QUASISTATIC EVOLUTION IN NON-ASSOCIATIVE PLASTICITY REVISITED

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ABSTRACT. The mathematical treatment of evolutionary non-associative elasto-plasticity is still in its infancy. In particular, all existence results thus far rely on a spatially mollified stress admissibility constraint. Further, the evolution is formulated in a rescaled time from which it is very difficult to infer any useful information on the "real" time evolution. We propose a causal spatio-temporal mollification of the stress admissibility constraint that, while no more far-fetched than a purely spatial one, produces a more elegant and complete evolution for such models, and this in the "real" time variable.

1. INTRODUCTION

In the past few years, small strain elasto-plasticity has seen a (modest) rebirth because of a change of perspective. While in the seminal work of P.-M. Suquet [22], elasto-plastic evolution was seen as the limit of visco-plastic evolutions as the viscosity parameter tends to 0, the evolution is now preferably viewed [3] as an energy conserving variational evolution in the spirit of A. Mielke's general framework [16].

Elasto-plastic evolution of a homogeneous elasto-plastic material occupying a volume $\Omega \subset \mathbb{R}^n$, with Hooke's law (elasticity tensor) A and subject to a time-dependent loading process with, say, $f(t)$ as body loads, $g(t)$ as surface loads on a part Γ_n of $\partial\Omega$, and $w(t)$ as displacement loads (hard device) on the complementary part Γ_d of $\partial\Omega$ results in the following system where $Eu(t)$ denote the infinitesimal strain at t, that is, the symmetric part of the spatial gradient of the displacement field $u(t)$ at t, $\sigma(t)$ is the Cauchy stress tensor at time t, and $e(t)$ and $p(t)$ (a deviatoric symmetric matrix) are the elastic and plastic strain at t:

- Kinematic compatibility: $Eu(t) = e(t) + p(t)$ in Ω and $u(t) = w(t)$ on Γ_d;
- Equilibrium: $\operatorname{div} \sigma(t) + f(t) = 0$ in Ω and $\sigma(t)\nu = g(t)$ on Γ_n, where ν denotes the outer unit normal to $\partial\Omega$;
- Constitutive law: $\sigma(t) = Ae(t)$ in Ω;
- Stress constraint: $\sigma_D(t) \in K$, where σ_D is the deviatoric part of the Cauchy stress σ, and K is the admissible set of stresses (a convex and compact subset of deviatoric $n \times n$ matrices);
- Flow rule: $\dot{p}(t) = 0$ if $\sigma_D(t) \in \operatorname{int} K$, while $\dot{p}(t)$ belongs to the normal cone to K at $\sigma_D(t)$ if $\sigma_D(t) \in \partial K$.

The corresponding variational evolution, as discussed in [3], formally consists in the following four-pronged formulation, for $t \in [0, T]$,

- Kinematic compatibility: $Eu(t) = e(t) + p(t)$ in Ω and $u(t) = w(t)$ on Γ_d;
- Global stability: The triplet $(u(t), e(t), p(t))$ globally minimizes

$$\frac{1}{2}\int_\Omega A\eta : \eta \, dx + \int_\Omega H(q - p(t)) \, dx$$

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among all admissible triplets (v,η,q), where $H(p)$ is the support function of K, *i.e.*, $H(p) := \sup\{\sigma_D : p \ : \ \sigma_D \in K\}$;

- Constitutive law: $\sigma(t) = Ae(t)$ in Ω;
- Energy balance:

$$\frac{d\mathcal{E}}{dt}(t) = \int_{\Gamma_d} (\sigma(t)\nu)\cdot \dot w(t)\, d\mathcal{H}^{n-1} - \int_\Omega \dot f(t)\cdot u(t)\, dx - \int_{\Gamma_n} \dot g(t)\cdot u(t)\, d\mathcal{H}^{n-1},$$

where

$$\mathcal{E}(t) := \frac12 \int_\Omega Ae(t) : e(t)\, dx + \int_0^t \int_\Omega H(\dot p(s))\, dx\, ds - \int_\Omega f(t)\cdot u(t)\, dx - \int_{\Gamma_n} g(t)\cdot u(t)\, d\mathcal{H}^{n-1}.$$

Unfortunately that formulation seems ill suited to deal with a slew of plastic models that share a common feature, non-associativity. The concerned material behavior ranges from that of a soil (in, *e.g.*, Cam-Clay plasticity or Drucker-Prager plasticity) to that of a metal when nonlinear kinematic hardening is introduced (Armstrong-Frederick type models[1]).

In any case, those non-associative models are thought to lie beyond the reach of any kind of variational formulation. However, it has been recently shown, first for Cam-Clay plasticity in [4, 5], then for Drucker-Prager plasticity in [2], and finally for Armstrong-Frederick type materials in [11], that this view is not correct and that a variational formulation can be brought forth for a rather generic model of non-associative elasto-plasticity.

However that formulation is plagued by several defects. First it cannot proceed if the set of admissible stresses is not bounded in all directions in the space of symmetric matrices. But this is not so unless a cap is imposed on the admissible stresses in some directions [2]. Then, the variational formulation introduces a set of admissible stresses that depends on the actual stress. The analysis does not seem to handle that dependence very well because the actual stress is not continuous. The way out consists in mollifying the actual stress in the formulation [4].

