}{\partial t}(0) = v_0^\varepsilon \end{aligned},$$

with

$$\begin{cases} u_0^\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(\mathbb{R}^N) \\ v_0^\varepsilon \rightharpoonup v_0 & \text{weakly in } L^2(\mathbb{R}^N) \end{cases},$$

and let us assume, for simplicity sake, that

$$\text{supp } \mathrm{u}_0^\varepsilon, \text{supp } \mathrm{v}_0^\varepsilon \subset \text{ a fixed compact K.}$$

Take $\Omega \supset K'(\text{compact}) \supset K$ and solve the wave equation on $\Omega \times [0,T]$ with T chosen such that $\text{supp } \mathrm{u}^\varepsilon(\mathrm{t}) \subset \mathrm{K}'$; this is always possible if T is small enough because the wave equation has finite speed of propagation, no greater than $\sqrt{\frac{\beta}{\alpha}}$. Thus, on the time intervall $[0,T]$, u^ε is equivalently defined as the solution of the same wave equation, formulated on Ω with DIRICHLET boundary conditions.

Elementary energy estimates imply that u^ε is uniformly bounded in

$$\mathcal{E}_T := L^\infty(0,T; H_0^1(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega)),$$

so that, for a subsequence still indexed by ε,

$$u^\varepsilon \rightharpoonup u, \text{ weakly in } \mathcal{E}_T,$$

with u solution of the same wave equation. Furthermore, AUBIN's classical compactness lemma (see e.g. [12]) implies that

$$\begin{cases} u^\varepsilon(0) \longrightarrow u(0) & \text{in } L^2(\Omega) \\ \rho(x)\dfrac{\partial u^\varepsilon}{\partial t}(0) \longrightarrow \rho(x)\dfrac{\partial u}{\partial t}(0) & \text{in } H^{-1}(\Omega) \end{cases},$$

so that $u(0) = u_0$ and $\dfrac{\partial u}{\partial t}(0) = v_0$. Summing up, u solves the same wave equation with u_0 and v_0 as initial data.

Further, by virtue of energy conservation,

$$\begin{aligned} E^{(\varepsilon)}(t) &:= 1/2 \int_\Omega \Big\{\rho(x)(\frac{\partial u^{(\varepsilon)}}{\partial t})^2 + k(x)|\operatorname{grad} u^{(\varepsilon)}|^2\Big\}\, dx \\ &= E_0^{(\varepsilon)} := 1/2 \int_\Omega \Big\{\rho(x)(v_0^{(\varepsilon)})^2 + k(x)|\operatorname{grad} u_0^{(\varepsilon)}|^2\Big\}\, dx. \end{aligned}$$

Thus,

$$\liminf E^\varepsilon(t) \geq E(t),$$

with equality if and only if

$$\begin{cases} u_0^\varepsilon \longrightarrow u_0, \text{ in } H^1(\mathbb{R}^N) \\ v_0^\varepsilon \longrightarrow v_0, \text{ in } L^2(\mathbb{R}^N) \end{cases}.$$

Generically, some energy has been lost during the limit process. Quantifying that loss *pointwise* is the goal of our study.

Setting

$$v^\varepsilon := u^\varepsilon - u \tag{1.2}$$

reduces the analysis to

$$\rho \frac{\partial^2 v^\varepsilon}{\partial t^2} - \operatorname{div}(k \operatorname{grad} v^\varepsilon) = 0 \tag{1.3}$$

with initial conditions

$$\begin{aligned} v^\varepsilon(0) = u_0^\varepsilon - u_0 := V_0^\varepsilon &\rightharpoonup 0, \text{ in } H^1(\mathbb{R}^N) \\ \frac{\partial v^\varepsilon}{\partial t}(0) = v_0^\varepsilon - v_0 := Z_0^\varepsilon &\rightharpoonup 0, \text{ in } L^2(\mathbb{R}^N), \end{aligned} \tag{1.4}$$

so that

$$v^\varepsilon \rightharpoonup 0, \text{ weakly in } \mathcal{E}_T.$$

The energy density

$$e^\varepsilon(t,x) := 1/2\Big\{\rho(x)(\frac{\partial v^{(\varepsilon)}}{\partial t})^2 + k(x)|\operatorname{grad} v^{(\varepsilon)}|^2\Big\}(t,x) \tag{1.5}$$

is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))$, so that a subsequence, still indexed by ε converges weak–* in $L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$ to $e(t,x) \in L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$, where $\mathcal{M}(\mathbb{R}^N)$ denotes the bounded RADON measures on $\mathbb{R}^N$.

The second section is devoted to the determination of the H–measure associated to $(\frac{\partial v^\varepsilon}{\partial t}, \operatorname{grad} v^\varepsilon)$ (see Theorem 2.14 below) and to the ensuing determination of $e(t,x)$. To this effect, H–measures are defined and their localization and transport properties are analyzed, then specialized to the adequate setting for the wave equation.

The third section briefly introduces semi–classical measures, as a convenient tool in the proof, in dimension $N = 3$, of a microlocal theorem (see Theorem 3.7 below), due to P. GÉRARD [5], which improves a classical result of compactness of P.L. LIONS [8]. The original result, specialized to a three–dimensional setting, is the following

Theorem 1.1. *If*

$$w^n \rightharpoonup 0, \text{ weakly in } H^1(\mathbb{R}^3),$$

while

$$\begin{cases} |\operatorname{grad} w^n|^2 & \rightharpoonup M, \text{ weak–* in } \mathcal{M}(\mathbb{R}^3) \\ (w^n)^6 & \rightharpoonup R, \text{ weak–* in } \mathcal{M}(\mathbb{R}^3) \end{cases},$$

then R has its support on the atoms of M.

The improved version is then applied to the solution v^ε of the wave equation with the help of the results of Section 2, and it yields the already mentioned generic (in time) compactness of $v^\varepsilon(x,t)$, hence of $u^\varepsilon(x,t)$, in $L^6(\mathbb{R}^3)$ (see Theorem 3.9 below).

In matters of notation, we will sometimes use Einstein's summation convention. The letter C will always denote a generic constant, so that for example $2C$ will be replaced by C. In dealing with the wave equation, we will denote the time variable, either by t or by x_0, as convenient, while the associated FOURIER variable will always be denoted by ξ_0; x will always mean the N–tuple $x_1, ..., x_N \in \mathbb{R}^N_x$, with associated FOURIER variable $\eta \in \mathbb{R}^N_\eta$(or S^{N-1}_η), while y will always mean the $(N+1)$–tuple $x_0, x_1, ..., x_N \in \mathbb{R}^{N+1}_y$, with associated FOURIER variable $\xi \in \mathbb{R}^{N+1}_\eta$(or S^N_η).Also,

$$\{\!\{p,q\}\!\} := \frac{\partial p}{\partial \xi_j}\frac{\partial q}{\partial y_j} - \frac{\partial p}{\partial y_j}\frac{\partial q}{\partial \xi_j}$$

will denote the Poisson bracket of two functions p, q of y and ξ. For any set A, χ_A will denote the characteristic function of that set. Finally, we will be somewhat loose in our writing of measures: for example by $\nu(x,\eta)$ we will denote a measure that leaves on the phase space $\mathbb{R}^N_x \times \mathbb{R}^N_\eta$(or S^{N-1}_η), while $\nu(x + t\eta)$ will denote the translate ν under $(x,\eta) \to (x - t, \eta)$; $\mathcal{L}^N(\xi)$ will denote the N–dimensional LEBESGUE measure on $\mathbb{R}^N$.

2. H–measures or microlocal defect measures

In a first subsection, we introduce elementary notions of the pseudo–differential calculus; the second subsection is devoted to the definition and basic properties of localization and transport of H–measures with a focus on the wave equation, while the third subsection particularizes the obtained results to the homogeneous wave equation.

2.1. **Pseudo–differential operators.** We first define a convenient class of pseudo–differential operators, namely

Definition 2.1.

$$S^m_M(\mathbb{R}^Q) := \Big\{ p(y,\xi) \in \mathcal{C}^\infty(\mathbb{R}^Q_y \times \mathbb{R}^Q_\xi; \mathbb{C}^{M^2}) : \text{ for any } K \subset \mathbb{R}^Q,$$

$$\text{and for any } n\text{–tuples } \gamma, \delta, \;\; |D^\gamma_y D^\delta_\xi p(y,\xi)| \le C(K,\gamma,\delta)(1 + |\xi|^{m-|\delta|}\Big\}.$$

Then the standard pseudo–differential operator P of order m, with symbol $\sigma(P) = p$, is defined as the mapping

$$\begin{cases} [\mathcal{C}_0^\infty(\mathbb{R}^Q)]^M & \to \quad [\mathcal{C}^\infty(\mathbb{R}^Q)]^M \\ u & \to \quad Pu(y) = 1/(2\pi)^Q \int_{\mathbb{R}^Q_\xi} e^{iy\cdot\xi} p(y,\xi)\hat{u}(\xi)d\xi \end{cases}$$

The mapping P extends as a mapping from $[H^t(\mathbb{R}^Q)]^M$ into $[H^{t-m}_{loc}(\mathbb{R}^Q)]^M$, and if $C(K,\gamma,\delta)$ is independent of K, then P maps $[H^t(\mathbb{R}^Q)]^M$ into $[H^{t-m}(\mathbb{R}^Q)]^M$. We denote the set of such mappings by $\Sigma^m_M(\mathbb{R}^Q)$.

For those readers that are somewhat unfamiliar with pseudo–differential operators, note that the above properties are easily established through various manipulations of the Fourier transform, together with application of the following Petree's inequality that holds true for any pair $(\xi, \xi') \in (\mathbb{R}^N)^2$ and any $s \in \mathbb{R}$:

$$(1 + |\xi|^2)^s \leq 2^{|s|}(1 + |\xi - \xi'|^2)^{|s|}(1 + |\xi'|^2)^s. \tag{2.1}$$

We further define

Definition 2.2.

$$\Psi^m_M(\mathbb{R}^Q) := \Big\{ P \in \Sigma^m_M(\mathbb{R}^Q) : \sigma(P)(y,\xi) = p^m(y,\xi)\chi(\xi) + p^{m-1}(y,\xi) :$$
$$p^m \in \mathcal{C}^\infty(\mathbb{R}^Q_y \times (\mathbb{R}^Q_\xi \setminus \{\xi = 0\}); \mathbb{C}^{M^2}) \text{ homogeneous of degree } m \text{ in } \xi,$$
$$\chi \in \mathcal{C}^\infty(\mathbb{R}^Q_\xi) \text{ with } \chi \equiv 0 \text{ in a neighborhood of } \xi = 0, p^{m-1} \in S^{m-1}_M(\mathbb{R}^Q) \Big\},$$

and $\Psi^m_{M,c}(\mathbb{R}^Q)$ is the subset of $\Psi^m_M(\mathbb{R}^Q)$ of y–compactly supported operators.

