Quasistatic brittle fracture seen as an energy minimizing movement

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This article presents an overview of the current state of the variational theory of quasistatic brittle fracture. It is shown that that theory, while departing only slightly from the classical theory of Griffith, alleviates many of the obstacles usually associated with quasistatic crack growth. The underlying mathematics are outlined, the various available results sketched, and the drawbacks discussed. Two numerical computations, well beyond the scope of the classical theory, are presented.

1 Introduction

The basic concepts of brittle fracture, established by A. GRIFFITH in the 1920's [28] and refined by various brilliant followers in his footstep, are very much entrenched in the collective psyche of the contemporary mechanician. Yet, the theory has been and continues to be plagued by major defects, most notably its inability to predict crack initiation, to follow the crack path, or to decide if the crack evolution is "stable".

As a result, a host of ad hoc remedies have been proposed, with unequal success. But, more often than not, those are viewed as additional ingredients, which should be appealed to whenever the mechanical environment becomes hostile. It is at times as if the crack had to carry its own toolbox.

This, in our opinion, runs contrary to the seminal tenet of continuum mechanics: scarcity of ingredients, abundance of results. In an effort to abide by this principle, J.J. MARIGO and I have proposed an economical theory that departs as little as possible from that elaborated by A. GRIFFITH. In doing so, we were inspired on the one hand by our own work on damage [23], but also very much guided by the impressive work on image segmentation initiated by D. MUMFORD & J. SHAH [32] and beautifully formalized and expanded by E. DE GIORGI and the school he created. E. DE GIORGI created the tools without which the variational formulation could not exist.

The model we propose is variational and only applies to quasistatic evolution. It does remove many of the obstacles associated to the classical theory; it is indeed very close in

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spirit to the original model proposed by A. GRIFFITH. But, of course, there is a price to pay: because it is based on global minimization, it poorly handles soft devices, as will be evidenced in the fifth section. This can be somewhat remedied, both within the model, and by weakening the minimization process, as will be further detailed in that section. But there too, there is a mathematical price to pay because local minimality (meta-stability) is a delicate topic, for which few tools are available.

Thus, I do not contend that we have the final word on quasistatic crack growth, but merely that a bit of flexibility goes a long way in what we deem to be a reasonable path that runs right along the field of classical fracture mechanics.

In these notes, I outline the model – increasingly referred to as the variational theory of brittle fracture – from its inception to its current status. To this effect, a first section is devoted to a review the classical model of brittle fracture, as formulated throughout the last eighty years by A. GRIFFITH [28], but also by e.g. G.I. BARENBLATT [7], G.R. IRWIN, [30] ... The reader might however encounter some difficulties, should she attempt to locate a similar presentation in the existing literature. A good reference in that direction is the book by H.D. BUI [14]. I then show how the time dependent model that J.J. MARIGO and I have proposed in [22] departs from that theory.

A second section discusses the time-discretization of that model, a natural step from the standpoint of any computations. I then indicate how and when the time continuous model can be seen as a limit of that discretization as the time step becomes smaller and smaller. I do not present complete proofs, because this is not intended as a mathematical paper, but rather try to point out the kind of obstacles one is confronted with when attempting a rigorous analysis. The provided references should satisfy the mathematically inclined reader.

A third section presents a near exhaustive compendium of available theoretical results, which can still be done at this stage, but may prove impossible, should the variational theory become more popular. A host of known facts or folklore can be recovered, clearly advocating the relevance of the model.

The variational theory is far from picture perfect, and the fourth section addresses the main unsolved issue and suggests possible remedies. The methods are still at an embryonic stage.

Finally, the last section develops two numerical computations. The first was presented in [12] and the second in [23]. The first computation follows a cack from its initiation to the ultimate failure of the sample.The variety of pathologies evidenced throughout the growth of the crack positions the first example far beyond the reach of the classical theory. Yet, the variational theory yields results that are, at the least, in qualitative agreement with the experiment. The second computation is a static one, yet it demonstrates that the variational framework can capture the occurrence of multiple cracks without importing any new ingredients into the theory.

2 Classical brittle fracture versus variational brittle fracture

As is customary in fracture mechanics, I develop the model in a two-dimensional setting, then generalize it to the spatial case.

Consider a linearly elastic material occupying a domain Ω with elasticity A, so that its constitutive behavior is described by the following stress-strain law:

$$\sigma(x) = Ae(u)(x), \text{ with } e(u) := 1/2(\nabla u + \nabla u^t).$$

The possible defects in that material at time t are assumed to be a crack $\gamma(t)$, which follows a preset crack path $\Gamma \supset \gamma(t)$. The preset path may go to the boundary of Ω, so as to allow debonding; thus $\Gamma \subset \overline{\Omega}$. I further assume that $\gamma(t)$ is connected, so that it can be seen as a curve parameterized by its arclength $l(t)$, that is that the knowledge of the crack length amounts to that of the crack itself. Equivalently, the knowledge of the crack tip is sufficient to describe the whole crack site. In other words, $\gamma(t)$ is in fact a function of $l(t)$,

$$\gamma(t) := \Gamma(l(t)).$$

Throughout this section, I assume that the only "loads" are time-dependent boundary displacements $U(t,x)$ on a part $\partial\Omega_d$ of $\partial\Omega$; one speaks of hard devices. This seemingly innocuous assumption is in fact fundamental, and the consideration of body or surface loads remains problematic in the proposed framework as noted at the onset of Section 5 below; see however the comments thereafter, which suggest a possible resolution of the dilemma posed by force loads.

It is actually simpler to view the field $U(t)$ as the trace on $\partial\Omega_d$ of a field, still denoted by $U(t)$, defined this time on all of $\mathbb{R}^2$. We further assume that $U(t)$ possesses all necessary smoothness in t.

Now, for a given crack length l (at time t), the displacement field $u(l,t)$ should be in static equilibrium with the loads at that time (this is the meaning of quasistatic). As is classical in linearized elasticity, this can be seen as a minimality statement for the elastic energy among kinematically admissible displacement fields. Thus, $u(l,t)$ minimizes

$$E(v,l) := \int_{\Omega\setminus\Gamma(l)} 1/2 Ae(v)\cdot e(v)dx, \tag{2.1}$$

among all v's such that $v = U(t)$ on $\partial\Omega_d \setminus \Gamma(l)$.

Of course, this is of no value as long as the actual crack length $l(t)$ is not determined. Assuming that E is smooth enough in l, one computes the energy release rate

$$G(l,t) = -\frac{\partial E}{\partial l}(u(l,t),l).$$

The evolution law for $l(t)$ suggested by A. GRIFFITH is as follows:

i. $\dfrac{dl}{dt}(t) \geq 0$ (the crack can only grow);

ii. $G(l(t),t) \leq k$ (k is the critical energy release rate);

iii. $\dfrac{dl}{dt}(t)(G(l(t),t) - k) = 0$ (the crack will not grow, unless the energy release rate is critical).

From now on, if $l(t)$ satisfies the evolution law, I set $u(t) := u(l(t),t)$, the minimizer for (2.1) at $l = l(t)$.

At this point, all classical ingredients of Griffith's theory of brittle fracture have been introduced. I will comment at a later stage the introduction proposed by G.I. BARENBLATT of a cohesive surface energy. In any case, from now onward, by *Griffith's model*, I mean quasistatic equilibrium (2.1), together with items $i - iii$ of the evolution law above.

Remark 2.1 As is well known, Griffith's criterion, in spite of its longevity, is plagued by at least three major defects listed and commented below.