Finally, it proves impossible to carry through the analysis in real time because of a lack of Lipschitz estimates in time (see, *e.g.*, [4]). The remedy is to rescale time (essentially using the energy as the new time), so as to recover Lipschitz estimates. That idea, borrowed from [17, 18], but whose germ is in [8], yields an elasto-plastic evolution in rescaled time. It is then extremely difficult to recover a real time evolution; this is the object of [5] which only partially succeeds in such an endeavor.

In this paper, we propose to demonstrate that such a rescaling is unnecessary in all models of non-associative elasto-plasticity that have been considered thus far, provided that the mollification of the actual stress is (slightly) modified. In lieu of a spatial convolution, we suggest a space-time convolution, the time part of that convolution only involving the past so that causality is preserved.

The paper is organized as follows. A first section is devoted to the generic formulation of a non-associative cap model in a framework that will be palatable to the subsequent analysis. The second section addresses the existence of a visco-plastic evolution. Viscosity type approximations for quasi-static elasto-plastic evolution problems have been used many times, starting with the pioneering work of P.-M. Suquet [22]; more recently, approximations that combine viscosity and dynamics have also been proposed [6]. In all cases, the approximation rested on a time-incremental process whereas, in our setting, existence cannot be secured through such a process. Rather, we use a faster and, in our opinion, more elegant fixed point argument. In Section 4 we prove the main result of the paper, namely Theorem 4.1, which states an existence theorem in real time.

The obtained results conform to our expectation that one should indeed recover an evolution in real time, at least in the Drucker-Prager context, because the solution in the spatially homogeneous

[1]The Armstrong-Frederick model is the simplest plastic model that phenomenologically captures the so-called Bauschinger effect, a kind of hysteretic behavior often observed in metals under cyclic loadings [12].

case has been shown in [13, Theorem 5] to be continuous in time, at least for particular classes of sets of admissible stresses.

Finally, we limit the loading process to a hard device, that is, to a displacement $w(t)$ acting on the entirety of the boundary $\partial\Omega$ of our domain. This is certainly a simplifying assumption because it alleviates in particular the need for safe load conditions on the loads $f(t)$ and $g(t)$ and renders duality much simpler. Those can become at times a thorny issue in plasticity. We are confident that the analysis remains the same if general loads were to be incorporated into the evolution.

2. Description of the model

In this section, we provide an overview of the type of models that can be addressed.

2.1. **The relevant model.** The context is that of small strains. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set occupied by a homogeneous elasto-plastic material. We denote by $u(t) : \Omega \to \mathbb{R}^n$ the displacement field at time t and by $Eu(t) := (Du(t) + Du^T(t))/2$ the strain tensor. As is usual in small deformations plasticity, the strain tensor is *additively* decomposed as

$$Eu(t) = e(t) + p(t),$$

where $e(t)$ and $p(t)$ stand for the elastic and plastic strains, respectively. This is part of what will be referred to as *kinematic compatibility.* The constitutive equation which relates the (Cauchy) stress tensor σ to the elastic part e of the linearized strain is also assumed to be linear, *i.e.*,

$$\sigma(t) = Ae(t),$$

where A is the Hooke tensor. At equilibrium, and if no volume forces are applied to the sample, the stress satisfies

$$\operatorname{div} \sigma(t) = 0 \quad \text{in } \Omega.$$

It is also constrained to remain in a σ-dependent compact convex subset $K(\sigma(t))$ of the set $\mathbb{M}^{n\times n}_{sym}$ of $n \times n$ symmetric matrices, that is,

$$\sigma(t) \in K(\sigma(t)).$$

The behavior of the plastic strain is governed by the following flow rule: denoting by $\dot{p}$ the time derivative of p,

$$\dot{p}(t) \in \partial I_{K(\sigma(t))},$$

where I_K denotes the indicator function of the set K.

Finally, as announced in the introduction, the material is subject to a hard loading device; in other words, a Dirichlet boundary condition $u(t, x) = w(t, x)$ is imposed on $\partial\Omega$.

Our goal is to obtain a triplet $(u(t, x), e(t, x), p(t, x))$ such that

$$\begin{cases} Eu(t, x) = e(t, x) + p(t, x), \\ \sigma(t, x) = Ae(t, x), \\ \operatorname{div} \sigma(t, x) = 0, \\ \sigma(t, x) \in K(\sigma(t, x)), \\ \dot{p}(t, x) \in \partial I_{K(\sigma(t,x))}, \end{cases}$$

together with the Dirichlet boundary condition. Of course, we know from prior works on plasticity that we cannot expect the boundary condition to be satisfied because plastic strains may develop at the boundary, so that, as seen later, we will have to replace that condition by

$$p(t, x) = (w(t, x) - u(t, x)) \odot \nu(x) \quad \text{on } \partial\Omega.$$

Throughout, the symbol $\odot$ stands for the symmetrized tensor product, while ν denotes the outer unit normal to $\partial\Omega$.

When $K(\sigma)$ is independent of σ, the model is *associative*, and the plastic strain rate obeys the usual normality flow rule (see [14])

$$\dot{p} \in N_K(\sigma),$$

where $N_K(\sigma)$ is the normal cone to K at $\sigma \in K$.