Further, $\sigma^m(P) := p^m$ is then called the principal symbol of P and p^{m-1} the lower order symbol of P.

Remark 2.3. Note that $\sigma^m(P)$ is uniquely determined whenever $P \in \Psi^m_M(\mathbb{R}^Q)$ while changing χ only modifies P by a smoothing operator($\in \cap_m \Sigma^m_M(\mathbb{R}^Q)$).

We now give, without proof, the two properties of pseudo–differential operators that we will use later (see [7] for a proof); note that the first lemma deals with principal symbols whereas the second one deals with the full symbols.

Lemma 2.4. *If P^* denotes the adjoint operator to $P \in \Psi^m_{M,c}(\mathbb{R}^Q)$, and $Q \in \Psi^n_{M,c}(\mathbb{R}^Q)$,*

(1) *$P^* \in \Psi^m_{M,c}$ and $\sigma^m(P^*) = \overline{\sigma^m(P)}^t =: \sigma^m(P)^*$;*
(2) *$PQ \in \Psi^{m+n}_{M,c}(\mathbb{R}^Q)$ and*
$\sigma^{m+n}(PQ) = \sigma^m(P)\sigma^n(Q)$.

Lemma 2.5. *If $P \in \Psi^m_{M,c}, Q \in \Psi^n_{M,c}(\mathbb{R}^Q)$,*

(1) $\sigma(P^*) - \sigma(P)^* - 1/i \sum_1^Q \dfrac{\partial^2 \sigma(P)^*}{\partial y_j \partial \xi_j} \in S^{m-2}_M(\mathbb{R}^Q)$;

(2) $\sigma(PQ) - \sigma(P)\sigma(Q) - 1/i \sum_1^Q \dfrac{\partial \sigma(P)}{\partial \xi_j}\dfrac{\partial \sigma(Q)}{\partial y_j} \in S^{m+n-2}_M(\mathbb{R}^Q)$. *Thus, if $\sigma(P)$ commutes with $\sigma(Q)$, then $[P,Q] := PQ - QP \in S^{m+n-1}_M(\mathbb{R}^Q)$ with symbol $1/i\{\!\{\sigma(P), \sigma(Q)\}\!\}$.*

2.2. **H–measures.** We begin with a definition/lemma for H–measures.

Lemma 2.6. *Let $V^\varepsilon \in [L^2(\mathbb{R}^Q)]^M$ be such that $V^\varepsilon \rightharpoonup O$. There exists a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$ and a $M \times M$ matrix μ_{ij} of* Radon *measures on $\mathbb{R}^Q_y \times S^{Q-1}_\xi$ such that, for any $P \in \Psi^0_{M,c}(\mathbb{R}^Q)$,*

$$\lim_{\varepsilon'} \int_{\mathbb{R}^Q_y} PV^{\varepsilon'}.\overline{V}^{\varepsilon'} dx =: \lim_{\varepsilon'} < PV^{\varepsilon'}, V^{\varepsilon'} >= \int_{\mathbb{R}^Q_y \times S^{Q-1}_\xi} \sigma^0(P)_{ij} d\mu_{ij} =:< \mu, \sigma^0(P) > .$$

Furthermore, μ is non–negative and hermitian, i.e.,

$$\begin{cases} \mu_{ij} = \overline{\mu}_{ji} \\ \sum_{i,j=1}^Q \mu_{ij} c_i \overline{c}_j \geq 0, \ c \in \mathbb{C}^Q. \end{cases}$$

Sketch of proof. Take $P \in \Psi^0_{M,c}(\mathbb{R}^Q)$. Since $V^\varepsilon \overset{[L^2(\mathbb{R}^Q)]^M}{\rightharpoonup} 0$, for a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$,

$$< \mu, P >:= \lim_{\varepsilon'} < PV^{\varepsilon'}, V^{\varepsilon'} > .$$

A simple diagonalization process, together with Stone–Weierstrass' theorem, imply that μ can be defined for any element $P \in \Psi^0_{M,c}(\mathbb{R}^Q)$. But, if $\sigma^0(P) = \sigma^0(Q)$, $P, Q \in \Psi^0_{M,c}(\mathbb{R}^Q)$, then $P - Q$ maps $L^2(\mathbb{R}^Q)$ into $H^1_c(\mathbb{R}^Q)$, so that Rellich's theorem implies that $< \mu, P - Q >= 0$. Thus $< \mu, P >$ only depends upon $\sigma^0(P)$, and we can redefine $< \mu, P >$ as $< \mu, \sigma^0(P) >$, a linear functional on $\mathcal{C}^\infty_0(\mathbb{R}^Q_y \times S^{Q-1}_\xi; \mathbb{C}^{M^2})$.

Let us show the additional properties of μ. The hermitian character of μ is evident from its definition. To demonstrate that μ is a non-negative matrix of Radon measures, we consider, for any scalar valued $\sigma^0(P) \geq 0 \in \mathcal{C}^\infty_0(\mathbb{R}^Q_y \times S^{Q-1}_\xi)$, the quantity $< \mu, \sigma^0(P) c \otimes \overline{c} >$, $c \in \mathbb{C}^M$. Introduce $\Phi \in \mathcal{C}^\infty_0(\mathbb{R}^Q_y \times S^{Q-1}_\xi)$, with $\Phi \geq 0$ and $\equiv 1$ on $supp(\sigma^0(P))$ and consider, for $\delta > 0$, the element $b_\delta := \sqrt{\sigma^0(P) + \delta\Phi}$ which belongs to $\mathcal{C}^\infty_0(\mathbb{R}^Q_y \times S^{Q-1}_\xi)$. Then, if $B_\delta \in \Psi^0_{M,c}(\mathbb{R}^Q)$ is defined as any element such that $\sigma^0(B_\delta) = b_\delta d \otimes \overline{d}$ with

$$d := \frac{c}{\sqrt{\sum_{k=1,\ldots,M} |c_k|^2}},$$

Lemma 2.4 and the very definition of μ imply that

$$0 \leq < \mu, \sigma^0(B^*_\delta B_\delta) >=< \mu, (\sigma^0(P) + \delta\Phi) c \otimes \overline{c} >$$

or still,

$$< \mu, \sigma^0(P) c \otimes \overline{c} > \geq -\delta < \mu, \Phi c \otimes \overline{c} > .$$

The result is obtained upon letting $\delta \searrow 0$, because a non-negative linear functional on $\mathcal{C}^\infty_0(\mathbb{R}^Q_y \times S^{Q-1}_\xi)$ is a non-negative Radon measure. ¶

Remark 2.7. If $P \in \Psi^0_{M,c}(\mathbb{R}^Q)$ has a principal symbol of the form $\sigma^0(P) = a(x)b(\xi)$, define, for any $V \in [L^2(\mathbb{R}^Q)]^M$ (and any χ smooth cut–off around $\xi = 0$),

$$\tilde{P}V(y) := \int_{\mathbb{R}^Q_\xi} \chi(\xi) b(\xi) \widehat{aV}(\xi) e^{iy.\xi} d\xi;$$

it is then a simple exercize that uses e.g. (2.1) to show that

$$PV^\varepsilon - \tilde{P}V^\varepsilon \overset{[L^2(\mathbb{R}^Q)]^M}{\rightarrow} 0.$$

Consequently, an equivalent definition of μ for such symbols is (see [13])

$$< \mu, \sigma^0(P) >= \lim_{\varepsilon'} < \tilde{P}V^{\varepsilon'}, V^{\varepsilon'} > .$$

As a corollary, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^Q_y)$, $\varphi^2\mu$ is the H–measure associated to $\varphi V^{\varepsilon'}$.

Example 2.8. We now give two examples where the H–measure is explicitly computable.

- *Periodic oscillations:* Assume that $v \in L^2(\mathcal{T}), \int_{\mathcal{T}} v dy = 0$ where $\mathcal{T}$ is the unit Q–dimensional torus and define the oscillating sequence $v^\varepsilon(y) := v(\frac{y}{\varepsilon}) \overset{L^2_{loc}(\mathbb{R}^Q)}{\rightharpoonup} 0$. For $\varphi \in \mathsf{S}(\mathbb{R}^Q)$ — the set of rapidly decreasing functions — look at the H–measure μ_φ associated to (a subsequence of) $\{v^\varepsilon\varphi\}$. Then, the whole sequence actually admits a H–measure μ_φ defined as

  $$\mu_\varphi = \sum_{k\neq 0} |\hat{v}_k|^2 \varphi(y) dy \otimes \delta_{\frac{k}{|k|}}(\xi),$$

  where $\hat{v}_k$ is the k^{th}–Fourier coefficient of v.

  Indeed, assume that $\hat{\varphi}$ has compact support. Then, $\widehat{\varphi v^\varepsilon} = \sum_{k\neq 0} \hat{v}_k \hat{\varphi}(\xi - \frac{k}{\varepsilon})$ is such that, for $k \neq k'$ and ε small enough, the supports of $\hat{\varphi}(\xi - \frac{k}{\varepsilon})$ and of $\hat{\varphi}(\xi - \frac{k'}{\varepsilon})$ are disjoint.

  Defining $\sigma^0(P) := \varphi^2(y) b(\frac{\xi}{|\xi|})$ with $b \in \mathcal{C}^\infty(S^{Q-1})$, Plancherel's identity yields

  $$< \tilde{P}v^\varepsilon, v^\varepsilon >= \int_{\mathbb{R}^Q_\xi} \chi(\xi) b(\frac{\xi}{|\xi|}) (\sum_{k\neq 0} |\hat{v}_k|^2 |\hat{\varphi}(\xi - \frac{k}{\varepsilon})|^2 d\xi,$$

  so that, upon performing the change of variables $\eta := \xi - \frac{k}{\varepsilon}$, we obtain

  $$\lim_\varepsilon < \tilde{P}v^\varepsilon, v^\varepsilon >= (\sum_{k\neq 0} |\hat{v}_k|^2 b(\frac{k}{|k|})) \int_{\mathbb{R}^Q_\eta} |\hat{\varphi}(\eta|^2 d\eta.$$

  The result follows through a density argument.
- *Concentrations:* Assume that $f \in L^2(\mathbb{R}^Q)$ and define the concentrating sequence $v^\varepsilon(y) := \varepsilon^{-\frac{Q}{2}} f(\frac{y-z}{\varepsilon}) \overset{L^2(\mathbb{R}^Q)}{\rightharpoonup} 0$. Then it is easily shown that the whole sequence admits an H–measure μ with

  $$\mu = \delta_z(y) \otimes [1/(2\pi)^Q \int_0^\infty |\hat{f}(t\xi)|^2 t^{Q-1} dt] d\xi.$$

Remark 2.9. When, as in the examples above, the *whole* sequence admits an H–measure, we say that *the sequence is pure.*

We now derive a localization lemma for the support of an H–measure.