First, the crack path must be preset, since a single equation cannot possibly predict the crack tip path.

Then, crack initiation is impossible if e.g. $l(0) = \dfrac{\partial E}{\partial l}(\cdot, 0) = 0$. For example, a pre–cracked half–plane $x_1 > 0$, with a pre–crack of length l orthogonal to the boundary at $x_2 = 0$ is considered; at $x_2 = \pm\infty$, a mode I load , that is an increasing vertical normal stress, of opposite sign on each side of $x_2 = 0$ and of magnitude t is applied. Then, it can be shown that the crack will increase when t is of the order of $1/\sqrt{l}$ (see [30]), so that, as $l \searrow 0^+$, the magnitude of the load required to further advance the crack becomes infinite. In all fairness, this second defect is viewed as an asset by some who argue in favor of the introduction of an initial defect; our own bias is contrary because it is our opinion that one should first explore all ramifications of a model before adding extraneous complements to that model.

Last, smooth crack growth is impossible if e.g. $G(l, t) \nearrow_t$ at $l = l(t)$, a situation which can arise in many examples (see e.g. [22], Section 4).

Various remedies have been proposed on a case by case basis, but there is, to my knowledge, no unified way of curing all three defects at once, besides that proposed hereafter.

I submit that Griffith's model can be rewritten in a more palatable manner as detailed below. Define

$$\mathcal{E}(v, l) := E(v, l) + kl.$$

First, quasistatic equilibrium, together with item ii of the evolution law is easily seen, through elementary variations, to be equivalent to

$$D\mathcal{E}(u(t), l(t)).\begin{pmatrix} v - u(t) \\ l - l(t) \end{pmatrix} \geq 0,\ v = U(t) \text{ on } \partial\Omega_d \setminus \Gamma(l), \quad l \geq l(t), \tag{2.2}$$

where $D\mathcal{E}$ is the Fréchet differential – assumed to exist – of $\mathcal{E}(v, l)$ with respect to v, l. In turn, item iii of the evolution law reads as

$$\frac{d}{dt}\left[\mathcal{E}(u(t), l(t))\right] = \int_{\Omega\setminus\Gamma(t)} Ae(u(t)).e(\frac{\partial U}{\partial t}(t))\, dx. \tag{2.3}$$

(Recall that $U(t)$ has been defined on all of $\mathbb{R}^2$.) The balance law (2.3) may be viewed as equivalent to Clausius-Duhem inequality, provided that the dissipation at t is identified with $kl(t)$.

In conclusion, our equivalent definition of *Griffith's model* is as follows: *any evolution* $u(t), l(t)$ *that satisfies (2.2),(2.3), together with the irreversible growth condition on the length of the crack* (item i in the evolution law).

Now, note that (2.2) is a first order optimality condition for $u(t), l(t)$ to be a one-sided local minimizer for $\mathcal{E}(\cdot, \cdot)$ among all $l \geq l(t)$, $v = U(t)$ on $\Omega \setminus \Gamma(l)$. A first and slight departure from *Griffith's model* would be to assume local minimality in lieu of (2.2). Unfortunately, such a modified stability criterion raises several issues: firstly, the notion of locality is distance dependent and there is no guiding principle that dictates the proper choice of that distance; then, local minimizers of non–convex functionals are poorly understood in dimensions greater than 1. See however [17] for a one-dimensional study of local minimizers and also Subsection 5 below. Consequently, we arbitrarily strengthen the postulate by requiring that local minimizers actually be *global* minimizers. In doing so, we can significantly enlarge the set of admissible cracks. In fact, we can *drop the preset path assumption* altogether, since global minimization will act as a drastic selection criterion. We define, for all compact sets Γ with finite Hausdorff measure , the total energy

$$\mathcal{E}(v,\Gamma) := \int_{\Omega\setminus\Gamma} 1/2 Ae(v)\cdot e(v)dx + k\mathcal{H}^1(\Gamma\setminus\partial\Omega_d^c);$$

note that there is no surface energy paid for the part of the crack that lives on the traction free part of the boundary $\partial\Omega_d^c := \partial\Omega\setminus\partial\Omega_d$, as it should be. We are thus led to the following

Variational Evolution 2.2 *At each time* t*,*

i. One-sided minimality: $(u(t),\Gamma(t))$ *minimizes* $\mathcal{E}(v,\Gamma)$ *among all admissible* (v,Γ)*, that is*

$$\begin{cases} v = U(t) \text{ on } \partial\Omega_d\setminus\Gamma \\ \Gamma\supset\Gamma(t) \end{cases}$$

ii. Non-dissipativity: $\mathcal{E}(u(t),\Gamma(t))$ *is absolutely continuous in* t *and satisfies*

$$\frac{d}{dt}\mathcal{E}((u(t),\Gamma(t)) = \int_{\Omega\setminus\Gamma(t)} Ae(u(t)).e(\frac{\partial U}{\partial t}(t))\,dx.$$

Let me stress again that the variational evolution above – referred to as the *strong variational evolution* for the remainder of the paper – only differs from Griffith's model in that the necessary first order optimality condition for local minimality (2.2) is replaced by a statement of global minimality. This makes all the difference!

Our goal is two-fold: a thorough investigation of the consequences of such an evolution on the one hand, and an existence result that will guarantee existence on the other.

My focus here is the second point and I merely indicate in the remarks below two consequences of the variational evolution, referring the reader to [22] for a detailed study of the mechanical implications of that evolution.

Remark 2.3 Crack initiation is automatically triggered in finite time, at least for proportional loads, that is for fields $U(t)$ of the form tU, U being a fixed displacement field. Indeed, as long as $\Gamma(t)\equiv\emptyset$,

$$\mathcal{E}(u(t),\emptyset) = t^2 E(u(1),\emptyset) \nearrow_t \infty,$$

since $u(1)\neq 0$, hence $E(u(1),\emptyset)\neq 0$. But cutting away $\partial\Omega_d$ costs at most $k\mathcal{H}^1(\partial\Omega_d)$ in surface energy, while the associated minimal elastic energy is null. Thus, the competitor $0,\partial\Omega_d$ has $k\mathcal{H}^1(\partial\Omega_d)$ as total energy, definitely less than that of $u(t),\emptyset$ that goes to ∞ with t. This is in striking contrast with Griffith's model, as already noted in Remark 2.1 above.

Remark 2.4 Singularities, which play a pivotal role in Griffith's model [30] are also of paramount importance in the strong variational evolution 2.2, as demonstrated in Section 4.4 of [22]. In a two-dimensional setting with an isotropic material of stiffness tensor A, say that, in the neighborhood of a point x_0, the elastic solution is of the form

$$u(x) = r^\alpha v(\theta) + \hat{u}(x),$$

where (r, θ) denotes the polar coordinates with pole x_0, $0 < \alpha < 1$ and $\hat{u} \in W^{2,2}(\Omega)$. The restriction on α ensures the finiteness of the bulk energy and the non $W^{2,2}$-regularity of u, i.e., its singular character, provided that $v \neq 0$. The point x_0 could be a crack tip, a boundary point where a change in boundary condition occurs, or a non-smooth point of the boundary. If $v \equiv 0$, then x_0 is a regular point, which we can equivalently formalize by setting $\alpha = 1$.