However, whenever significant volume variations accompany the plastic deformation, the principle of maximum plastic work is no longer valid and thus the associative flow rule should be abandoned in favor of a non-associative model. Various works [2, 4, 5, 7, 11] have recently tackled the issue of non-associativity and shown that many non-associative models can actually be viewed as "regular" models of elasto-plasticity, provided that the set of admissible stresses is allowed to depend on the stress field σ.

We next define the dissipation potential $H : \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ as the support function of $K(\sigma)$, that is,

$$H(\sigma, p) = \max_{\tau \in K(\sigma)} \tau : p,$$

and note that, for a fixed σ, it is convex, subadditive, and positively one-homogeneous in p. By standard convex analysis, the inclusion $\dot{p} \in \partial I_{K(\sigma)}(\sigma)$ is equivalent to $\sigma \in \partial_2 H(\sigma, \dot{p})$, where $\partial_2 H(\sigma, \dot{p})$ denotes the subdifferential of $\xi \mapsto H(\sigma, \xi)$ at $\dot{p}$. Thus, an equivalent formulation of the problem is

$$\begin{cases} Eu(t,x) = e(t,x) + p(t,x), \\ p(t,x) = (w(t,x) - u(t,x)) \odot \nu(x) \quad \text{on } \partial\Omega, \\ \sigma(t,x) = Ae(t,x), \\ \operatorname{div} \sigma(t,x) = 0, \\ \sigma(t,x) \in \partial_2 H(\sigma(t,x), \dot{p}(t,x)). \end{cases}$$

Note that the last condition implies the stress constraint $\sigma(t,x) \in K(\sigma(t,x))$ by the positive homogeneity of $\xi \mapsto H(\sigma, \xi)$.

In what follows, we will assume that:

(H0) The map $\xi \mapsto H(x, \xi)$ is convex and positively one-homogeneous;
(H1) The map H is continuous over $\mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym}$;
(H2) There exist $0 < \alpha_H < \beta_H < +\infty$ such that

$$B(0, \alpha_H) \subset K(\sigma) \subset B(0, \beta_H) \quad \text{for every } \sigma \in \mathbb{M}^{n\times n}_{sym}, \tag{2.1}$$

or still

$$\alpha_H |p| \le H(\sigma, p) \le \beta_H |p| \quad \text{for every } \sigma, p \in \mathbb{M}^{n\times n}_{sym}; \tag{2.2}$$

(H3) There exists a constant $C_H > 0$ such that

$$|H(\sigma_1, p) - H(\sigma_2, p)| \le C_H |p| |\sigma_1 - \sigma_2| \quad \text{for any } \sigma_1, \sigma_2, p \in \mathbb{M}^{n\times n}_{sym};$$

(H4) There exists a constant $C'_H > 0$ such that

$$|P_{K(\sigma_1)}(\tau) - P_{K(\sigma_2)}(\tau)| \le C'_H |\sigma_1 - \sigma_2| \quad \text{for any } \sigma_1, \sigma_2, \tau \in \mathbb{M}^{n\times n}_{sym},$$

where $P_{K(\sigma)}$ denotes the minimal distance projection onto the convex set $K(\sigma)$.

We introduce the perturbed dissipation potential $H_\varepsilon : \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym} \to [0, +\infty)$ defined, for each $\varepsilon > 0$, as

$$H_\varepsilon(\sigma, p) := H(\sigma, p) + \frac{\varepsilon}{2} |p|^2.$$

The convex conjugate $H^*_\varepsilon : \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym} \to [0, +\infty)$ of H_ε with respect to the second variable is defined by

$$H^*_\varepsilon(\sigma, \tau) := \sup_{p \in \mathbb{M}^{n\times n}_{sym}} \big\{ \tau : p - H_\varepsilon(\sigma, p) \big\}.$$

Using standard convex analysis, see [20, Theorem 16.4], one can show that

$$H^*_\varepsilon(\sigma,\tau) = \frac{|\tau - P_{K(\sigma)}(\tau)|^2}{2\varepsilon}.$$

In particular, H^*_ε is differentiable with respect to the second variable, and its partial derivative is given by

$$N_\varepsilon(\sigma,\tau) = \partial_2 H^*_\varepsilon(\sigma,\tau) = \frac{\tau - P_{K(\sigma)}(\tau)}{\varepsilon}.$$

Note that, since $0 \in K(\sigma)$ by (2.1), we have

$$|N_\varepsilon(\sigma,\tau)| \leq \frac{1}{\varepsilon}|\tau|,$$

and this implies that, for any σ, τ_1 and $\tau_2 \in \mathbb{M}^{n\times n}_{sym}$,

$$|H^*_\varepsilon(\sigma,\tau_1) - H^*_\varepsilon(\sigma,\tau_2)| \leq \frac{1}{\varepsilon}(|\tau_1| + |\tau_2|)|\tau_1 - \tau_2|.$$

Actually, N_ε is Lipschitz continuous. Indeed, we have the following result.

Lemma 2.1. *Let $C''_H := C'_H + 2$, where C'_H is the constant in (H4). Then*

$$|N_\varepsilon(\sigma_1,\tau_1) - N_\varepsilon(\sigma_2,\tau_2)| \leq \frac{C''_H}{\varepsilon}\left(|\sigma_1 - \sigma_2| + |\tau_1 - \tau_2|\right)$$

for any σ_1, σ_2, τ_1 and $\tau_2 \in \mathbb{M}^{n\times n}_{sym}$.