Lemma 2.10. *Consider* $V^\varepsilon \overset{[L^2(\mathbb{R}^Q)]^M}{\rightharpoonup} 0$ *with associated H–measure* μ*. If, for some* $R \in \Psi^m_M(\mathbb{R}^Q)$*,* $RV^\varepsilon \in$ compact of $[H^{-m}_{loc}(\mathbb{R}^Q)]^M$*, then*

$$\sigma^m(R)\mu = 0,$$

or, in indices,

$$\sum_{j=1}^{M} \sigma^m(R)_{ij}\mu_{jq} = 0,\ 1 \le i \le M, 1 \le q \le M.$$

Proof. Assume $M = 1$ for simplicity. Consider, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_y^Q), b(\xi) \in S_\xi^{Q-1}$, the pseudo–differential operator $T \in \Psi_{1,c}^{-m}(\mathbb{R}^Q)$ with associated principal symbol $\varphi(y)b(\frac{\xi}{|\xi|})|\xi|^{-m}$.

Then, $TRV^\varepsilon \in$ compact of $[L^2_{loc}(\mathbb{R}^Q)]$, thus $< TRV^\varepsilon, V^\varepsilon > \to 0$, or in other words $< \mu, \sigma^m(R)|\xi|^{-m}b(\xi)\varphi(y) >= 0$. The arbitrariness of the choice of φ and b yields the lemma. ¶

First application to the wave equation. In the framework of Section 1 (see (1.2) introduce

$$V^\varepsilon := (\frac{\partial v^\varepsilon}{\partial t}, \operatorname{grad} v^\varepsilon) \overset{L^\infty(\mathbb{R}_t;[L^2(\mathbb{R}^Q)]^M)}{\rightharpoonup} 0$$

and remark that such a sequence will fit within the H–measure framework, provided we multiply it by a smooth compactly supported function of t which is identically 0 if $|t|$ is large enough. In the remainder of the paper we will do as if there were no such multiplier in dealing with the solution to the wave equation, favoring simplicity over rigor.

The H–measure associated to (a subsequence of) V^ε is a $N \times N$ matrix μ. Let us apply the localization lemma. First, we express the commutativity of mixed derivatives, namely

$$\frac{\partial V_j^\varepsilon}{\partial y_i} = \frac{\partial V_i^\varepsilon}{\partial y_j},\ i, j = 0, ..., N.$$

Thanks to the localization lemma, we obtain

$$\xi_i\mu_{jk} = \xi_j\mu_{ik},\ i, j, k = 0, ..., N.$$

Thus $\mu_{ik} = \xi_i\nu_k,\ i, k = 0, ..., N$, and since μ is Hermitian, i.e. $\mu_{ik} = \overline{\mu}_{ki},\ \nu_i = \xi_i\nu,\ i = 0, ..., N$ with ν a non-negative scalar Radon measure. Thus

$$\mu = (\xi \otimes \xi)\ \nu. \tag{2.2}$$

Then, we write the actual wave equation, namely

$$\rho(x)\frac{\partial V_0^\varepsilon}{\partial x_0} - \sum_{j=1,N} \frac{\partial}{\partial x_j}(k(x)V_j^\varepsilon) = 0,$$

or still

$$\rho(x)\frac{\partial V_0^\varepsilon}{\partial x_0} - k(x)\sum_{j=1,N}\frac{\partial V_j^\varepsilon}{\partial x_j} = \sum_{j=1,N}\frac{\partial k(x)}{\partial x_j}V_j^\varepsilon.$$

The right hand–side of the previous equality belongs to a bounded set of $L^2(\mathbb{R}^{N+1})$, hence to a compact set in $H^{-1}_{loc}(\mathbb{R}^{N+1})$ and the localization theorem applies yielding

$$\rho(x)\xi_0\mu_{0k} - k(x)\sum_{j=1,...,N}\xi_j\mu_{jk} = 0,\ k = 0, ..., N,$$

or still in view of (2.2) and because not all ξ_k may cancel at the same time ($|\xi| = 1$),

$$q(x, \xi)\nu = 0. \tag{2.3}$$

where

$$q(x,\xi) := \rho(x)\xi_0^2 - k(x) \sum_{j=1,\dots,N} \xi_j^2. \tag{2.4}$$

Thanks to (2.2), the limit of (a subsequence of) the energy density e^ε defined in (1.5) is given by

$$\lim_{\varepsilon} e^\varepsilon = 1/2 \int_{S_\xi^N} (\rho(x)\xi_0^2 + k(x) \sum_{i=1,\dots,N} \xi_i^2) d\nu(y,\xi). \tag{2.5}$$

It thus remains to determine ν subject to the support restriction (2.3). This will be performed with the help of the following transport lemma specialized to the context of the wave equation. Note that, as emphasized in the introduction, this lemma is the outstanding feature which permits a complete characterization of H–measures as solutions of a transport equation, in contrast to YOUNG measures which are not constrained by any type of partial differential equation.

Lemma 2.11. *Consider* $V^\varepsilon \overset{[L^2(\mathbb{R}^Q)]^M}{\rightharpoonup} 0$ *with associated H–measure* μ. *Consider also* $R \in \Psi_M^m(\mathbb{R}^Q)$ *such that*

- $\sigma^m(R)$ *is self–adjoint;*
- *the lower order symbol of* P *defines an element in* $\Psi_M^{m-1}(\mathbb{R}^Q)$ *with principal symbol* $\sigma^{m-1}(R)$,

and assume that $RV^\varepsilon \in$ compact of $[H_{loc}^{-m+1}(\mathbb{R}^Q)]^M$.

Then, for any $a \in \mathcal{C}_0^\infty(\mathbb{R}_y^Q \times S_\xi^{Q-1})$,

$$< \mu, \{\!\{\sigma^m(R), a\}\!\} + \Big[i\Big((\sigma^{m-1}(R))^* - \sigma^{m-1}(R)\Big) + \frac{\partial^2 \sigma^m(R)}{\partial y_j \partial \xi_j} + (m-1)\xi_j \frac{\partial \sigma^m}{\partial y_j}\Big] a >= 0.$$

Remark 2.12. Note that the hypotheses on R are automatically satisfied if R is a differential operator with self–adjoint higher order terms.

Proof. Take an arbitrary $A \in \Psi_{1,c}^0(\mathbb{R}^Q)$, with $a := \sigma^0(A)$, and define $Q := (-\Delta)^{-\frac{m-1}{2}} R$, so that, according to Lemma 2.4(2), $Q \in \Psi_{M,c}^1(\mathbb{R}^Q)$ and $\sigma^1(Q) = |\xi|^{-(m-1)}\sigma^m(R)$. Actually, according to Lemma 2.5(2),

$$\sigma(Q) = |\xi|^{-(m-1)}\sigma(R) + 1/i \frac{\partial}{\partial \xi_j}(|\xi|^{-(m-1)}) \frac{\partial \sigma(R)}{\partial y_j} + q'(\in S_M^{-1}(\mathbb{R}^Q)),$$

hence, according to Lemma 2.5(1),

$$\sigma(Q^*) = |\xi|^{-(m-1)}\sigma(R)^* - \frac{1}{i}\frac{\partial}{\partial \xi_j}(|\xi|^{-(m-1)}) \frac{\partial \sigma(R)^*}{\partial y_j} + \frac{1}{i}\frac{\partial^2}{\partial y_j \partial \xi_j}(|\xi|^{-(m-1)}\sigma(R)^*) + q''$$

with $q'' \in S_M^{-1}(\mathbb{R}^Q)$). But, by hypothesis, $\sigma^m(R)$ is self–adjoint and the lower order symbol of P defines an element in $\Psi_M^{m-1}(\mathbb{R}^Q)$ with principal symbol $\sigma^{m-1}(R)$, thus

$Q^* - Q \in \Psi^0_M(\mathbb{R}^Q)$ with principal symbol

$$\begin{aligned}\sigma^0(Q^*-Q) &= |\xi|^{-(m-1)}\Big((\sigma^{m-1}(R))^* - \sigma^{m-1}(R)\Big)\\ &\quad -\frac{2}{i}\frac{\partial}{\partial\xi_j}(|\xi|^{-(m-1)})\frac{\partial\sigma^m(R)}{\partial y_j} + \frac{1}{i}\frac{\partial^2}{\partial y_j\partial\xi_j}(|\xi|^{-(m-1)}\sigma^m(R))\\ &= |\xi|^{-(m-1)}\Big((\sigma^{m-1}(R))^* - \sigma^{m-1}(R)\Big)\\ &\quad +\frac{2}{i}(m-1)|\xi|^{-(m+1)}\xi_j\frac{\partial\sigma^m(R)}{\partial y_j} + \frac{1}{i}\frac{\partial^2}{\partial y_j\partial\xi_j}(|\xi|^{-(m-1)}\sigma^m(R))\end{aligned}$$

or still,

$$\begin{aligned}\sigma^0(Q^*-Q) &= |\xi|^{-(m-1)}\Big((\sigma^{m-1}(R))^* - \sigma^{m-1}(R)\Big)\\ &\quad +\frac{1}{i}(m-1)|\xi|^{-(m+1)}\xi_j\frac{\partial\sigma^m(R)}{\partial y_j} + \frac{1}{i}|\xi|^{-(m-1)}\frac{\partial^2\sigma^m(R)}{\partial y_j\partial\xi_j}.\end{aligned} \tag{2.6}$$

Now, application of Lemma 2.5(2) to A and Q implies, since a commutes with $\sigma(Q)$, that $[A,Q] \in \Sigma^0_M(\mathbb{R}^Q)$, with $\sigma([A,Q]) = 1/i\{\!\{a,\sigma(Q)\}\!\}$, and consequently since $A \in \Psi^0_{1,c}(\mathbb{R}^Q)$ and $Q \in \Psi^1_{M,c}(\mathbb{R}^Q)$, that $[A,Q]$ actually belongs to $\Psi^{0,c}_M(\mathbb{R}^Q)$ with principal symbol