We assume henceforth in this remark that the crack can only extend from x_0 with a small crack Γ of length l. It is formally shown in [31] that the bulk energy

$$E(u, \Gamma) := 1/2 \int_{\Omega \setminus \Gamma} Ae(u) \cdot e(u)\, dx$$

expands as

$$E(u, \Gamma) = E(u, \emptyset) - K l^{2\alpha} + o(l^{2\alpha}),$$

where $K > 0$ – except maybe if $v \equiv 0$ – depends on the shape of the crack Γ, on α, but not on l. Taking that expansion for granted and applying an ever increasing load, we obtain the following:

(i) If x_0 exhibits a strong singularity ($\alpha < 1/2$), then the crack growth is progressive, with zero initiation time;

(ii) If x_0 exhibit a weak singularity ($\alpha > 1/2$), then the crack growth is brutal – a crack of finite length appears at a given time – with a nonzero, finite initiation time;

(iii) If the singularity is a $\sqrt{r}$-singularity, then the crack growth has a nonzero initiation time,

(iv) If the point x_0 is regular, then, either there is no crack growth, or the crack growth is brutal with a nonzero, finite initiation time.

As we see, the behavior of possible cracks changes drastically, depending on the strength of the singularity at the point under investigation. In particular, initiation inside a sample is always brutal. For a numerical confirmation of some of the items listed above, the reader is invited to examine the computations presented in Section 6 below.

Note that there is nothing that prevents an extension of the variational evolution to higher dimensions; in dimension N, this merely requires to replace the set of test line-cracks by test surface-cracks, that is sets Γ with $\mathcal{H}^{N-1}(\Gamma) < \infty$. Similarly, we can extend the variational evolution to a non-linear setting, provided static equilibrium is still expressed as a minimization problem. Thus, we propose by analogy a variational evolution in hyperelasticity. Here the quadratic energy is replaced by an energy density W, which is now a function of the gradient of the deformation, still denoted by u. Upon setting

$$\mathcal{E}(v, \Gamma) := \int_{\Omega \setminus \Gamma} W(\nabla u)\, dx + k\mathcal{H}^{N-1}(\Gamma \setminus \partial\Omega_d^c),$$

We obtain the following

Variational Evolution 2.5 *At each time t,*

i. One-sided minimality: $(u(t), \Gamma(t))$ *minimizes* $\mathcal{E}(v, \Gamma)$ *among all admissible* (v, Γ)*, that is*

$$\begin{cases} v = U(t) \text{ on } \partial\Omega \setminus \Gamma \\ \Gamma \supset \Gamma(t) \end{cases}$$

ii. Non-dissipativity: $\mathcal{E}(u(t), \Gamma(t))$ *is absolutely continuous in* t *and satisfies*

$$\frac{d}{dt}\mathcal{E}((u(t), \Gamma(t)) = \int_{\Omega\setminus\Gamma(t)} DW(\nabla u(t)).\nabla(\frac{\partial U}{\partial t}(t))\, dx.$$

Paradoxically, this latter setting is easier to handle mathematically, as will be demonstrated in the next section. The main reason is that it is more convenient to view the crack sets as the loci of the possible jumps of the displacement and/or deformation fields. But then, the natural functional space from the variational standpoint is a subspace of the space of $u \in L^1(\Omega; \mathbb{R}^N)$ with either ∇u, or $e(u)$ that are bounded Radon measures. It so happens that the former space, $BV(\Omega; \mathbb{R}^N)$, is much better known than is symmetrized counterpart, $BD(\Omega; \mathbb{R}^N)$.

3 Towards existence: time discretization

Here, we perform a time discretization of the variational evolution. This, we do for two reasons: first, because it is hopefully a way to attain existence for the time-continuous evolution, as the time step goes to 0; then, because it is what any numerical scheme will end up doing.

Consider $I_n := \{t_0^n, t_1^n, ..., t_{k(n)}^n\}$, with $0 = t_0^n \le t_1^n, ..., \le t_{k(n)}^n := T$ ($k(n) \nearrow_n \infty$ and $\Delta_n := \max_i\{t_{i+1}^n - t_i^n\} \overset{n}{\searrow} 0$) a nested sequence of times and define $U_0^n, ..., U_{k(n)}^n$ to be a discretization of the "load". We adopt as discrete evolution the following scheme:

Variational Evolution 3.1 *Setting* $\Gamma_{-1}^n = \emptyset$*, the pair* u_i^n, Γ_i^n *minimizes* $\mathcal{E}(\cdot, \cdot)$ *among all* (v, Γ) *such that*

$$\begin{cases} v = U_i^n := U(t_i^n) \text{ on } \partial\Omega \setminus \Gamma \\ \Gamma \supset \Gamma_{i-1}^n \end{cases}.$$

As the discretization step $\Delta_n \searrow 0$, we clearly (formally) recover item i of the variational evolutions 2.2 or 2.5; more surprisingly, item ii is also recovered: this will be discussed in greater details below.

The discrete variational evolution 3.1 is meaningless, unless its existence is established. This is a long story in itself, with intimate connections to the problem of image segmentation, and, specifically, to D. MUMFORD & J. SHAH's approach to image segmentation [32]. For our purpose, it is enough to distinguish two sets of results.

Whenever v is scalar-valued – the generalization to possibly higher dimensions of the anti-plane shear setting –, the most fruitful approach appeals to a weak formulation of the discrete variational evolution. Introduce

$$\Gamma_{i-1}^n := \bigcup_{j=0}^{i-1} S(u_j)$$

and

$$\mathcal{E}_i^n(v) := \int_\Omega W(\nabla v)\, dx + k\mathcal{H}^{N-1}\left(S(v) \setminus \left(\Gamma_{i-1}^n \cup \partial\Omega_d^c\right)\right), \tag{3.1}$$

where v is any function in $SBV(\Omega)$. Recall that $SBV(\Omega)$ is the subset of $BV(\Omega)$ of all elements $v \in L^1(\Omega)$, such that their distributional derivative Dv reads as

$$Dv = \nabla v \mathcal{L}^N + (v^+ - v^-)\nu \mathcal{H}^{N-1} \lfloor S(v).$$

In the formula above, ∇v is the density of the Lebesgue part of the measure Dv, $S(v)$ is the complement of the Lebesgue points of v, a rectifiable set with normal $\nu(x)$ at a point $x \in S(v)$, across which v jumps from $v^-(x)$ to $v^+(x)$ (see e.g. [4]). The discrete variational evolution becomes

Variational Evolution 3.2 *u_i^n minimizes $\mathcal{E}_i^n$ among all $v \in SBV(\Omega)$ such that $v = U_i^n$ on $\mathbb{R}^N \setminus \overline{\Omega}$.*

Then, the discrete crack at time t_i^n is $\Gamma_i^n := \Gamma_{i-1}^n \cup S(u_i^n)$.

Thanks to a compactness theorem of L. AMBROSIO [2], the discrete variational evolution 3.2 is easily shown to have a (possible non unique) solution, at least provided that the energy density is convex with p-growth for some $1 < p < \infty$, that is

$$a(|F|^p - 1) \leq W(F) \leq b(|F|^p + 1), F \in \mathbb{R}^N, \tag{3.2}$$

for some $0 < a < b < \infty$, and also that the boundary data are smooth enough, that is

$$U_i^n \in L^\infty(\Omega) \cap W^{1,p}(\mathbb{R}^N).$$

Remark 3.3 Whenever v is vector-valued, but the energy is still a function of ∇v – the hyperelastic case – existence is more involved. The true setting of hyperelasticity, with the well thought of constraint that $W(F) \nearrow \infty$, whenever $\det F \searrow 0^+$, is beyond the current scope of the analysis. Even when W satisfies (3.2), together with the usual quasiconvexity assumptions, existence is complicated in particular by the lack of supremum bound on v. It is necessary to introduce a certain class of surface and/or body loads, so as to ensure compactness of the minimizing sequences. In [18], G. DAL MASO, R. TOADER and I present a detailed analysis of the general ”hyperelastic” case (and prove existence of a variational evolution of the type 2.5).