Proof. By definition of N_ε and since the projection is 1-Lipschitz continuous,

$$|N_\varepsilon(\sigma,\tau_1) - N_\varepsilon(\sigma,\tau_2)| \leq \frac{2}{\varepsilon}|\tau_1 - \tau_2|.$$

On the other hand, by (H4) we have

$$|N_\varepsilon(\sigma_1,\tau) - N_\varepsilon(\sigma_2,\tau)| \leq \frac{C'_H}{\varepsilon}|\sigma_1 - \sigma_2|,$$

so that we obtain the thesis. □

As a final note, given $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, we define the set

$$\mathcal{K}(\sigma) := \{\tau \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \tau(x) \in K(\sigma(x)) \text{ for a.e. } x \in \Omega\}.$$

Then, if $\tau \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$,

$$\|N_\varepsilon(\sigma,\tau)\|_2 = \frac{\mathrm{dist}_2(\tau,\mathcal{K}(\sigma))}{\varepsilon}, \tag{2.3}$$

where, for any closed set $\mathcal{C} \subset L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\mathrm{dist}_2(\tau,\mathcal{C})$ is the L^2-distance from τ to $\mathcal{C}$.

2.2. **Mathematical setting.** Throughout the paper, Ω is a bounded connected open set in $\mathbb{R}^n$ with Lipschitz boundary. The Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure are denoted by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$, respectively.

We use standard notation for Lebesgue and Sobolev spaces. In particular, for $1 \leq p \leq \infty$, the L^p-norms of the various quantities are denoted by $\|\cdot\|_p$. The space $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ is that of all $\mathbb{M}^{n\times n}_{sym}$-valued finite Radon measures on $\overline{\Omega}$, and the norm in that space is denoted by $\|\cdot\|_1$. By the Riesz Representation Theorem, $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ can be identified with the dual of $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. Finally, $BD(\Omega)$ stands for the space of functions with bounded deformations on Ω, *i.e.*, $u \in BD(\Omega)$ if $u \in L^1(\Omega;\mathbb{R}^n)$ and $Eu \in \mathcal{M}(\Omega;\mathbb{M}^{n\times n}_{sym})$ (the space of all $\mathbb{M}^{n\times n}_{sym}$-valued finite Radon measures on Ω), where $Eu := (Du + Du^T)/2$ and Du is the distributional derivative of u. We refer to [23] for general properties of that space.

Let A be a fourth order Hooke tensor satisfying the usual symmetry properties $A_{ijkh} = A_{jikh} = A_{khij}$ for every i, j, k, $h \in \{1, \ldots, n\}$, and

$$\alpha_A |\xi|^2 \le A\xi : \xi \le \beta_A |\xi|^2, \tag{2.4}$$

for some $0 < \alpha_A \le \beta_A < +\infty$ and every $\xi \in \mathbb{M}^{n\times n}_{sym}$. We define, for any $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, the elastic energy as

$$\mathcal{Q}(e) := \frac{1}{2} \int_\Omega Ae : e \, dx.$$

If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $p \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ we define the functionals

$$\mathcal{H}(\sigma, p) := \int_\Omega H(\sigma, p) \, dx, \qquad \mathcal{H}_\varepsilon(\sigma, p) := \int_\Omega H_\varepsilon(\sigma, p) \, dx,$$

while, if $\sigma \in \mathcal{C}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and $p \in \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ the first functional is defined as

$$\mathcal{H}(\sigma, p) := \int_{\overline{\Omega}} H\Big(\sigma, \frac{p}{|p|}\Big) \, d|p|,$$

where $p/|p|$ denotes the Radon-Nikodým derivative of p with respect to its variation measure $|p|$.

Remark 2.2. The following lower semi-continuity results hold: If $\{\sigma_k\} \subset \mathcal{C}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, $\{p_k\} \subset \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, $\sigma_k \to \sigma$ uniformly in $\overline{\Omega}$, and $p_k \rightharpoonup p$ weakly* in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, then

$$\mathcal{H}(\sigma, p) \le \liminf_{k\to+\infty} \mathcal{H}(\sigma_k, p_k).$$

Indeed, hypothesis (H3) implies that

$$|\mathcal{H}(\sigma, p_k) - \mathcal{H}(\sigma_k, p_k)| \le C_H \|\sigma - \sigma_k\|_\infty \|p_k\|_1 \to 0,$$

since $\{p_k\}$ is bounded in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$. Hence

$$\liminf_{k\to+\infty} \mathcal{H}(\sigma_k, p_k) = \liminf_{k\to+\infty} \mathcal{H}(\sigma, p_k).$$

Since $(x, \xi) \mapsto H(\sigma(x), \xi)$ is (lower semi-)continuous, while $\xi \mapsto H(\sigma(x), \xi)$ is convex and positively one-homogeneous, we infer from the Reshetnyak Lower Semi-continuity Theorem (see, *e.g.*, [1, Theorem 2.38]) that $\liminf_k \mathcal{H}(\sigma, p_k) \ge \mathcal{H}(\sigma, p)$. ¶

When letting the viscosity parameter tend to 0, we will only obtain weak convergence in L^2 of the approximating σ-sequence, and convergence in the space of measures of the approximating p-sequence.