(2.7)

$$\begin{aligned}\sigma^0([A,Q]) &= 1/i\{\!\{a,\sigma^1(Q)\}\!\} = 1/i\{\!\{a,|\xi|^{-(m-1)}\sigma^m(R)\}\!\}\\ &= 1/i\Big(|\xi|^{-(m-1)}\{\!\{a,\sigma^m(R)\}\!\} + (m-1)|\xi|^{-(m+1)}\xi_j\frac{\partial a}{\partial y_j}\sigma^m(R)\Big)\ .\end{aligned}$$

We now apply the definition of μ to the sequence $<[A,Q]V^\varepsilon, V^\varepsilon>$. Noting that $QV^\varepsilon \in$ compact of $[L^2_{loc}(\mathbb{R}^Q)]^M$, we immediately conclude that, as $\varepsilon \searrow 0$,

$$\begin{cases} <AQV^\varepsilon, V^\varepsilon> \to 0\\ <AV^\varepsilon, QV^\varepsilon> \to 0\end{cases}$$

so that,

$$\lim_\varepsilon <[A,Q]V^\varepsilon, V^\varepsilon> = \lim_\varepsilon <(Q^*-Q)AV^\varepsilon, V^\varepsilon>,$$

or still, appealing to (2.6,2.7),

$$<\mu, 1/i\Big(|\xi|^{-(m-1)}\{\!\{a,\sigma^m(R)\}\!\} + (m-1)|\xi|^{-(m+1)}\xi_j\frac{\partial a}{\partial y_j}\sigma^m(R)\Big)> =$$

$$\begin{aligned}&<\mu, \Big[|\xi|^{-(m-1)}\Big((\sigma^{m-1}(R))^* - \sigma^{m-1}(R)\Big) + \frac{1}{i}(m-1)|\xi|^{-(m+1)}\xi_j\frac{\partial\sigma^m(R)}{\partial y_j}\\ &+\frac{1}{i}|\xi|^{-(m-1)}\frac{\partial^2\sigma^m(R)}{\partial y_j\partial\xi_j}\Big]a>\ .\end{aligned}$$

The localization lemma 2.10 implies that $\sigma^m(R)\mu = 0$, so that the last term in the left hand–side of the previous equality disappears; furthermore, $|\xi| \equiv 1$ on the support of μ. Rearranging terms, we are left with the transport equation stated in the lemma. ¶

Second application to the wave equation. To analyze the transport properties of the measure ν introduced in (2.2), it is more convenient to introduce the $(N+1)$–vector $W^\varepsilon := (\sqrt{\rho}V^\varepsilon, \sqrt{k}V^\varepsilon_i)$ with associated H–measure

$$\pi = (L(x)\xi \otimes L(x)\xi)\nu \tag{2.8}$$

where $L(x) := \begin{pmatrix} \sqrt{\rho} & 0 & ... & 0 \\ 0 & \sqrt{k} & ... & ... \\ ... & ... & ... & 0 \\ 0 & ... & 0 & \sqrt{k} \end{pmatrix}$. Then it is easily checked that the commutativity properties of the mixed derivatives, together with the wave equation, now expressed in those new variables, become

$$iRW^\varepsilon = 0$$

with

$$iR := \begin{pmatrix} \frac{\partial}{\partial t} & -\frac{1}{\sqrt{\rho}}\frac{\partial(\sqrt{k}\ .)}{\partial x_1} & ... & -\frac{1}{\sqrt{\rho}}\frac{\partial(\sqrt{k}\ .)}{\partial x_3} \\ -\sqrt{k}\frac{\partial(\frac{1}{\sqrt{\rho}}\ .)}{\partial x_1} & \frac{\partial}{\partial t} & 0 & ... \\ ... & ... & ... & ... \\ -\sqrt{k}\frac{\partial(\frac{1}{\sqrt{\rho}}\ .)}{\partial x_3} & ... & 0 & \frac{\partial}{\partial t} \end{pmatrix}.$$

Note that $R \in \Psi^{1,c}_M(\mathbb{R}^Q)$ with

$$\sigma^1(R) = \begin{pmatrix} \xi_0 & -\xi_1\sqrt{\frac{k}{\rho}} & ... & -\xi_3\sqrt{\frac{k}{\rho}} \\ -\xi_1\sqrt{\frac{k}{\rho}} & \xi_0 & 0 & 0 \\ ... & ... & ... & ... \\ -\xi_3\sqrt{\frac{k}{\rho}} & 0 & 0 & \xi_0 \end{pmatrix}. \tag{2.9}$$

Further, the lower order symbol defines an element in $\Psi^{0,c}_M(\mathbb{R}^Q)$ with (full) symbol

$$\sigma^0(R) = \begin{pmatrix} 0 & \frac{i}{\sqrt{\rho}}\frac{\partial\sqrt{k}}{\partial x_1} & ... & \frac{i}{\sqrt{\rho}}\frac{\partial\sqrt{k}}{\partial x_3} \\ i\sqrt{k}\frac{\partial\frac{1}{\sqrt{\rho}}}{\partial x_1} & 0 & 0 & 0 \\ ... & ... & ... & ... \\ i\sqrt{k}\frac{\partial\frac{1}{\sqrt{\rho}}}{\partial x_3} & 0 & 0 & \xi_0 \end{pmatrix}. \tag{2.10}$$

In view of (2.9,2.10), the transport Lemma 2.11 straightforwardly yields

$$< \pi, \{\!\{\sigma^1(R), a\}\!\} >= 0,$$

a relation we now explicit with the help of (2.8). The computation is straightforward, but a bit tedious. It only uses the localization equation (2.3), and is safely left to the reader.

We finally obtain the following transport equation:

$$< \nu, \xi_0\{\!\{q, a\}\!\} >= 0, \quad \text{for any } a \in \mathcal{C}_0^\infty(\mathbb{R}_y^{N+1} \times S_\xi^N). \tag{2.11}$$

Consequences 2.13. The following four remarks complete the study of the H–measure for the wave equation:

i. Since the support of ν lies in the zero set of $q(x,\xi)$ on S^N_ξ, and in view of (1.1),

$$\{\xi \in S^N_\xi : \xi_0 = 0\} \subset (supp\ \nu)^c$$

so that we are at liberty to choose a in (2.11) of the form $\frac{1}{\xi_0}\zeta(\xi)a$ where $\zeta \in \mathcal{C}^\infty(S^N_\xi)$ is such that $\zeta(0,\xi_1,....,\xi_N) = 0$ and $\zeta \equiv 1$ on *supp* ν. Then, since q does not depend upon $x_0 = t$ or ξ_0,

$$0 =< \nu, \xi_0 \{\!\{q, \frac{1}{\xi_0}\zeta a\}\!\} >=< \nu, \{\!\{q, \zeta a\}\!\} >=< \nu, \{\!\{q, a\}\!\} >$$

and the transport equation reads more simply as

$$< \nu, \{\!\{q, a\}\!\} >= 0, \quad \text{for any } a \in \mathcal{C}^\infty_0(\mathbb{R}^{N+1}_y \times S^N_\xi); \tag{2.12}$$

ii. We have referred several times to (2.11,2.12) as "transport equations". In the case of the wave equation, it is actually shown in [3], Section 3.2.1, that ν remains constant along the projections onto $\mathbb{R}^{N+1}_y \times S^N_\xi$ of the integral curves of

$$\begin{cases} \dfrac{d\overline{y}_i}{ds} = \dfrac{\partial(q/|\xi|)}{\partial \xi_i}(\overline{y}, \overline{\xi}) \\ \dfrac{d\overline{\xi}_i}{ds} = -\dfrac{\partial(q/|\xi|)}{\partial y_i}(\overline{y}, \overline{\xi}) \end{cases} \quad \Big(|\xi| := (\sum_0^N \xi_j^2)^{1/2}\Big)$$

with initial conditions

$$\begin{cases} \overline{y}(0) = y \\ \overline{\xi}(0) = \xi \end{cases} \quad \text{with } q(y,\xi) = 0,\ \xi \in S^N_\xi.$$

The principle leading to the above statement is simple. Assume for a moment that the weak form (2.12) was for test functions $a \in \mathcal{C}^\infty_0(\mathbb{R}^{N+1}_y \times \mathbb{R}^{N+1}_\xi)$. Then, a simple integration by parts would immediately yield

$$\{\!\{q, \nu\}\!\} = 0$$

and ν would then be constant along the integral curves of

$$\begin{cases} \dfrac{d\overline{y}_i}{ds} = \dfrac{\partial q}{\partial \xi_i}(\overline{y}, \overline{\xi}) \\ \dfrac{d\overline{\xi}_i}{ds} = -\dfrac{\partial q}{\partial y_i}(\overline{y}, \overline{\xi}) \end{cases}$$

with

$$\begin{cases} \overline{y}(0) = y \\ \overline{\xi}(0) = \xi. \end{cases}$$

The difficulty in our context is that the ξ variable only lives on the sphere S^N_ξ;

iii. The transport equation (2.12) only makes sense once the initial value of ν is determined. This is not completely immediate because the H–measure associated to the initial conditions (1.4) *a priori* leaves in $\mathbb{R}^N_x \times S^{N-1}_\eta$; there is thus a reduction of *2* dimensions. The following theorem is established in [3] (see Corollary 3.1):

Theorem: *If $\tilde\nu^{\pm}$ are the H–measures associated to (a subsequence of) $\{v^\varepsilon_{\pm 0}\}$ with $v^\varepsilon_{\pm 0} := \sqrt{\rho} Z^\varepsilon_0 \pm i\sqrt{k}|D_x| V^\varepsilon_0$ ($V^\varepsilon_0, Z^\varepsilon_0$ defined in (1.4)), then*

$$\nu(t=0) = \frac{1}{4\rho\xi_0^2}(\tilde\pi^+ + \tilde\pi^-)$$

where, for any $\phi \in \mathcal{C}^\infty_0(\mathbb{R}^N_y \times S^N_\xi)$,

$$< \tilde\pi^{\pm}, \phi >:= \int_{\mathbb{R}^N_x \times S^{N-1}_\eta} \phi\Big(x, \pm(\frac{k(x)}{\rho(x)+k(x)})^{1/2}, (\frac{\rho(x)}{\rho(x)+k(x)})^{1/2}\eta_1, ...\Big) d\tilde\nu^{\pm}.$$

We will not establish this here, but merely note that the above definition of $\tilde\pi^{\pm}$ permits to transform the H–measures $\tilde\nu^{\pm}$ that live on $\mathbb{R}^N_x \times S^{N-1}_\eta$ into measures on $\mathbb{R}^N_\xi \times S^N_\xi$, which is precisely what we need to obtain a meaningful initial condition for ν. Also note that $|D_x|$ is the element of $\Psi^1_1(\mathbb{R}^N)$ with principal symbol $|\xi|$.

iv. According to Remark 3.10 in [3], any point in the support of ν can be reached by a unique projection of an integral curve, and that integral curve intersects $t = 0$.