Unfortunately, the weak formulation fails when dealing with symmetrized gradients, because the right space becomes $SBD(\Omega)$ – the space of $v \in L^1(\Omega; \mathbb{R}^N)$ with $e(v)$ a bounded Radon measure – and the necessary estimates for compactness fail [8]. The weak bounded variation setting is no longer appropriate. As of yet, the only case where this obstacle has been circumvented is that of plane elasticity (N=2); see [15]. The method consists in proving existence for the strong discrete variational evolution (3.1). This in turn requires a restrictive assumption on the possible cracks Γ, namely that their number of connected components remain a priori bounded. The deciding result is then Golab's theorem (see e.g. [15]), which states that a sequence of connected compact sets in $\mathbb{R}^2$ with uniformly bounded length admits a subsequence that converges in the sense of the Hausdorff distance to a connected compact set with length less than or at most equal to the liminf of the lengths.

The case of three dimensional linear elasticity is completely open at present.

In any case, we interpolate the discrete solutions u_i^n, Γ_i^n at time t_i as follows:

$$\begin{cases} u^n(t) := u_i^n \\ \Gamma^n(t) := \Gamma_i^n \quad , t \in [t_i^n, t_{i+1}^n), \\ U^n(t) := U_i^n \end{cases}$$

noting that, in view of the definition of I_n, $U^n(t) = U(t),\ t \in I_\infty := \bigcup_n I_n$, as soon as n is large enough.

Now that we have at our disposal the time discrete variational evolution, it remains to pass to the limit in the time step Δ_n, hoping to recover some version of the variational evolution (2.2), resp. (2.5), in the limit. To convince the reader of the non-trivial character of such an undertaking, I illustrate the kind of pathologies that could arise on a simple, yet instructive, example. It is that of the Neumann sieve [33]. I recall that a Neumann sieve situation occurs when boundaries close up at a critical speed that creates channels of non–zero capacity in the domain. For example, consider $\Omega = (-1,1)^2$ and assume, in a linear antiplane shear setting, that, at a given time, the crack Γ_n is a "sieve" of the form

$$\Gamma_n = \Big\{(0,y) : y \notin \bigcup_{p=0,\ldots,n} (\pm p/n - \exp^{-n}, \pm p/n + \exp^{-n})\Big\}$$

for a boundary load

$$u = 0, \text{ resp. } 1, \text{ on } \{x = -1\}, \text{ resp. } \{x = 1\}. \tag{3.3}$$

Then the displacement u_n satisfies

$$-\Delta u_n = 0 \text{ on } \Omega_n := (-1,1)^2 \setminus \Gamma_n,$$

with

$$\begin{cases} \dfrac{\partial u_n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_n \setminus \{x = \pm 1\} \\ u = 0 & \text{on} \quad \partial\Omega_n \cap \{x = -1\} \\ u = 1 & \text{on} \quad \partial\Omega_n \cap \{x = 1\}. \end{cases}$$

It can then be shown that $u_n \to u$ strongly in $L^2(\Omega)$, with $\Omega = [(-1,0) \cup (0,1)] \times (-1,1)$ and u the solution to

$$-\Delta u = 0 \text{ on } \Omega,$$

with

$$\begin{cases} \dfrac{\partial u}{\partial y} = 0 & \text{on} \quad \partial\Omega \cap \{y = \pm 1\} \\ u = 0 & \text{on} \quad \{-1\} \times (-1,1) \\ u = 1 & \text{on} \quad \{1\} \times (-1,1) \\ \dfrac{\partial u}{\partial x} = \mu[u] & \text{on} \quad \{0\} \times (-1,1), \end{cases}$$

where $\mu[u] > 0$. Hence u_n does not converge to the solution

$$w = \begin{cases} 0 \ \text{ on } (-1,0) \times (-1,1) \\ 1 \ \text{ on } (0,1) \times (-1,1) \end{cases}$$

of the Neumann problem on $\Omega \setminus \Gamma$, with $\Gamma = \{0\} \times (-1, 1)$.

The Neumann sieve must be prevented, if one strives to recover a brittle model in the limit of the discretization. A possible path consists in prohibiting cracks with too many connected components, exactly as in the case of planar linearized elasticity. Then, one-sided minimality in the sense of item i in the variational evolution (2.2) is obtained, thanks to an adaptation of a result of A. CHAMBOLLE &F. DOVERI [16] and of D. BUCUR & N. VARCHON [13], which states– in particular – that, if Ω is a Lipschitz two dimensional domain and $\{\Gamma^n\}_n$ is a uniformly bounded in length sequence of compact connected sets that converges for the Hausdorff distance to Γ, the solution to a Neumann problem of the form

$$\begin{cases} -\Delta v_n + v_n &= f \text{ in } \quad \Omega \setminus \Gamma^n \\ \dfrac{\partial v_n}{\partial \nu} &= 0 \text{ on } \partial[\Omega \setminus \Gamma^n], \end{cases}$$

is such that $v_n, \nabla v_n \stackrel{n}{\longrightarrow} v, \nabla v$, strongly in $L^2(\Omega)$, with v solution to

$$\begin{cases} -\Delta v + v &= v \text{ in } \quad \Omega \setminus \Gamma \\ \dfrac{\partial v}{\partial \nu} &= 0 \text{ on } \partial[\Omega \setminus \Gamma], \end{cases}$$

Deriving that adaptation, G. DAL MASO & R. TOADER prove in [19] the existence of a solution to the variational evolution 2.2, *under the restriction that the cracks have an a priori set number of connected components*. In turn, A. CHAMBOLLE in [15] proves an analogous result for plane elasticity, where, as noted before, connectedness seems to be the only viable assumption.

In fact, the Neumann sieve phenomenon cannot occur because, for n large enough, u_n, Γ_n is not a minimizer for

$$\mathcal{E}(v, \Gamma) = \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla v|^2 \, dx + \mathcal{H}^1(\Gamma), \Gamma \supset \Gamma_n.$$

with the boundary conditions (3.3). Indeed, by lower semi-continuity,

$$\liminf_n \mathcal{E}(u_n, \Gamma_n) \geq \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx + 1,$$

with u defined above. Now, u has non zero bulk energy $\frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx$, say α, so that, for n large enough,

$$\mathcal{E}(u_n, \Gamma_n) \geq 1 + \frac{\alpha}{2}.$$

But $\mathcal{E}(w, \Gamma) = 1$, a strictly smaller value, while $\Gamma \supset \Gamma_n$; hence, for n large enough, it is better to close the holes of the sieve and to take the crack to be Γ. This remark is what prompted C.J. LARSEN and I to hope for a stability result for one-sided minimality under refinement of the time step.