Unfortunately, the Reshetnyak Lower Semi-continuity Theorem is false when H fails to be (lower semi)-continuous, so that we are pretty much forced to restrict our analysis to continuous stresses; but continuity is not preserved under L^2-weak convergence, which is the best we can hope for the various sequences of stresses that will enter the formulation. Consequently, the analysis will soon grind to a halt for lack of lower semi-continuity of H. This is why it was first proposed in [4] to perform a regularization of σ in the definition of $\mathcal{K}(\sigma)$. That was achieved by introducing a convolution kernel ρ and replacing $\mathcal{K}(\sigma)$ by $\mathcal{K}(\sigma * \rho)$ defined below.

We fix $\rho \in \mathcal{C}^1_c(\mathbb{R}^n)$ and set, for $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$,

$$x \in \overline{\Omega} \mapsto (\sigma * \rho)(x) := \int_\Omega \rho(x - y)\sigma(y) \, dy.$$

The convolution $\sigma * \rho$ defines an element in $\mathcal{C}^1(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$.

Note that, with that definition of the convolution, if $\sigma_k \rightharpoonup \sigma$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, then, in particular,

$$\sigma_k * \rho \to \sigma * \rho \quad \text{uniformly on } \overline{\Omega}.$$

With this kind of regularization, it proved possible to establish the following existence theorem for a quasi-static evolution in a rescaled time setting, see, *e.g.*, [2, Theorem 4.1 and page 289]. We extend the boundary datum $w \in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$ to $w(T)$ for $t \geq T$.

Theorem 2.3. *Let* $w \in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$ *and let* $(u_0,e_0,p_0) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ *be such that*

$$Eu_0 = e_0 + p_0 \quad \text{in } \Omega,$$
$$p_0 = (w(0) - u_0) \odot \nu\, \mathcal{H}^{n-1} \quad \text{on } \partial\Omega,$$

and

$$\operatorname{div} \sigma_0 = 0 \quad \text{in } \Omega, \qquad \sigma_0 \in \mathcal{K}(\sigma_0 * \rho),$$

where $\sigma_0 := Ae_0$. *Then, there exist* $\overline{T} > 0$ *and a mapping* $[0,\overline{T}] \ni s \mapsto (u^\circ(s), e^\circ(s), p^\circ(s), t^\circ(s))$ *such that*

$$u^\circ : [0,\overline{T}] \to BD(\Omega) \text{ is strongly continuous and a.e. weakly}^* \text{ differentiable;}$$
$$e^\circ : [0,\overline{T}] \to L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \text{ is strongly continuous and a.e. differentiable;}$$
$$p^\circ : [0,\overline{T}] \to \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) \text{ is 1-Lipschitz continuous;}$$
$$t^\circ : [0,\overline{T}] \to [0,+\infty) \text{ is nondecreasing and 1-Lipschitz continuous, with } t^\circ(\overline{T}) \geq T.$$

Further, setting $\sigma^\circ := Ae^\circ$, *the following properties are satisfied:*

Initial condition: $(u^\circ(0), e^\circ(0), p^\circ(0), t^\circ(0)) = (u_0, e_0, p_0, 0)$;

Kinematic compatibility: *For every* $s \in [0,\overline{T}]$,

$$Eu^\circ(s) = e^\circ(s) + p^\circ(s) \quad \text{in } \Omega,$$
$$p^\circ(s) = (w(t^\circ(s)) - u^\circ(s)) \odot \nu\, \mathcal{H}^{n-1} \quad \text{on } \partial\Omega;$$

Equilibrium condition: *For every* $s \in [0,\overline{T}]$,

$$\operatorname{div} \sigma^\circ(s) = 0 \quad \text{in } \Omega;$$

Partial stress constraint: *For every* $s \in [0,\overline{T}] \setminus U^\circ$,

$$\sigma^\circ(s) \in \mathcal{K}(\sigma^\circ(s) * \rho),$$

where $U^\circ := \{s \in (0,\overline{T}] : t^\circ$ *is constant in a neighborhood of* $s\}$;

Energy equality: *For every* $S \in [0,\overline{T}]$,

$$\mathcal{Q}(e^\circ(S)) + \int_0^S \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s))\, ds + \int_0^S \|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho))\, ds$$
$$= \mathcal{Q}(e^\circ(0)) + \int_0^S \int_\Omega \sigma^\circ(s) : E\dot{w}^\circ(s)\, dx\, ds.$$

Interpreting this evolution in "real" time is a non trivial task that was partially undertaken in [5]. But much information is lost because it may be so that there is no interval of finite length lying entirely outside U°. Also, as mentioned before, there is a certain degree of arbitrariness in the rescaling which immediately begs the question of the dependence of the jump times (the image of U° through t°) upon said rescaling.

Our goal in what follows is to show that a slightly different regularization completely alleviates the need for rescaling in time and permits to obtain a real time evolution with well defined jump times.