Coalescing items i and iii of Consequences 2.13, we finally obtain the following

Theorem 2.14. *The H–measure associated to (a subsequence of)*

$$V^\varepsilon := (\frac{\partial v^\varepsilon}{\partial t}, \operatorname{grad} v^\varepsilon)$$

associated to v^ε defined in (1.2) is of the form $(\xi\otimes\xi)\nu$, where ν lives on the zero set of $q(x,\xi)$ defined in (2.4). Furthermore ν remains constant along the projections onto $\mathbb{R}^{N+1}_y \times S^N_\xi$ of the integral curves of

$$\begin{cases} \dfrac{d\overline{y}_i}{ds} = \dfrac{\partial(q/|\xi|)}{\partial\xi_i}(\overline{y},\overline{\xi}) \\ \dfrac{d\overline{\xi}_i}{ds} = -\dfrac{\partial(q/|\xi|)}{\partial y_i}(\overline{y},\overline{\xi}) \end{cases} \quad \Big(|\xi| := (\sum_0^N \xi_j^2)^{1/2}\Big)$$

with initial conditions

$$\begin{cases} \overline{y}(0) = y \\ \overline{\xi}(0) = \xi \end{cases} \quad \textit{with } q(y,\xi) = 0,\ \xi \in S^N_\xi.$$

These are also called bicharacteristic strips. Finally,

$$\nu(t=0) = \frac{1}{4\rho\xi_0^2}(\tilde\pi^+ + \tilde\pi^-)$$

where $\tilde\pi^{\pm}$ has been defined in item iii of Consequences 2.13.

The limit of the energy density is then given by (2.5).

Particular Case 2.15. We conclude this section with an investigation of the particular case of the homogeneous wave equation, that is of the case where $k(x) = \rho(x) \equiv 1$. In such a case the bicharacteristic strips are easily determined. We obtain

$$\begin{cases} \overline{\xi} = \xi \\ \xi_0\overline{x}_i + \xi\overline{t} = \xi_0 x_i,\ 1 \le i \le N \end{cases} \quad (\xi_0^2 = \sum_1^N \xi_i^2,\ |\xi| = 1).$$

Then the H–measure ν is explicitly computable by virtue of Theorem 2.14; we get:

$$\nu(t,x,\xi_0,\eta) = \frac{1}{4\xi_0^2}(\tilde{\pi}^+ + \tilde{\pi}^-)(x + \frac{\eta}{\xi_0}t, \xi_0, \eta),$$

or still, in terms of $\tilde{\nu}^{\pm}$,

(2.13)

$$\nu(t,x,\xi_0,\eta) = 2^{\frac{N-2}{2}}\Big\{\tilde{\nu}^+(x+\sqrt{2}\eta t, \sqrt{2}\eta)\delta_{\xi_0=\frac{1}{\sqrt{2}}} + \tilde{\nu}^-(x-\sqrt{2}\eta t, \sqrt{2}\eta)\delta_{\xi_0=-\frac{1}{\sqrt{2}}}\Big\}.$$

Since there is nothing special about time $t = 0$, the H–measure ν is also

$$\begin{aligned}\nu(t,x,\xi_0,\eta) \quad &= 2^{\frac{N-2}{2}}\Big\{ \quad \tilde{\nu}^+_{t_0}(x+\sqrt{2}\eta(t-t_0), \sqrt{2}\eta)\delta_{\xi_0=\frac{1}{\sqrt{2}}} \\ &\qquad +\tilde{\nu}^-_{t_0}(x-\sqrt{2}\eta(t-t_0), \sqrt{2}\eta)\delta_{\xi_0=-\frac{1}{\sqrt{2}}}\Big\},\end{aligned} \tag{2.14}$$

where $\tilde{\nu}^{\pm}_{t_0}$ are the H–measures associated to $\dfrac{\partial v^\varepsilon(t_0)}{\partial t} \pm i|D_x|v^\varepsilon(t_0)$. Comparison of (2.13) and (2.14) immediately implies that

$$\tilde{\nu}^{\pm}_t(x,\eta) = \tilde{\nu}^{\pm}(x \mp \eta t, \eta). \tag{2.15}$$

We denote by ν_t the measure on $\mathbb{R}^N_x \times S^{N-1}_\eta$ such that, for a subsequence of v^ε, and any $A \in \Psi^0_{1,c}(\mathbb{R}^N)$,

$$< \nu_t, \sigma^0(A) >= 1/2 \lim_\varepsilon \Big\{ < A\frac{\partial v^\varepsilon(t)}{\partial t}, \frac{\partial v^\varepsilon(t)}{\partial t} > + < A\frac{\partial v^\varepsilon(t)}{\partial x_i}, \frac{\partial v^\varepsilon(t)}{\partial x_i} > \Big\}. \tag{2.16}$$

For almost any given time t, the existence of such a subsequence is a direct consequence of the $L^\infty(0,T; H^1_0(\mathbb{R}^N)) \cap W^{1,\infty}(0,T; L^2(\mathbb{R}^N))$–bound on v^ε, together with Lemma 2.6. It can actually be shown that the subsequence can be picked *independently of* $t \in [0,T]$ and that the convergence in (2.16) is locally uniform in time (see [5], Proposition 4.4).

A simple computation would show that

$$< \nu_t, \sigma^0(A) >= 1/4 \lim_\varepsilon \Big(\sum_\pm < A(\frac{\partial v^\varepsilon(t)}{\partial t} \pm i|D_x|v^\varepsilon(t)), \frac{\partial v^\varepsilon(t)}{\partial t} \pm i|D_x|v^\varepsilon(t) > \Big)$$

In view of (2.15), this also reads as

$$\nu_t(x,\eta) = 1/4\Big\{\tilde{\nu}^+(x-\eta t, \eta) + \tilde{\nu}^-(x+\eta t, \eta)\Big\}, \tag{2.17}$$

a relation that will be used in Subsection 3.3 below.

3. A compactness theorem for the wave equation

In a first subsection, we briefly introduce semi–classical measures as a tool for the further investigation of the solution to the wave equation investigated in the particular case 2.15. The following subsection is devoted to the proof of a microlocal compactness theorem of P. Gérard (Theorem 3.7 below). As a corollary, we obtain in the short third subsection the compactness result (Theorem 3.9) anounced in the introduction.

3.1. **Semi–classical measures.** Whenever the problem under consideration exhibits a characteristic scale, say ε, it is of special interest to investigate the oscillations that take place at that scale. In the sake of simplicity, we only consider scalar problems.

To this effect, we consider the regularizing standard pseudo–differential operators $A(\varepsilon D) \in \cap_m \Sigma_1^m(\mathbb{R}^Q)$ with symbols $\sigma(A)(y, \varepsilon\xi)$, where $\sigma(A)(y,\xi) \in \mathsf{S}(\mathbb{R}^Q_y \times \mathbb{R}^Q_\xi)$. Note that

$$u^\varepsilon \text{ bounded in } L^2(\mathbb{R}^Q) \Longrightarrow A(\varepsilon D)u^\varepsilon \text{ bounded in } L^2(\mathbb{R}^Q).$$

An existence lemma analogous to Lemma 2.6 holds true.

Lemma 3.1. *Let* $v^\varepsilon \overset{L^2(\mathbb{R}^Q)}{\rightharpoonup} v$. *There exists a subsequence* $\{\varepsilon'\} \subset \{\varepsilon\}$ *and a non-negative* Radon *measure* m *on* $\mathbb{R}^Q_y \times \mathbb{R}^Q_\xi$ *such that, for any* $\sigma(A) \in \mathsf{S}(\mathbb{R}^Q_y \times \mathbb{R}^Q_\xi)$,

$$\lim_{\varepsilon'} \int_{\mathbb{R}^Q_y} A(\varepsilon' D) v^{\varepsilon'} \overline{v}^{\varepsilon'} dx =: \lim_{\varepsilon'} < A(\varepsilon' D)v^{\varepsilon'}, v^{\varepsilon'} >= \int_{\mathbb{R}^Q_y \times R^Q_\xi} \sigma(A) dm =:< m, \sigma(A) > .$$

Remark 3.2. In contrast to the definition of H–measures, the weak limit of v^ε is not taken to be 0. In problems involving semi–classical measures, the weak limit of the oscillating field is not always easily identifiable.

Usually, we a priori know that v^ε oscillates at the scale of ε; that is for example the case when both v^ε and $\varepsilon D v^\varepsilon$ are bounded in $L^2(\mathbb{R}^Q)$ (think of $v^\varepsilon(x) := v(\frac{x}{\varepsilon}), v \in L^2(\mathcal{T})$). In such a case, we label the sequence ε–oscillatory; this gives rise to the following

Definition 3.3. *The sequence* $v^\varepsilon \overset{L^2(\mathbb{R}^Q)}{\rightharpoonup} v$ *is called* ε*–oscillatory if*

$$\lim_{R \nearrow \infty} \limsup_{\varepsilon} \int_{|\xi| \geq R/\varepsilon} |\widehat{\varphi v^\varepsilon}(\xi)|^2 d\xi = 0, \ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^Q).$$

Then, the following result holds true:

Lemma 3.4. *Assume that* $v^\varepsilon \overset{L^2(\mathbb{R}^Q)}{\rightharpoonup} v$ *admits* m *as semi-classical measure.*

i. *If* v^ε *is* ε*–oscillatory, then the measure limit of* $(v^\varepsilon)^2$ *is given by* $\int_{\mathbb{R}^Q_\xi} dm(y,\xi)$;

ii. *If* $m(\mathbb{R}^Q \times \{0\}) = 0$, *then* $v = 0$ *and if, further* v^ε *is* ε*–oscillatory,* v^ε *admits a H–measure defined, for any* $A \in \Psi^0_{1,c}(\mathbb{R}^Q)$, *as*

$$< \mu, \sigma^0(A) >= \int_{\mathbb{R}^Q_y \times (\mathbb{R}^Q_\xi \setminus \{0\})} \sigma^0(A)(y, \frac{\xi}{|\xi|}) dm(y,\xi); \ \textit{and}$$

iii. *If* $v^\varepsilon \overset{L^2(\mathbb{R}^Q)}{\rightharpoonup} 0$ *and admits both a H-measure and a semi–classical measure (it is both pure and* ε*–pure), then*

$$< \mu, \sigma^0(A) > \geq \int_{\mathbb{R}^Q_y \times (\mathbb{R}^Q_\xi \setminus \{0\})} \sigma^0(A)(y, \frac{\xi}{|\xi|}) dm(y,\xi).$$

Proof. The proof of item i. is as follows. Take $\varphi, \psi \in [\mathcal{C}_0^\infty(\mathbb{R}^Q)]^2$ with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in a neighborhood of $\xi = 0$ and consider $\sigma(A) := \varphi^2(y)\psi(\frac{\xi}{R})$, $\sigma(\Psi) := \psi(\xi)$. It is easily checked (in the spirit of Remark 2.7) that φv^ε admits $\varphi^2(y)m$ as semi–classical measure. Denote the measure limit of (a subsequence of) $(v^\varepsilon)^2$ by M; then, with obvious notation,

$$< m(y,\xi), \varphi^2(y)\psi(\frac{\xi}{R}) > = < M(y), \varphi^2(y) > - \lim_{\varepsilon} < 1 - \Psi(\varepsilon D/R)\varphi v^\varepsilon, \varphi v^\varepsilon > .$$

But, in view of the definition of ψ and since v^ε is ε–oscillatory, the last term in the above equality tends to 0 as $R \nearrow \infty$. Lebesgue dominated convergence implies that the left hand–side of the same equality goes to $< m(y,\xi), \varphi^2(y) >$ as $R \nearrow \infty$, which proves item i.