I now wish to give the reader a rough idea of the challenge at hand in the antiplane shear case (that where u is scalar-valued). Recalling the one-sided minimality in 3.1, $u^n(t)$ satisfies in particular

$$\frac{1}{2}\int_\Omega W(\nabla u^n(t))\,dx \le \frac{1}{2}\int_\Omega W(\nabla v)\,dx + \mathcal{H}^{N-1}(S(v)\setminus(S(u^n(t))\cup\partial\Omega_d^c)). \quad (3.4)$$

We say that $u^n(t)$ is a *minimizer for its own jump set*. It is easily shown that, provided that

$$U \in L^\infty(0,T;L^\infty(\mathbb{R}^N))\cap W^{1,1}(0,T;W^{1,p}(\mathbb{R}^N)), \quad (3.5)$$

then

$$\begin{cases} u^n(t) & \text{is bounded in} \quad L^\infty(\Omega) \\ \nabla u^n(t) & \text{is bounded in} \quad L^p(\Omega;\mathbb{R}^N) \\ \mathcal{H}^{N-1}(S(u^n(t)) & \text{is bounded in} \quad \mathbb{R}, \end{cases}$$

from which it is deduced, thanks to Ambrosio's compactness theorem, that a (a priori t-dependent) subsequence of $u^n(t)$ converges in SBV to some $u(t)$, by which I mean that

$$\begin{cases} u^n(t) \overset{n}{\rightharpoonup} u(t), \text{ weak-* in } L^\infty(\Omega) \\ \nabla u^n(t) \overset{n}{\rightharpoonup} \nabla u(t), \text{ weak in } L^p(\Omega;\mathbb{R}^N) \\ \mathcal{H}^{N-1}(S(u(t))) \le \liminf_n \mathcal{H}^{N-1}(S(u^n(t))). \end{cases} \quad (3.6)$$

If one-sided minimality is to be obtained in the limit, then $u(t)$ should in particular be a minimizer for its own jump set. In view of (3.6), which implies the lower semi-continuity of the left hand side of (3.4) by a result of L. AMBROSIO [3], the result would follow easily, provided that

$$\limsup_n \mathcal{H}^{N-1}(S(v)\setminus S(u^n(t))) \le \mathcal{H}^{N-1}(S(v)\setminus S(u(t))).$$

That such might not be the case is easily demonstrated by considering v such that $S(v)\subset S(u(t))$, while $u^n(t)$ is such that $S(u^n(t))\cap S(u(t)) = \emptyset$ (which would surely happen if $S(u^n)\subset K_n$, with $K_n\cap K=\emptyset$ and the Hausdorff distance from K_n to K goes to 0); we would then get $\mathcal{H}^{N-1}(S(v))=0$!

This indicates that one cannot hope to prove stability of one-sided minimality without modification of the test fields v. This is in essence what our jump transfer theorem [21], Section 2, does. Specifically, that theorem says that, for u^n, u satisfying (3.6), and any $v\in SBV(\Omega)$, there exists a sequence $v^n\in SBV(\Omega)$, with

$$\begin{cases} v^n(t) \overset{n}{\longrightarrow} v, \text{ strongly in } L^1(\Omega) \\ \nabla v^n \overset{n}{\longrightarrow} \nabla v, \text{ strongly in } L^p(\Omega;\mathbb{R}^N) \\ \mathcal{H}^{N-1}(S(v)\setminus S(u(t))) \le \limsup_n \mathcal{H}^{N-1}(S(v^n)\setminus S(u^n(t))). \end{cases}$$

Stability of one-sided minimality, which is easily derived, for all t's in I_∞, with the help of the jump transfer theorem, is but the first hurdle to overcome. The solution $u(t)$, $\Gamma(t) := \bigcup_{\{s\le t: s\in I_\infty\}} S(u(s))$, has to be extended to all t's in $[0,T]$. This can be done fairly simply in the scalar valued case(see [21], Section 3.2), but becomes much more delicate in the vectorial case because the minimizers of $\int_\Omega W(\nabla v)\;dx$ over $\{v \in SBV(\Omega) : S(v) \subset \Gamma\}$, for $\Gamma, \mathcal{H}^{N-1}$-rectifiable set, are not unique . The argument necessitates the introduction of a weak notion of set convergence, that of σ^p-convergence ([18], Section 4.1), which I will not describe here.

Then, one should prove non-dissipativity, that is:

$$\int_\Omega W(\nabla u(t))\;dx + k\mathcal{H}^{N-1}(\Gamma(t)) = \int_0^t \int_\Omega DW(\nabla u(t)).\nabla(\frac{\partial U}{\partial t}(s))\;dxds$$
$$+ \int_\Omega W(\nabla u(0))\;dx + k\mathcal{H}^{N-1}(\Gamma(0)).$$

Once again, this is not such an easy task. The upper inequality is obtained from an analogous inequality for the discrete variational evolution. The key result (especially in the vector-valued case) is that

$$DW(\nabla u^n(t)) \rightharpoonup DW(\nabla u(t)), \text{ weakly in } L^{p'}(\Omega; \mathbb{R}^{N^{(2)}}),$$

which allows to pass to the limit in the upper inequality ([18] Section 4.3).

The lower inequality is a corollary of one-sided minimality for $u(t)$, but it requires, either the t-continuity of $DW(\nabla u(t))$ in L^p (essentially true in the scalar case) [21], or a subtle approximation result of Lebesgue integrals by appropriate Riemann sums ([18] Section 4.4).

Finally, convergence (for a subsequence) of the discrete bulk energy to the continuous bulk energy, and of the discrete surface energy to the continuous surface energy is easily obtained.

Existence of a (weak) variational evolution for the SBV-version of 2.2 or 2.5 is now ascertained. Remark that, in general, $u(t)$ and $\Gamma(t)$ have no special properties in time; in other words, it is not even clear that they are measurable maps. Similarly, it is the sum of the bulk and surface energy that is absolutely continuous in time, but the bulk energy can suddenly jump down, and the crack ”length” jump up, at a given time. This is the occurrence of brutal cracking, an instantaneous extension of the crack; this particular incident is happily encompassed in the model that we champion, in striking contrast to what would happen within the perimeter of Griffith’s model.

Summarizing, existence of a variational evolution is guaranteed in the following cases:

- antiplane shear, linear of non-linear, subjected to a time-varying hard device [21], [18];
- hyperelasticity, without the determinant condition (no frame indifference), subjected to a time-varying hard device and to appropriate body and/or surface forces (no linear loads!)[18];
- two-dimensional linearized elasticity, , subjected to a time-varying hard device and with the ”connectedness” restriction [15].

I emphasize that, if no ”connectedness” restriction are imposed on the test cracks, then the above outlined argument only shows existence of a weak variational evolution; it has yet to be proved that $\Gamma(t)$ is a compact set, so that the classical elasticity problem – with $u \in W^{1,p}(\Omega \setminus \Gamma(t); \mathbb{R}^{(N)})$ – is well posed on $\Omega \setminus \Gamma(t)$. This would require a regularity result à la E. DE GIORGI, M. CARRIERO, A. LEACI [20], that is that $\mathcal{H}^{N-1}\left(\overline{\Gamma(t)} \setminus \Gamma(t)\right) = 0$, a formidable task in the current setting.

4 Further results

In this section, we list most of the available theoretical results that have been derived in the past few years with the help of the variational evolution. The approximations results are especially noteworthy, because they are at the root of the available numerical methods. The numerical aspects of the variational evolution are varied and fascinating; I would do them injustice, if I tried to squeeze a short numerical section in this paper and I prefer to refer the reader to [12] for a detailed account of some of those. I will merely present two illustrative computations in Section 6 below.

A first class of results attempts to derive Griffith's classical criterion (item ii in Griffith's model) from the variational evolution, assuming the crack path to be smooth. This is first achieved in the two-dimensional "connected" case in [19], Section 8, then, for a flexural plate model, in [1], Section 7. Note that, to the best of my knowledge, the latter result is new; it gives an expression for the energy release rate in terms of the coefficients of the singular part of the displacement field, solution to a fourth order problem.