We thus introduce $\rho \in \mathcal{C}^1_c(\mathbb{R}^{n+1})$ with $0 \le \rho(t,x) \le 1$ and $\int_0^\infty \int_{\mathbb{R}^n} \rho(t,x)\,dx\,dt = 1$. Set, for $\sigma \in L^\infty(-\infty,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$,

$$(t,x) \in (-\infty,T] \times \overline{\Omega} \mapsto (\sigma * \rho)(t,x) := \int_{-\infty}^t \int_\Omega \rho(t-s,x-y)\sigma(s,y)\,dy\,ds.$$

The convolution $\sigma * \rho$ defines an element in $\mathcal{C}((-\infty,T]\times\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) \cap L^\infty(-\infty,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. Also remark that, with the above definition of the convolution, if

$$\sigma_k \rightharpoonup \sigma \quad \text{weakly}^* \text{ in } L^\infty(-\infty,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),$$

then, in particular,

$$\sigma_k * \rho \to \sigma * \rho \quad \text{uniformly on } [0,T]\times\overline{\Omega}. \tag{2.5}$$

Indeed, we clearly have pointwise convergence. On the other hand, the functions $\sigma_k * \rho$ are equicontinuous on $[0,T]\times\overline{\Omega}$, so that the thesis follows from the Ascoli-Arzelà Theorem.

For now, we address in the next section the visco-plastic regularization.

3. The visco-plastic model

We propose in this short section to establish existence of the solution to the visco-plastic regularization. In contrast with the case of a space only regularization, it is not possible to proceed through a time incremental process because of the temporal non-local dependence of $\sigma * \rho$ on σ. Rather our proof will be based on an actually more direct and faster fixed point argument.

Consider a boundary displacement $w \in H^1(\Omega;\mathbb{R}^n)$. We set

$$A_{\rm reg}(w) := \Big\{(v,\eta,q) \in H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \\ Ev = \eta + q \text{ a.e. in } \Omega,\ v = w\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\Big\}. \tag{3.1}$$

The main result of this section is the following existence result for the non-associative visco-plastic evolution.

Theorem 3.1. *Let $w \in W^{1,\infty}(0,T;H^1(\Omega,\mathbb{R}^n))$, let $(u_0,e_0,p_0) \in A_{\rm reg}(w(0))$ be such that $\operatorname{div}\sigma_0 = 0$ in Ω, where $\sigma_0 := Ae_0$, and let $\varepsilon > 0$. Then, there exists a unique triplet $(u_\varepsilon(t),e_\varepsilon(t),p_\varepsilon(t)) \in A_{\rm reg}(w(t))$ with*

$$u_\varepsilon \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n)), \quad e_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \quad p_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),$$

such that, setting

$$\sigma_\varepsilon(t) := \begin{cases} Ae_\varepsilon(t) & \text{for } 0 < t \le T, \\ Ae_0 & \text{for } t \le 0, \end{cases}$$

the following conditions are satisfied:

Initial condition: $(u_\varepsilon(0),e_\varepsilon(0),p_\varepsilon(0)) = (u_0,e_0,p_0)$;

Kinematic compatibility: *For every $t \in [0,T]$,*

$$Eu_\varepsilon(t) = e_\varepsilon(t) + p_\varepsilon(t) \quad \text{a.e. in } \Omega,$$
$$u_\varepsilon(t) = w(t) \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega;$$

Equilibrium condition: *For every $t \in [0,T]$,*

$$\operatorname{div}\sigma_\varepsilon(t) = 0 \quad \text{in } \Omega;$$

Regularized non-associative flow rule: *For a.e. $t \in [0,T]$,*

$$\dot p_\varepsilon(t) = N_\varepsilon((\sigma_\varepsilon * \rho)(t),\sigma_\varepsilon(t)) \quad \text{for a.e. } x \in \Omega,$$

or equivalently,

$$\sigma_\varepsilon(t) - \varepsilon\dot p_\varepsilon(t) \in \partial_2 H((\sigma_\varepsilon * \rho)(t),\dot p_\varepsilon(t)) \quad \text{for a.e. } x \in \Omega.$$

In particular, $\varepsilon\|\dot p_\varepsilon(t)\|_2 = \mathrm{dist}_2(\sigma_\varepsilon(t), \mathcal{K}((\sigma_\varepsilon * \rho)(t)))$.

We call such a triplet a non-associative visco-plastic solution.

Proof. Let

$$\mathcal{A} : L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})) \to L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$$

be the operator that maps $p \in L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ into $\mathcal{A}(p) \in L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ defined as follows: for a.e. $t \in [0,T]$

$$\mathcal{A}(p)(t) := Eu(t) - p(t),$$

where $u(t) \in H^1(\Omega;\mathbb{R}^n)$ is the solution of $\operatorname{div} A(Eu(t) - p(t)) = 0$ in Ω, $u(t) = w(t)$ on $\partial\Omega$. Let

$$\mathcal{K}_\varepsilon : L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})) \to L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$$

be the operator defined as follows: given $e \in L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, we first extend e to $(-\infty,0)$ by setting $e(t) := e_0$ for every $t < 0$; we then define $\mathcal{K}_\varepsilon(e) \in L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ by

$$\mathcal{K}_\varepsilon(e)(t) := p_0 + \int_0^t N_\varepsilon((Ae) * \rho(s), Ae(s))\,ds$$

for a.e. $t \in [0,T]$. In view of Lemma 2.1, (2.4), and the properties of ρ, we have that

$$\begin{aligned}\|\mathcal{K}_\varepsilon(e_1)(t) - \mathcal{K}_\varepsilon(e_2)(t)\|_2 &\le \frac{C''_H C_\rho}{\varepsilon}\int_0^t \big(\|\sigma_1 - \sigma_2\|_{L^\infty(0,t;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} + \|\sigma_1(s) - \sigma_2(s)\|_2\big)\,ds \\ &\le \beta_A \frac{C''_H C_\rho}{\varepsilon}\, t\, \|e_1 - e_2\|_{L^\infty(0,t;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}, \qquad (3.2)\end{aligned}$$

where C_ρ is a constant depending on ρ and we have set $\sigma_i := Ae_i$, $i = 1,2$. By classical elliptic estimates we have

$$\|\mathcal{A}(p_1)(t) - \mathcal{A}(p_2)(t)\|_2 \le \Big(1 + \frac{\beta_A}{\alpha_A}\Big)\|p_1(t) - p_2(t)\|_2. \qquad (3.3)$$