To prove ii., we consider $\sigma(A_n)(\xi) \geq 0 \in \mathcal{C}_0^\infty(\mathbb{R}^Q)$. Then, invoking Plancherel's identity,

$$< (A_n(\varepsilon D)v^\varepsilon, v > = \int_{\mathbb{R}^Q_\xi} \sigma((A_n)(\varepsilon\xi)\hat{v}^\varepsilon \overline{\hat{v}}\, d\xi \overset{\varepsilon \to 0}{\rightharpoonup} \sigma((A_n)(0)\|v\|^2_{L^2(\mathbb{R}^Q)}.$$

But, appealing to Cauchy–Schwartz inequality,

$$\begin{aligned} \limsup_\varepsilon < (A_n(\varepsilon D)v^\varepsilon, v > &\leq \lim_\varepsilon < (A_n(\varepsilon D)v^\varepsilon, v^\varepsilon >^{1/2} \lim_\varepsilon < (A_n(\varepsilon D)v, v >^{1/2} \\ &= < m, \sigma((A_n) >^{1/2} \sigma((A_n)^{1/2}(0)\|v\|_{L^2(\mathbb{R}^Q)}, \end{aligned}$$

so that

$$\sigma((A_n)^{1/2}(0)\|v\|_{L^2(\mathbb{R}^Q)} \leq < m, \sigma((A_n) >^{1/2} .$$

Choosing $\sigma(A_n)(\xi) \overset{n}{\to} \begin{cases} 0, \ \xi \neq 0 \\ 1, \ \xi = 0 \end{cases}$, we obtain ii. since m does not charge $\{\xi = 0\}$.

Assume now that v^ε is ε–oscillatory. Take $a(\xi) \in \mathcal{C}^\infty(\mathbb{R}^Q_\xi)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^Q_y)$. Consider $\chi(\xi) \in \mathcal{C}^\infty(\mathbb{R}^Q_\xi)$ with $\chi \equiv 0$ in a neighborhood of $\xi = 0$ and $\chi \equiv 1$ for $|\xi| \geq 1$. Define $\sigma^0(A)(y,\xi) := \varphi^2(y)a(\frac{\xi}{|\xi|})$ as the (principal) symbol of an element $A \in \Psi^0_{1,c}(\mathbb{R}^Q)$; then, according to Remark 2.7,

(3.1)

$$\begin{aligned} < A\varphi v^\varepsilon, \varphi v^\varepsilon > &= \int_{\mathbb{R}^Q_\xi \setminus \{0\}} a(\frac{\xi}{|\xi|})\chi(\frac{\xi}{r})\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi \\ &= \int_{\mathbb{R}^Q_\xi \setminus \{0\}} a(\frac{\varepsilon\xi}{\varepsilon|\xi|})\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi + \\ &\quad \int_{\mathbb{R}^Q_\xi \setminus \{0\}} a(\frac{\varepsilon\xi}{\varepsilon|\xi|})(\chi(\frac{\xi}{r}) - 1)\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi \\ &= \int_{\mathbb{R}^Q_\xi \setminus \{0\}} a(\frac{\varepsilon\xi}{\varepsilon|\xi|})\chi(\varepsilon r\xi)\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi + \\ &\quad \int_{\mathbb{R}^Q_\xi \setminus \{0\}} \chi(\tfrac{\xi}{\eta})a(\tfrac{\varepsilon\xi}{\varepsilon|\xi|})(1 - \chi(\varepsilon r\xi))\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi + \\ &\quad \int_{\mathbb{R}^Q_\xi \setminus \{0\}} (1 - \chi(\tfrac{\xi}{\eta}))a(\tfrac{\varepsilon\xi}{\varepsilon|\xi|})(1 - \chi(\varepsilon r\xi))\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi + \\ &\quad \int_{\mathbb{R}^Q_\xi \setminus \{0\}} a(\frac{\varepsilon\xi}{\varepsilon|\xi|})(\chi(\frac{\xi}{r}) - 1)\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi. \end{aligned}$$

Since v^ε is ε–oscillatory,

$$\limsup_{r\searrow 0}\limsup_{\varepsilon}\int_{\mathbb{R}^Q_\xi}\chi(\varepsilon r\xi)|\widehat{\varphi v^\varepsilon}|^2(\xi)d\xi=0,$$

so that the first term in the right hand side of the last equality in (3.1) tends to 0 as ε then r tend to 0.

As $\varepsilon\searrow 0$, the second term in the right hand–side of the last equality in (3.1) tends to $<\varphi^2(y)m(y,\xi),(1-\chi(r\xi))a(\frac{\xi}{|\xi|})\chi(\frac{\xi}{\eta})>$. Thus, because by assumption $m(\mathbb{R}^Q\times\{0\})=0$, the dominated convergence theorem yields

$$\begin{aligned}&\lim\nolimits_{\eta\searrow 0}\lim\nolimits_{r\searrow 0}\lim_{\varepsilon}\int_{\mathbb{R}^Q_\xi}\chi(\frac{\xi}{\eta})a(\frac{\varepsilon\xi}{\varepsilon|\xi|})(1-\chi(\varepsilon r\xi))\widehat{\varphi v^\varepsilon}(\xi)\overline{\widehat{\varphi v^\varepsilon}}(\xi)d\xi=\\ &\lim_{\eta\searrow 0}<\varphi^2(y)m(y,\xi),a(\frac{\xi}{|\xi|})\chi(\frac{\xi}{\eta})>=<\varphi^2(y)m(y,\xi),a(\frac{\xi}{|\xi|})\chi_{\{\xi\neq 0\}}>=\\ &<\varphi^2(y)m(y,\xi),a(\frac{\xi}{|\xi|})>.\end{aligned}\tag{3.2}$$

The third and fourth terms in that equality can be bounded from above by

$$C\int_{\mathbb{R}^Q_\xi\setminus\{0\}}|\chi(\frac{\xi}{\eta})-1|\ (\text{resp. }|\chi(\frac{\xi}{r})-1|)|\widehat{\varphi v^\varepsilon}|^2(\xi)d\xi.$$

If we denote by $\hat{R}_\varphi$ the measure limit of (a subsequence of) $|\widehat{\varphi v^\varepsilon}|^2(\xi)$, we obtain

$$\limsup_{r\searrow 0}\lim_{\varepsilon}C\int_{\mathbb{R}^Q_\xi\setminus\{0\}}|\chi(\frac{\xi}{r})-1||\widehat{\varphi v^\varepsilon}|^2(\xi)d\xi\le\int_{\mathbb{R}^Q_\xi\setminus\{0\}}\chi_{\{\xi=0\}}d\hat{R}_\varphi(\xi)=0.$$

Collecting the various limits above, we conclude that the limit of the left hand–side of (3.1) is that computed in (3.2), that is

$$\lim_{\varepsilon}<A\varphi v^\varepsilon,\varphi v^\varepsilon>=<\varphi^2(y)m(y,\xi),a(\frac{\xi}{|\xi|})>.\tag{3.3}$$

If the principal symbol of A is of the general form $a(y,\xi)\in\mathcal{C}^\infty_0(\mathbb{R}^Q_y\times S^{Q-1}_\xi)$, an approximation of $\sup\{a,0\}$ and $\sup\{-a,0\}$ in $\mathcal{C}^0_0(\mathbb{R}^Q_y\times S^{Q-1}_\xi)$ by symbols of the form $\varphi^2(y)a(\xi)$ implies that (3.3) still holds for such symbols, that is that v^ε admits a H–measure μ given by

$$<\mu,\sigma^0(A)>=<m(y,\xi),a(y,\frac{\xi}{|\xi|})>,$$

hence the second part of item ii.

The proof of iii. is implicit in the proof of the second part of item ii. above. ¶

Remark 3.5. Note that in the course of deriving the first part of the previous lemma we have actually shown that the measure–limit M of $(v^\varepsilon)^2$ satisfies

$$M\ge\int_{\mathbb{R}^Q_\xi}dm(y,\xi).$$

Remark 3.6. Localization and transport results of the type obtained in the previous section can be analogously derived within the framework of semi–classical measures (see [6], (3.12) and Proposition 3.5*)*.

3.2. **An improvement of** P.L. LIONS **' compactness result.** As mentioned in the introduction, the microlocal tools of H and semi–classical measures are fundamental in P. GÉRARD's following improvement (see [5], Corollary 5) of P.L. LIONS' compactness Theorem 1.1 which we now state and prove in a three–dimensional setting for simplicity sake.

Theorem 3.7. *Let* $w^n \rightharpoonup 0$, *weakly in* $H^1(\mathbb{R}^3)$.

Assume that $\operatorname{grad} w^n$ *admits a H–measure* $(\xi \otimes \xi)\mu$ *such that*

$$\mu(y,\xi) \perp \delta(y-z) \otimes d\sigma(\xi),\ \forall z \in \mathbb{R}^3,$$

where σ *is the superficial* LEBESGUE *measure on* S^2_ξ.

Then,

$$w^n \to 0,\ \textit{strongly in } L^6_{loc}(\mathbb{R}^3).$$

We offer a complete proof of the theorem; the proof provides an elegant application of the interplay between semi–classical and H–measures displayed in Lemma 3.4, together with a rather striking use of a SOBOLEV–BESOV type imbedding.