In a different direction, A. GIACOMINI & M. PONSIGLIONE investigate the interaction between homogenization and fracture evolution. For ε-dependent bulk energies with uniform p-growth, and ε-dependent surface energies of the brittle type, that is energies of the form $\int_{S(v)} k^\varepsilon(x,\nu) d\mathcal{H}^{N-1}$, with a uniform bound from above and below on the k^ε's, they prove that the variational evolution, for a given ε, "tends to" that of the brittle material that would have for bulk energy the homogenized bulk energy, and for surface energy, the homogenized surface energy. There is thus no interplay between homogenization and the variational crack evolution.

Consider an elastic material with a cohesive – à la Barenblatt type – surface energy , that is an energy of the form

$$\int_{S(v)} \varphi([v])\; d\mathcal{H}^{N-1},\; \varphi : \varphi(0) = 0 \nearrow \varphi(\infty) = 1 \text{ on } [0,+\infty], \varphi \text{ concave}, \varphi'(0) < +\infty. \tag{4.1}$$

Set $a := \varphi'(0)$; then $\varphi(s) \leq as$ for all $s \in [0,+\infty[$. It is folklore in the fracture community that cohesive surface energies give rise to brittle behavior as the size of the sample becomes large. This issue is addressed by A. GIACOMINI in [25]. The consideration of cohesive type surface energies introduces an additional difficulty. Indeed, the functional

$$\frac{1}{2}\int_\Omega |\nabla v|^2\; dx + \int_{S(v)\setminus\partial\Omega_d^c} \varphi(|[v]|)\; d\mathcal{H}^{N-1},$$

with φ as above, is not lower semi-continuous on $SBV(\Omega)$. It needs to be relaxed [10]; its lower semi-continuous envelope is the following functional, *defined this time on all of* $BV(\Omega)$:

$$\int_\Omega \psi(|\nabla v|)\; dx + \int_{S(v)\setminus\partial\Omega_d^c} \varphi(|[v]|)\; d\mathcal{H}^{N-1} + a|D_c v|, \tag{4.2}$$

where $D_c v$ denotes the Cantor part of the measure Dv and

$$\psi(t) := \begin{cases} \frac{1}{2}t^2,\; |t| \leq a \\ \frac{1}{2}a^2 + a(|t| - a),\; |t| > a. \end{cases}$$

In the case of crack evolution, this needs to be modified, so as to account for irreversibility: if, at time t_i^n, the crack set is Γ_i^n , then we consider

$$\inf\left\{\frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \int_{(S(v)\cup\Gamma^n_{i-1})\setminus\partial\Omega_d^c}\varphi(|[v]|\vee g^n_{i-1})\,d\mathcal{H}^{N-1}\right\},$$

and set

$$g_i^n := |[u_i^n]|\vee g^n_{i-1},\quad \Gamma_i^n := \Gamma^n_{i-1}\cup S(u_i^n).$$

First, it is not obvious that the relaxation process and time stepping commute; second, even if it is so, recovering the variational evolution for the relaxed functional in $BV(\Omega)$ from a time discretization – currently the only available method – is an open problem. A. GIACOMINI considers a sequence of homothetically increasing domains $h\Omega$, with adequate scaling of the boundary displacements $U^h(t,x) := h^{\frac{1}{2}}U(t,x/h)$, then a sequence of discrete times that tends to 0 as the size parameter h increases. Rescaling the problem back to Ω, he shows that this double approximation gives rise, as $h\nearrow\infty$ to the variational evolution 2.2 discussed in Section 3. Folklore becomes fact.

The most numerous results concern various approximations of the variational evolutions 2.2. Their shared starting point is the approximation of the Mumford-Shah functional for image segmentation, a huge field in itself. Adaptation of those to the problem at hand has been circumscribed to two methods. The first derives from the work of L. AMBROSIO & TORTORELLI [5], [6]. The idea is to approximate in the sense of Γ-convergence the functional

$$1/2\int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}(S(v))$$

by a two-field functional

$$\int_\Omega\left(\frac{1}{2}(w^2+o(\varepsilon))|\nabla v|^2\,dx + \varepsilon|\nabla w|^2 + \frac{1}{4\varepsilon}(w-1)^2\right)\,dx,\quad 0\le w\le 1, \qquad (4.3)$$

the main point being that minimizers $v^\varepsilon, w^\varepsilon$ for the second functional – upon adding suitable zeroth order terms – will converge to minimizers for the first one. Specifically,

$$v^\varepsilon\to u,\ \text{strongly in } L^2(\Omega),\ \varepsilon|\nabla w^\varepsilon|^2+\frac{1}{4\varepsilon}(w^\varepsilon-1)^2\rightharpoonup \mathcal{H}^{N-1}\lfloor S(u),\ \text{weak-* as measures},$$

where u is a minimizer for the first functional. The crack set will in essence be the set where w^ε does not converge to 1. This idea is used and implemented for the discrete variational evolution by B. BOURDIN [12]; serious numerical issues have to be overcome for a successful implementation of such an algorithm. In [26], A. GIACOMINI proves the existence of a variational evolution for the approximation – at fixed ε – and shows that the resulting time-parameterized minimizing fields converge to a variational evolution of the (weak) type 2.2.

The second method operates directly at the finite element level. The idea is to look at a triangulation $\mathcal{T}_\varepsilon$ of Ω, where each triangle is roughly of size ε. Then, two options are available.

The first consists in refining the triangulation as follows: on each side of the triangles of $\mathcal{T}_\varepsilon$, a point is chosen, so that its distance to the two vertices of that side is between $a\varepsilon$ and $(1-a)\varepsilon$, $0<a<1/2$. Then, one considers piecewise-affine fields with possible side-discontinuities on each element of the nested new triangulation obtained by joining together all such points. This is the road traveled by A. GIACOMINI & M. PONSIGLIONE in [27]. At fixed a, ε the minimization is carried out on the resulting triangulation at discrete times (parameterized by δ), then all parameters – δ, ε, a – are made to tend to 0, and the (weak) variational evolution 2.2 is recovered, at least in the antiplane shear case.

In the second formulation, a concave, non-decreasing, continuous, non-negative function f with slope 1 at $t = 0$ and limit 1 at $t = \infty$ is introduced. The method consists, for a fixed triangulation $\mathcal{T}_\varepsilon$, in using an approximation of the Mumford-Shah functional of the form

$$G_\varepsilon(v, T) := \sum_{T \in \mathcal{T}_\varepsilon} |T \cap \Omega| 1/h_T f(h_T |\nabla v_T|^2),$$

h_T being the smallest side-length of T and ∇v_T being the constant value of the gradient of u, with u continuous and affine on each $T \in \mathcal{T}_\varepsilon$. This approximation is shown by B. BOURDIN & A. CHAMBOLLE in [11] to Γ-converge to the Mumford-Shah functional, as $\varepsilon \searrow 0$; the resulting numerical algorithm compares favorably to that based on the Ambrosio-Tortorelli approximation, notably because it is much faster. In the context of brittle fracture, a variant of this is proposed by M. NEGRI in [34].

The above mentioned results span, in a nutshell, the range of available results related to the variational evolution presented in Section 3 and demonstrate its scope and numerical adaptability.