Combining (3.2) and (3.3) we get

$$\|(\mathcal{A}\circ\mathcal{K}_\varepsilon)(e_1) - (\mathcal{A}\circ\mathcal{K}_\varepsilon)(e_2)\|_{L^\infty(0,t;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))} \le \Big(1 + \frac{\beta_A}{\alpha_A}\Big)\beta_A \frac{C''_H C_\rho}{\varepsilon}\, t\, \|e_1 - e_2\|_{L^\infty(0,t;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))}.$$

Thus for t small enough, $\mathcal{A}\circ\mathcal{K}_\varepsilon$ is a strict contraction on $L^\infty(0,t;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. Hence there exists a unique fixed point $e_\varepsilon \in L^\infty(0,t;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ and, consequently, there exists a unique triplet $(u_\varepsilon(s), e_\varepsilon(s), p_\varepsilon(s)) \in A_{\mathrm{reg}}(w(s))$ satisfying for a.e. $s \in (0,t)$ the kinematic compatibility, the equilibrium condition and the equality

$$p_\varepsilon(s) = p_0 + \int_0^s N_\varepsilon((Ae_\varepsilon) * \rho(r), Ae_\varepsilon(r))\,dr.$$

But this implies in turn that $p_\varepsilon \in W^{1,\infty}(0,t;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, hence the additional time regularity on $(u_\varepsilon(s), e_\varepsilon(s), p_\varepsilon(s))$, and that the regularized non-associative flow rule is satisfied. We deduce that the kinematic compatibility and the equilibrium condition hold for every time $s \in [0,t]$, as well as the initial condition. Since the interval $(0,t)$ on which the fixed point argument holds, does not depend upon the initial conditions on the triplet $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$, we can iterate the argument and obtain a solution over the whole interval $(0,T)$. □

We end this subsection with two propositions stating some useful properties of visco-plastic regularized evolutions. The proof of the first proposition is very close to that of [4, Theorem 3.4] and is omitted here.

Proposition 3.2. *Under the assumptions of Theorem 3.1, let* $t \mapsto (u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in A_{\rm reg}(w(t))$ *with*

$$u_\varepsilon \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n)), \quad e_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \quad p_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),$$

satisfy the initial condition, the kinematic compatibility, and the equilibrium condition in Theorem 3.1. Then $t \mapsto (u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$ *satisfies the regularized non-associative flow rule in Theorem 3.1 if and only if the following two conditions are satisfied:*

1. Modified Stress Constraint*:* $\sigma_\varepsilon(t) - \varepsilon\dot p_\varepsilon(t) \in \mathcal{K}((\sigma_\varepsilon * \rho)(t))$ *for a.e.* $t \in [0,T]$*, or equivalently, since* $K((\sigma*\rho)(t)) = \partial_2 H((\sigma*\rho)(t),0)$*,*
$$\sigma_\varepsilon(t,x) - \varepsilon\dot p_\varepsilon(t,x) \in \partial_2 H((\sigma_\varepsilon * \rho)(t,x),0) \quad \text{for a.e. } (t,x) \in [0,T]\times\Omega;$$
2. Energy equality*: for every* $t \in [0,T]$
$$\mathcal{Q}(e_\varepsilon(t)) + \int_0^t \mathcal{H}((\sigma_\varepsilon*\rho)(s), \dot p_\varepsilon(s))\,ds + \varepsilon\int_0^t \|\dot p_\varepsilon(s)\|_2^2\,ds = \mathcal{Q}(e_0) + \int_0^t\int_\Omega \sigma_\varepsilon(s) : E\dot w(s)\,dx\,ds,$$
or equivalently,
$$\mathcal{Q}(e_\varepsilon(t)) + \int_0^t \mathcal{H}((\sigma_\varepsilon*\rho)(s), \dot p_\varepsilon(s))\,ds + \int_0^t \|\dot p_\varepsilon(s)\|_2 \,\mathrm{dist}_2(\sigma_\varepsilon(s), \mathcal{K}((\sigma_\varepsilon*\rho)(s)))\,ds$$
$$= \mathcal{Q}(e_0) + \int_0^t\int_\Omega \sigma_\varepsilon(s) : E\dot w(s)\,dx\,ds.$$

In view of the energy equality in Proposition 3.2 and of (2.2), (2.3), (2.4), we immediately obtain the following

Proposition 3.3. *Let* $t \mapsto (u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$ *be a visco-plastic regularised evolution according to Theorem 3.1. Then*

$$\sup_{t\in[0,T]} \|e_\varepsilon(t)\|_2 \le C_T, \quad \sup_{t\in[0,T]} \|\sigma_\varepsilon(t)\|_2 \le C_T, \quad \int_0^T \|\dot p_\varepsilon(s)\|_1\,ds \le C_T, \quad \varepsilon\int_0^T \|\dot p_\varepsilon(s)\|_2^2\,ds \le C_T, \tag{3.4}$$

hence also

$$\|p^\varepsilon\|_{BV([0,T];L^1(\Omega;\mathbb{M}^{n\times n}_{sym}))} \le C_T, \tag{3.5}$$

where C_T *is an* ε*-independent constant.*

4. The evolution

In this section we propose to pass to the limit in the visco-platic evolution obtained in Theorem 3.1, as the viscosity parameter ε goes to 0.