Proof. The following imbedding estimate is pivotal in the subsequent proof:

Lemma 3.8. *Define, for any* $1 < p < \infty$ *the space* B_p *of all* $f \in L^2(\mathbb{R}^3)$ *such that*

$$\|f\|_{B_p} := \sup_{k\in\mathbb{Z}} \|\widehat{\Delta_k f}\|_{L^p} < \infty$$

with $\widehat{\Delta_k f} := \chi_{\{2^k \le |\xi| < 2^{k+1}\}}(\xi)\hat{f}(\xi)$.

Then, for any $1 < p \le \frac{6}{5}$, *there exists* $0 < s \le 1$ *and a constant* C *such that, for any* $u \in H^s(\mathbb{R}^3) \cap B_p$,

$$\|u\|_{L^6} \le C\|D^s u\|^{1/3}_{L^2} \|u\|^{2/3}_{B_p}.$$

The actual value of s *is* $\frac{6(p-1)}{p}$.

Proof of Lemma. Define, for any $A > 0$, $\widehat{u_{>A}}(resp.\ \widehat{u_{\le A}}) := \chi_{\{|\xi| > (resp.\ \le) A\}}\hat{u}(\xi)$ and write $u = u_{\le A} + u_{>A}$. Then,

$$\|u_{\le A}\|_{L^\infty} \le C\|\widehat{u_{\le A}}\|_{L^1} \le C \sum_{k(\in\mathbb{Z})\le k(A)} \|\chi_{\{2^k\le|\xi|<2^{k+1}\}}\hat{u}\|_{L^1},$$

where $2^{k(A)} \le A < 2^{k(A)+1}$. Now, by Hölder's inequality, for $p > 1$,

$$\|\chi_{\{2^k\le|\xi|<2^{k+1}\}}\hat{u}\|_{L^1} \le C2^{3k(\frac{p-1}{p})}\Big(\int_{2^k\le|\xi|\le 2^{k+1}} |\hat{u}|^p d\xi\Big)^{1/p}.$$

thus,

$$\|u_{\le A}\|_{L^\infty} \le CA^{3(\frac{p-1}{p})}\|u\|_{B_p}.$$

Now,

$$\|u\|^6_{L^6} = 6\int_0^\infty \lambda^5 \mathcal{L}^3(\{|u| > \lambda\})d\lambda.$$

Set, for any $\lambda > 0$, $A(\lambda) := \Big(\dfrac{\lambda}{2C\|u\|_{B_p}}\Big)^{\frac{p}{3(p-1)}}$, so that $\|u_{\le A(\lambda)}\|_{L^\infty} \le \frac{\lambda}{2}$. Then,

$$\mathcal{L}^3(\{|u| > \lambda\}) \le \mathcal{L}^3(\{|u_{>A(\lambda)}| > \lambda/2\}) \le \frac{4}{\lambda^2}\||u_{>A(\lambda)}\|^2_{L^2} = \frac{4}{\lambda^2}\|\widehat{u_{>A(\lambda)}}\|^2_{L^2}.$$

Consequently,

$$\|u\|^6_{L^6} \le 24\int_0^\infty \lambda^3\|\widehat{u_{>A(\lambda)}}\|^2_{L^2}d\lambda,$$

or still, invoking Fubini's theorem,

$$\|u\|_{L^6}^6 \leq 24 \int_0^\infty \Big(\int_0^{4C|\xi|^{\frac{3(p-1)}{p}} \|u\|_{B_p}} \lambda^3 d\lambda \Big) |\hat{u}|^2(\xi) d\xi,$$

which finally implies that

$$\|u\|_{L^6}^6 \leq C \|u\|_{B_p}^4 \|D^{\frac{6(p-1)}{p}} u\|_{L^2}^2;$$

hence the lemma. ¶

We now address the proof of Theorem 3.7. To this effect, we define

$$\rho(y,\xi) := (y, \frac{\xi}{|\xi|})$$

and apply item iii. of Lemma 3.4 to the current setting for any subsequence $\{n'\}$ of $\{n\}$ such that $\operatorname{grad} w^{n'}$ admits a semi–classical measure $(\xi \otimes \xi)m$ and also such that $|\operatorname{grad} w^{n'}|^2$ admits a measure limit M. Thus, because by assumption

$$\mu(y,\xi) \perp \delta(y-z) \otimes d\sigma(\xi),\ z \in \mathbb{R}^3,$$

a fortiori

$$\rho(\chi_{\{\xi \neq 0\}} m(y,\xi)) \perp \rho(\chi_{\{\xi \neq 0\}} \delta(y-z) \otimes \mathcal{L}^3(\xi)),\ z \in \mathbb{R}^3,$$

or still, since $\mathcal{L}^3(\xi) \perp \delta_{\xi=0}$,

$$m(y,\xi) \perp \delta(y-z) \otimes \mathcal{L}^3(\xi). \tag{3.4}$$

Consider now, for $\theta \in \mathcal{C}_0^\infty(B(0,1))$ with $0 \leq \theta(y) \leq 1,\ \theta(0) = 1$,

$$z_\delta^n(y) := \theta(\frac{y-z}{\delta}) w^n(y).$$

Now,

$$\limsup_n \|z_\delta^n\|_{B_p} = \limsup_n \sup_k \|\widehat{\Delta_k z_\delta^n}\|_{L^p}$$

and, at fixed δ, $\widehat{z_\delta^n} \overset{L^2_{loc}(\mathbb{R}^3)}{\rightarrow} 0$ because $w^n \overset{L^2(\mathbb{R}^3)}{\rightharpoonup} 0$ and $\theta(\frac{y-z}{\delta})$ has compact support. Thus, each term $\|\widehat{\Delta_k z_\delta^n}\|_{L^p},\ p \leq 2$ goes to 0 with n and

$$\limsup_n \|z_\delta^n\|_{B_p} = \limsup_{n \nearrow \infty} \|\widehat{\Delta_{k_n} z_\delta^n}\|_{L^p}, \tag{3.5}$$

$\{k_n\}$ being a sequence that tends to ∞ with n.

We choose $p = \frac{6}{5}$.

Set $\varepsilon_n := 2^{-k_n}$ and consider a subsequence $\{n'\} \subset \{n\}$ (in all rigor, a subsequence that depends upon $\{\varepsilon_n\}$) such that $\operatorname{grad} w^{n'}$ admits a semi–classical measure $(\xi \otimes \xi)m$ (with associated k'_n). Then, applying Holder's inequality,

$$\begin{aligned} \|\widehat{\Delta_{k'_n} z_\delta^{n'}}\|_{L^{\frac{6}{5}}} &\leq C(\varepsilon_{n'})^{-1} \|\widehat{\Delta_{k'_n} z_\delta^{n'}}\|_{L^2} \\ &\leq C(\varepsilon'_n)^{-1} \Big(\int_{\mathbb{R}^3_\xi} \chi_{\{1 \leq \varepsilon_{n'}|\xi| < 2\}}(\xi) (\varepsilon_{n'})^2 |\xi|^2 |\widehat{z_\delta^{n'}}|^2 d\xi \Big)^{1/2}, \end{aligned} \tag{3.6}$$

or still,

$$\|\widehat{\Delta_{k'_n} z_\delta^{n'}}\|_{L^{\frac{6}{5}}} \leq C \Big(\int_{\mathbb{R}^3_\xi} \chi_{\{1 \leq \varepsilon_{n'}|\xi| < 2\}}(\xi) |\xi|^2 |\widehat{z_\delta^{n'}}|^2 d\xi \Big)^{1/2}. \tag{3.7}$$

Note that $\widehat{\operatorname{grad} z_\delta^{n'}} = (\theta(\frac{\cdot - z}{\delta})\widehat{\operatorname{grad} w^{n'}}(.))$+(a term that converges strongly to 0 in $L^2(\mathbb{R}^3)$), so that (3.7) also reads as

$$\|\widehat{\Delta_{k'_n} z_\delta^{n'}}\|_{L^{\frac{6}{5}}} \le C\Big(\int_{\mathbb{R}^3_\xi} \chi_{\{1\le \varepsilon_{n'}|\xi|<2\}}(\xi)|(\theta(\frac{\cdot - z}{\delta})\widehat{\operatorname{grad} w^{n'}}(.))|^2 d\xi\Big)^{1/2} + \omega(n), \tag{3.8}$$

with $\omega(n) \overset{n\nearrow\infty}{\to} 0$. Now, in the spirit of Remark 2.7, the previous inequality can be rewritten as

$$\|\widehat{\Delta_{k'_n} z_\delta^{n'}}\|_{L^{\frac{6}{5}}} \le C < \Theta_z(\varepsilon_{n'} D)\operatorname{grad} w^{n'}, \operatorname{grad} w^{n'} >^{1/2}, \tag{3.9}$$

where Θ_z is the pseudo–differential operator with symbol

$$\sigma(\Theta_z) := \theta(\frac{y-z}{\delta})\chi_{\{1\le|\xi|<2\}}(\xi).$$

At the expense of a smoothing of the symbol $\sigma(\Theta_z)$, we are in a position to compute the limit of the right hand–side of (3.9) in terms of the semi–classical limit m associated to $\operatorname{grad} w^{n'}$. Consequently, in view of (3.8), (3.5) becomes

$$\limsup_n \|\widehat{\Delta_{k_n} z_\delta^{n}}\|_{L^{\frac{6}{5}}} \le C < |\xi|^2 m(y,\xi), \theta(\frac{y-z}{\delta})\chi_{\{1\le|\xi|<2\}}(\xi) >^{1/2}, \tag{3.10}$$

which we rewrite as

$$\limsup_n \|\widehat{\Delta_{k_n} z_\delta^{n}}\|_{L^{\frac{6}{5}}} \le C\Big\{\int_{\mathbb{R}^3_\xi} d\nu^\delta(m)(\xi)\Big\}^{1/2} \tag{3.11}$$

with

$$\nu^\delta(m)(\xi) := |\xi|^2\chi_{\{1\le|\xi|<2\}}(\xi)\int_{\mathbb{R}^3_y}\theta(\frac{y-z}{\delta})dm(y,\xi).$$

Using the Radon –Nikodym theorem, we decompose $\nu^\delta(m)$ as

$$\nu^\delta(m) = \nu^\delta_{\mathcal{L}}(m)\mathcal{L}^3 + \nu^\delta_s(m),$$

where $\nu^\delta_{\mathcal{L}}(m)$ is the density of the Lebesgue–absolutely continuous part of $\nu^\delta(m)$ and $\nu^\delta_s(m)$ is the Lebesgue–singular part of that measure. We then consider, for any $\beta > 0$, $0 \le \zeta_{\mathcal{L}}, \zeta_s \le 1 \in \mathcal{C}^0(\mathbb{R}^3_\xi)$ such that:

. *supp* $\nu^\delta_{\mathcal{L}}(m) \subset \{\zeta_{\mathcal{L}} \equiv 1\}$;
. $\nu^\delta_s(m)\Big(\textit{supp}\ \zeta_{\mathcal{L}}\Big) < \beta$;
. *supp* $\nu^\delta_s(m) \subset \{\zeta_s \equiv 1\}$;
. $\int_{\mathbb{R}^3_\xi} \zeta_s d\xi < \beta$;
. $\zeta_{\mathcal{L}} + \zeta_s \equiv 1$.