5 A defect and its remedies

The main defect of the variational evolution – its generic inability to handle soft devices – has been alluded to several times already. This is obvious at the level of the discrete variational evolution. Indeed, the energy $\mathcal{E}(v, \Gamma)$ must be modified, so as to accomodate the work done by the loads, which I synthetically denote by $\mathcal{L}(t, v)$. Now, whenever $\mathcal{L}$ is linear in v and there is a pair v, Γ for which $\int_{\Omega \setminus \Gamma} W(\nabla v)\, dx = 0$, then

$$\mathcal{E}(\lambda v, \Gamma) \overset{\lambda}{\searrow} -\infty.$$

This will be the case for example when constant surface loads are applied on a part $\partial_f \Omega$ of the boundary.

Consequently, there is no minimum pair for $\mathcal{E}$!

In the framework of ”finite elasticity” (see Remark 3.3 above), this issue can be circumvented, at the expense of the exclusion of linear loads ([18], Section 3). The loads should be such, that the accompanying minimizers for $\mathcal{E}_i^n(v, \Gamma)$ (see 3.1) are uniformly bounded in some L^q-space, with $1 < q < \infty$. To that effect, they are assumed to satisfy a coercivity hypothesis of the form:

$$-\mathcal{L}(t, v) \geq \alpha \|v\|^q_{L_q} - \beta.$$

From a mechanical viewpoint, this ensures that, even if the body splits into several components, those all remain at finite distance from each other. Of course, such a ”trick” would be nonsensical in the linearized context.

It is then tempting to lay the blame on global minimization, the cornerstone of our model. A criterion of local minimality (meta-stability) would seem more appropriate. It immediately begs the question of what the measure of locality should be; which distance should one adopt? A natural distance could be that induced by the BV-norm. If doing so, then the one-dimensional results in [17] suggest that the cure is not strong enough, because elastic solutions are shown to remain local minimizers, independently of the loading level: initiation does not occur. A recent investigation by A. CHAMBOLLE, A. GIACOMINI & M. PONSIGLIONE establishes a similar result in linearized antiplane shear, provided that the elastic solution remains smooth enough, that is essentially when the singular part of that solution near a point of singularity grows like $|x|^\alpha$, with $\alpha > 1/2$.

Replacing global stability by meta-stability brings us even closer to the original Griffith's model and indeed, as emphasized before, the absence of crack initiation is generic in that model. Consequently, if crack initiation is to be retained as an essential ingredient of a rational theory for quasistatic brittle fracture, then Griffith's model has to be further modified. But the elastic (bulk) part of the energy cannot be challenged. Thus, Griffith's surface energy is the weak link.

In the very simple one-dimensional setting, we show in [17] that one should simultaneously abandon global stability in favor of meta-stability, and Griffith's surface energy in favor of a cohesive type energy. Then, a yield stress can be evidenced and soft devices behave like hard devices. The required tools for a similar analysis in dimensions greater than one are lacking at present.

6 Numerical implementation: two examples

I present in this final section the results of two numerical computations that illustrate the practical predictive power of the variational evolution.

The first computation, performed by B. BOURDIN and already presented in [12], is based on the Ambrosio-Tortorelli approximation detailed in Section 4.

A square elastic matrix is reinforced by a perfectly bonded rigid circular fiber: this is a plane stress problem. A uniform upwardly directed displacement field δ, represented by a black area in the following figures, is imposed on the upper side of the square; the remaining sides are traction-free.

The crack is represented by the red lines. The smearing of those lines is due to the approximation, since the colors indicate level sets for the additional variable w in the approximation (4.3); the red corresponds to w close to 0, $\{w = 0\}$ being the crack site.

As long as $\delta < .2$, the matrix remains purely elastic. At $\delta \sim .2$, a crack of finite length brutally appears slightly above the north pole of the inclusion (figure 1): it is not a boundary crack. The brutal character of the growth is conform to the results in [22], Section 4, where brutal crack growth with a finite initiation time is predicted in the absence of singular points for the purely elastic solution. When δ varies between $.2$ and $.32$, the crack progressively grows in the matrix (figure 2).

At $\delta \sim .32$, the right hand-side of the matrix is brutally cut (figure 3), which also agrees with the results in [22], Section 4. When δ varies between $.32$ and $.37$, the left part of the crack progressively grows (figure 4). Finally, at $\delta \sim .37$, the crack brutally severs the remaining filament of uncracked material (figure 5) and the sample is split into two parts.

Fig. 1 $\delta \sim .2$: A crack of finite length appears slightly above the interface.

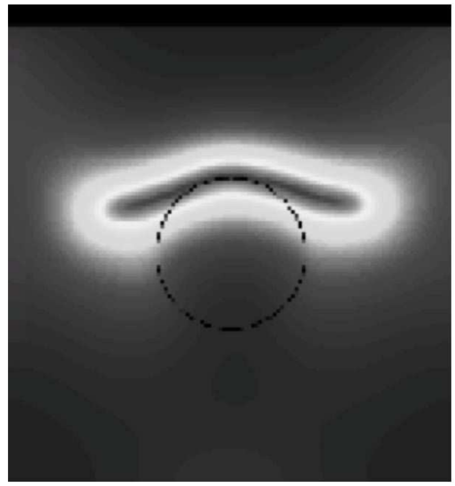

Fig. 2 The crack extends smoothly on both sides; the slight asymmetry is a byproduct of the corresponding asymmetry of the mesh.

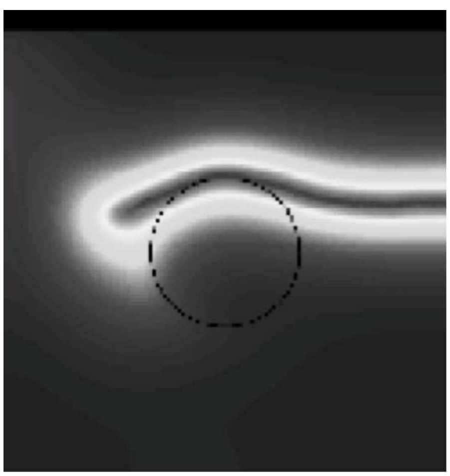

Fig. 3 $\delta \sim .32$: The crack brutally grows on one side; the choice of the side is a byproduct of the asymmetry of the mesh.

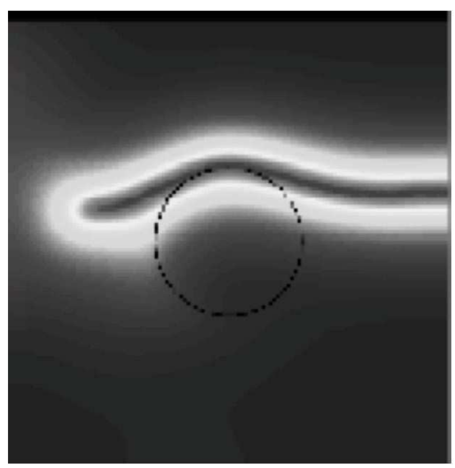

Fig. 4 The crack grows smoothly on the remaining side.

Remark 6.1 It is checked in [12] that Griffith's criterion (item ii in Griffith's model of Section 3) is verified during the progressive phases of the evolution.

Remark 6.2 If the symmetry breaking direction is purely numerical, the carefully planned asymmetry of the underlying mesh guarantees the reality of a computed lateral symmetry.

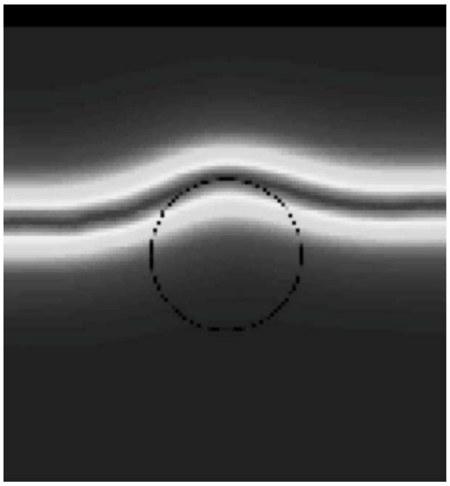

Fig. 5 $\delta \sim .37$: The crack splits the sample, leading to its failure.

I think it is fairly clear that the complexity of the computed behavior is well beyond the scope and analytical power of classical brittle fracture. As noted in [12], the qualitative agreement between our results and experimental observations is quite satisfactory (see [29]).

The second example is a traction experiment on a circular composite cylindrical shaft performed by F. BILTERYST in [9] and reported in [23]. The inner shaft is perfectly bonded to the outer hollow cylinder. The inner and outer material are each elastic, but the inner shaft is assumed unbreakable: k, the fracture toughness, is taken to be infinite there. A monotonically increasing displacement field is applied at each point of the end sections.

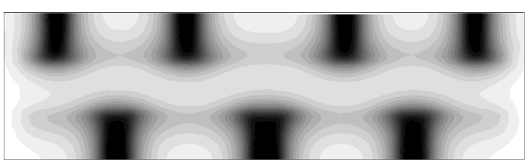

Fig. 6 Periodically distributed transverse cracks brutally appear.

Once again, the computation is based on Ambrosio-Tortorelli's approximation, which explains the numerical smearing of the crack. Figure 6 shows the instant at which a alternating set of periodically distributed cracks transverse cracks suddenly appears. Once again, this is in agreement with our theoretical predictions, yet such a result could not be attained by resorting to Griffith's model. Experimental observations [24] qualitatively validate the computation.

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References

[1] F. Acanfora, M. Ponsiglione, Quasi-static growth of brittle cracks in a linearly elastic flexural plate, to appear.

[2] L. Ambrosio, A compactness theorem for a new class of functions of bounded variation, Boll. Un. Mat. Ital. **3**-B, 857-881, (1989).

[3] L. Ambrosio, On the lower semicontinuity of quasi-convex functionals in SBV, Nonlinear Anal. **23** , 405-425, (1994).

[4] L. Ambrosio, E. Fusco, D. Pallara, Functions of bounded variation and free discontinuity problems (Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000).

[5] L. Ambrosio, V.M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, Comm. Pure Appl. Math. **43**, 999–1036 (1990).

[6] L. Ambrosio, V.M. Tortorelli, On the approximation of free discontinuity problems, Bollettino U.M.I. **7**-6, 105–123 (1992).

[7] G.I. Barenblatt, The mathematical theory of equilibrium cracks in brittle fracture, Adv. Appl. Mech. **7**, 55–129 (1962).

[8] G. Bellettini, A. Coscia, G. Dal Maso, Compactness and lower semicontinuity properties in $SBD(\Omega)$, Math. Z. **228**2, 337–351 (1998).

[9] F. Bilteryst, Une approche énergétique de la décohésion et de la multifissuration dans les composites, Thèse de Doctorat, Université Pierre et Marie Curie (2000).

[10] G. Bouchitté , A. Braides ,G. Buttazzo, Relaxation results for some free discontinuity problems, J. Reine Angew. Math. **458**, 1–18 (1995).

[11] B. Bourdin, A. Chambolle, Implementation of an adaptive finite-element approximation of the Mumford-Shah functional, Numer. Math. **85**, 609–646 (2000).

[12] B. Bourdin, G.A. Francfort, J.J. Marigo, Numerical experiments in revisited brittle fracture, J. Mech. Phys. Solids, **48**, 797–826 (2000).

[13] D. Bucur, N. Varchon, Boundary variation for the Neumann Problem, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **29**-4, 807–821 (2000).

[14] H. D. Bui, Mécanique de la rupture fragile (Masson, Paris, 1978).

[15] A. Chambolle, A density result in two-dimensional linearized elasticity and applications, Arch. Rat. Mech. Anal. **167**-3, 211–233 (2003).

[16] A. Chambolle, F. Doveri, Continuity of Neumann linear elliptic problems on varying two–dimensional bounded open sets, Comm. P.D.E. **22**, 811–840 (1997).

[17] M. Charlotte, G. Francfort, J.J. Marigo, L. Truskinovsky, Revisiting brittle fracture as an energy minimization problem: comparison of Griffith and Barenblatt surface energy models in: Proceedings of the Symposium on continuous damage and fracture, Cachan 2000, edited by A. Benallal (The Data Science Library, Elsevier, Paris, 2000), 7–12.

[18] G. Dal Maso, G.A. Francfort, R. Toader, Quasistatic crack growth in finite elasticity, Arch. Rat. Mech. Anal. **176**-2, 165–225 (2005).

[19] G. Dal Maso, R. Toader, A model for the quasi–static growth of brittle fracture: existence and approximation results, Arch. Rat. Mech. Anal. **162**, 101–135 (2002).

[20] E. De Giorgi, M. Carriero, A. Leaci, Existence theorem for a minimum problem with free discontinuity set, Arch. Rat Mech. Anal. **108**, 195–218 (1989).

[21] G.A. Francfort, C.J. Larsen, Existence and convergence for quasi-static evolution in brittle fracture, Comm. Pure Appl. Math. **56**, 1465-1500 (2003).

[22] G.A. Francfort, J.J. Marigo, Revisiting brittle fracture as an energy minimization problem, J. Mech. Phys. Solids **46**-8, 1319-1342 (1998).

[23] G.A. Francfort, J.J. Marigo, Stable damage evolution in a brittle continuous medium, Eur. J. Mech. A/Solids **12**- 2, 149–189 (1993).

[24] K.W. Garrett, J.E. Bailey, Multiple transverse fractures in 90 cross-ply laminates of a glass-fiber reinforced polyester, J. Material Sci. **12**, 157–168 (1977).

[25] A. Giacomini, Size effects on quasistatic growth of fractures, SIAM J. Math. Anal. **36**-6, 1887–1928 (2005).
[26] A. Giacomini, Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures, Calc. Var. Partial Differential Equations **22**-2, 129–172 (2005).
[27] A. Giacomini, M. Ponsiglione, A discontinuous finite element approximation of quasi-static growth of brittle fractures, Numer. Funct. Anal. Optim. **24**-7/8, 813–850 (2003).
[28] A. Griffith, The phenomena of rupture and flow in solids, Phil. Trans. Roy. Soc. London **CCXXI**-A, 163–198 (1920).
[29] D. Hull, An introduction to composite materials (Cambridge Solid State Science Series, Cambridge University Press, 1981).
[30] G.R. Irwin, Fracture, in: Handbuch der Physik, 6 (Springer Verlag, Berlin, 1957), 551–590.
[31] D. Leguillon, Calcul du taux de restitution de l'énergie au voisinage d'une singularité, C. R. Acad. Sci. Paris Série II **309**, 945–950 (1990).
[32] D. Mumford, J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems, Comm. Pure. Applied Math. **42**, 577-685 (1989).
[33] F. Murat, The Neumann sieve in: Nonlinear Variational Problems, edited by A. Marino, L. Modica, S. Spagnolo, M. Degiovanni (Research Notes in Mathematics, **127**, Pitman, London, 1985), 24–32.
[34] M. Negri, A finite element approximation of the Griffith's model in fracture mechanics, Numer. Math. **95**-4, 653–687 (2003).