Then, rewriting $\|\widehat{\Delta_{k_{n'}} z_\delta^{n'}}\|_{L^{\frac{6}{5}}}$ as $\|\widehat{\Delta_{k_{n'}} z_\delta^{n'}}(\xi)(\zeta_{\mathcal{L}}+\zeta_s)(\varepsilon^{n'}\xi)\|_{L^{\frac{6}{5}}}$, an argument identical to that of (3.6)–(3.11) would lead to

$$\begin{aligned}\limsup_n \|\widehat{\Delta_{k_n} z_\delta^{n}}\|_{L^{\frac{6}{5}}} \le\ & C\Big\{\Big(\textstyle\int_{\mathbb{R}^3_\xi}\zeta_{\mathcal{L}} d\nu^\delta(m)(\xi)\Big)^{\frac{3}{5}}\Big(\int_{\{1\le|\xi|<2\}}\zeta_{\mathcal{L}} d\xi\Big)^{\frac{2}{5}} \\ & + \Big(\textstyle\int_{\mathbb{R}^3_\xi}\zeta_s d\nu^\delta(m)(\xi)\Big)^{\frac{3}{5}}\Big(\int_{\{1\le|\xi|<2\}}\zeta_s d\xi\Big)^{\frac{2}{5}}\Big\}^{\frac{5}{6}},\end{aligned}$$

or still, in view of the properties of $\zeta_{\mathcal{L}}, \zeta_s$,

$$\begin{aligned}\limsup_n \|\widehat{\Delta_{k_n} z^n_\delta}\|_{L^{\frac{6}{5}}} &\leq C\Big\{\Big(\int_{\mathbb{R}^3_\xi} d\nu^\delta_{\mathcal{L}}(m)(\xi) + C\beta\Big)^{\frac{3}{5}} \\ &+\Big(\int_{\mathbb{R}^3_\xi} d\nu^\delta_s(m)(\xi) + C\beta\Big)^{\frac{3}{5}} \beta^{\frac{2}{5}}\Big\}^{\frac{5}{6}}.\end{aligned}$$

Letting $\beta \searrow 0$, we obtain the following refinement of (3.11):

$$\limsup_n \|\widehat{\Delta_{k_n} z^n_\delta}\|_{L^{\frac{6}{5}}} \leq C\Big\{\int_{\mathbb{R}^3_\xi} d\nu^\delta_{\mathcal{L}}(m)(\xi)\Big\}^{\frac{1}{2}}. \tag{3.12}$$

At this point, we recall (3.4) and apply it to m, which immediately implies that

$$\nu^\delta_{\mathcal{L}}(m) = \nu^\delta_{\mathcal{L}}(m\chi_{\{y\neq z\}}(y)). \tag{3.13}$$

Note that the above would not be true for $\nu^\delta(m)$ itself; just take, for ξ_0 such that $|\xi_0| = 3/2$, $m = \delta_{y\neq z} \otimes \delta_{\xi=\xi_0} \perp \delta(y-z) \otimes \mathcal{L}^3(\xi)$ (the reader can easily construct a sequence which admits such a measure as semi–classical measure), yet the associated $\nu^\delta(m)$ is $|\xi_0|^2\delta_{\xi=\xi_0}$, while $\nu^\delta(m\chi_{\{y\neq z\}}(y)) = 0$.

In any case, thanks to (3.13), we rewrite (3.12) as

$$\limsup_n \|\widehat{\Delta_{k_n} z^n_\delta}\|_{L^{\frac{6}{5}}} \leq C\Big\{\int_{\mathbb{R}^3_\xi} d\nu^\delta_{\mathcal{L}}(m\chi_{\{y\neq z\}})(\xi)\Big\}^{1/2},$$

and, *a fortiori*,

$$\limsup_n \|\widehat{\Delta_{k_n} z^n_\delta}\|_{L^{\frac{6}{5}}} \leq C < |\xi|^2\chi_{\{y\neq z\}} m(y,\xi), \theta(\frac{y-z}{\delta})\chi_{\{1\leq|\xi|<2\}}(\xi) >^{1/2}.$$

Appealing to Remark 3.5 immediately implies that the right hand–side of the previous inequality can be bounded from above by

$$C\int_{\mathbb{R}^3_y} \chi_{\{y\neq z\}}|\theta(\frac{y-z}{\delta})|^2 dM(y),$$

where we recall that M is the measure limit of $|\operatorname{grad} w^{n'}|^2$. Since $0 \leq \theta \leq 1$, we thus get that

$$\limsup_n \|\widehat{\Delta_{k_n} z^n_\delta}\|_{L^{\frac{6}{5}}} \leq CM(B(z,\delta) \setminus \{\delta\}).$$

Letting $\delta \searrow 0$ finally yields

$$\limsup_\delta \limsup_n \|\widehat{\Delta_{k_n} z^n_\delta}\|_{L^{\frac{6}{5}}} = 0,$$

hence, in view of (3.5),

$$\limsup_\delta \limsup_n \|z^n_\delta\|_{B_{\frac{6}{5}}} = 0. \tag{3.14}$$

We now apply Lemma 3.8 and obtain,

$$\limsup_\delta \limsup_n \|z^n_\delta\|_{L^6} \leq C\, 0 \times \limsup_\delta \limsup_n \|\operatorname{grad} z^n_\delta\|^{1/3}_{L^2}. \tag{3.15}$$

But,

$$\limsup_n \|\operatorname{grad} z^n_\delta\|^{1/3}_{L^2} \leq \limsup_n (\|\operatorname{grad} w^n\|_{L^2} + C/\delta\|w^n\|_{L^2(B(z,\delta))})^{1/3}_{L^2}$$

or still, by virtue of Rellich's theorem applied to w^n on $L^2(B(z,\delta))$,

$$\limsup_n \|\operatorname{grad} z^n_\delta\|^{1/3}_{L^2} \leq \limsup_n \|\operatorname{grad} w^n\|^{1/3}_{L^2}.$$

Thus, (3.15) actually reads as

$$\limsup_{\delta} \limsup_{n} \|z^n_\delta\|_{L^6} = 0,$$

that is, if R denotes the measure limit of (a subsequence of) $(w^n)^6$,

$$\limsup_{\delta} \int_{\mathbb{R}^3_y} |\theta(\frac{y-z}{\delta})|^6 dR(y) = R(\{z\}) = 0.$$

Since z is arbitrary, we conclude that R does not charge atoms; application of Theorem 1.1 then yields the desired result. ¶

3.3. **A compactness result for the homogeneous wave equation.** In this subsection, the results of the Particular case 2.15 and of Theorem 3.7 coalesce to produce a pointwise in time compactness result for the solution v^ε to the homogeneous wave equation.

The following theorem is due to P. GÉRARD (see [5], Theorem 9).

Theorem 3.9. *Let $V^\varepsilon_0 \rightharpoonup 0$, in $H^1(\mathbb{R}^N)$, $Z^\varepsilon_0 \rightharpoonup 0$, in $L^2(\mathbb{R}^N)$ with supp V^ε_0 and supp $Z^\varepsilon_0 \subset K$ (compact of $\mathbb{R}^3$). Consider the homogeneous wave equation*

$$\frac{\partial^2 v^\varepsilon}{\partial t^2} - \operatorname{div}(\operatorname{grad} v^\varepsilon) = 0$$

with initial conditions

$$v^\varepsilon(0) = u^\varepsilon_0 - u_0 := V^\varepsilon_0$$
$$\frac{\partial v^\varepsilon}{\partial t}(0) = v^\varepsilon_0 - v_0 := Z^\varepsilon_0.$$

Then the complement in $\mathbb{R}^+$ of the set $\{t \geq 0 : \|v^\varepsilon(t)\|_{L^6(\mathbb{R}^3)} \to 0\}$ is at most countable.

Proof. Because of the finite speed of propagation, *supp* $v^\varepsilon(t)$ lies in a compact subset of $\mathbb{R}^3$ for any $t \geq 0$. Assume that for some $t \geq 0$,

$$\limsup_{\varepsilon} \|v^\varepsilon(t)\|_{L^6(\mathbb{R}^3)} > 0. \tag{3.16}$$

Then, according to Theorem 3.7, a subsequence of $\{\operatorname{grad} w^\varepsilon(t)\}$ — possibly depending on t — admits a H–measure $\xi \otimes \xi \kappa_t$ and a point $x_t \in \mathbb{R}^3$ such that κ_t and $\delta(x - x_t) \otimes d\sigma(\eta)$ are not mutually singular.

Recall the Particular case 2.15. Since, obviously, $\kappa_t \leq \nu_t$, we *a fortiori* have that ν_t and $\delta(x-x_t) \otimes d\sigma(\eta)$ are not mutually singular, or, in other words, appealing to (2.17) that $1/4\Big\{\tilde{\nu}^+(x-\eta t, \eta) + \tilde{\nu}^-(x+\eta t, \eta)\Big\}$ and $\delta(x-x_t) \otimes d\sigma(\eta)$ are not mutually singular. Now, this means that either $\tilde{\nu}^+(x, \eta)$ and $\delta(x + \eta t - x_t) \otimes d\sigma(\eta)$ are not mutually singular, or that $\tilde{\nu}^-(x, \eta)$ and $\delta(x - \eta t - x_t) \otimes d\sigma(\eta)$ are not mutually singular.

Consequently, either $\tilde{\nu}^+(x, \eta)$ or $\tilde{\nu}^-(x, \eta)$ are not singular with respect to the superficial LEBESGUE measure on a sphere of center x_t and radius t. But, such superficial measures are pairwise mutually singular for distinct t's, whether the x_t's are distinct or not. Since a RADON measure cannot have a non–zero RADON – NYKODIM derivative with respect to more than a countable set of mutually singular measures, there cannot be more than a countable set of times t's for which (3.16) holds. ¶

Remark 3.10. The same result holds for the solution to the heterogeneous wave equation (1.3, 1.4) as could be derived at the expense of a revisiting of the Particular case 2.15 in the more general context of arbitrary ρ's and k's.

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