Because of the bounds in Proposition 3.3, we cannot expect to keep the L^2-regularity of the fields Eu and p when passing to the 0-viscosity limit and we thus have to redefine $A_{\rm reg}(w)$ from (3.1) as

$$A(w) := \Big\{(v,\eta,q) \in BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) :$$
$$Ev = \eta + q \text{ in } \Omega,\ q = (w-v)\odot\nu\mathcal{H}^{n-1} \text{ on } \partial\Omega\Big\}$$

with $w \in H^1(\Omega;\mathbb{R}^n)$. The interpretation of the boundary condition is that, if the displacement u does not match the prescribed boundary displacement w, then the loaded boundary can experience plastic slips.

We still keep a boundary datum $w \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n))$. The main result of the paper is the following existence result for the regularized quasistatic evolution model in non-associative plasticity.

Theorem 4.1. *Let* $w \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n))$ *and let* $(u_0,e_0,p_0) \in A(w(0))$ *be such that*

$$\operatorname{div}\sigma_0 = 0 \quad \text{in } \Omega, \qquad \sigma_0 \in \mathcal{K}((\sigma_0 * \rho)(0)), \tag{4.1}$$

where $\sigma_0(t) := Ae_0$ *for* $t \le 0$. *Then, there exists a mapping* $[0,T] \ni t \mapsto (u(t),e(t),p(t))$ *with* $p \in BV([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ *such that, setting*

$$\sigma(t) := \begin{cases} Ae(t) & \text{for } 0 < t \le T, \\ Ae_0 & \text{for } t \le 0, \end{cases}$$

the following properties are satisfied:

Initial condition: $(u(0),e(0),p(0)) = (u_0,e_0,p_0)$;

Kinematic compatibility: *For every* $t \in [0,T]$,

$$Eu(t) = e(t) + p(t) \quad \text{in } \Omega,$$
$$p(t) = (w(t)) - u(t)) \odot \nu\, \mathcal{H}^{n-1} \quad \text{on } \partial\Omega;$$

Equilibrium condition: *For every* $t \in [0,T]$,

$$\operatorname{div}\sigma(t) = 0 \quad \text{in } \Omega;$$

Partial stress constraint: *For every* $t \in [0,T] \setminus N$,

$$\sigma(t) \in \mathcal{K}((\sigma * \rho)(t)),$$

where

N *is the countable set of jumps of* $t \mapsto \operatorname{Var}(p;0,t) :=$

$$= \sup\Big\{ \sum_{i=1}^n \|p(t_i) - p(t_{i-1})\|_1 : \ t_0 = 0 \le t_1 \le \ldots \le t_n = t, \ n \in \mathbb{N}\Big\}; \tag{4.2}$$

Energy equality: *For every* $t \in [0,T]\backslash N$,

$$\mathcal{Q}(e(t)) + \int_{[0,t]\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p| = \mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma(s) : E\dot w(s)\, dx\, ds, \tag{4.3}$$

where $|p|$ *is the variation measure associated with* p *viewed as a measure on* $[0,T]\times\overline{\Omega}$;

Regularity: *The maps* $t \mapsto u(t)$ *and* $t \mapsto e(t)$ *are continuous at all points of* $[0,T]\setminus N$ *in the strong topology of* $BD(\Omega)$ *and of* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, *respectively, while* $|\dot p|$ *does not charge* $\{t\}\times\overline{\Omega}$ *for* $t \in [0,T]\setminus N$;

Condition at jumps: *for every* $t \in N$

$$\mathcal{Q}(e(t+)) - \mathcal{Q}(e(t-)) + \int_{\{t\}\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p| = 0, \tag{4.4}$$

where $e(t-)$ *and* $e(t+)$ *denote the left and the right limit of the map* $t \mapsto e(t)$ *at time* t *with respect to the strong topology of* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$.

Proof. According to [4, Lemma 5.1], there exists a sequence $\{u_0^\varepsilon\}$ in $H^1(\Omega;\mathbb{R}^n)$ such that $u_0^\varepsilon = w(0)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $u_0^\varepsilon \to u_0$ strongly in $L^1(\Omega;\mathbb{R}^n)$, and $Eu_0^\varepsilon \rightharpoonup Eu_0$ weakly* in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. Setting $p_0^\varepsilon := Eu_0^\varepsilon - e_0$, we have that $(u_0^\varepsilon, e_0, p_0^\varepsilon) \in A_{\rm reg}(w(0))$, $u_0^\varepsilon \rightharpoonup u_0$ weakly* in $BD(\Omega)$, and $p_0^\varepsilon \rightharpoonup p_0$ weakly* in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. By Theorem 3.1 for every $\varepsilon > 0$ there exists a unique non-associative visco-plastic solution $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$ with initial datum $(u_0^\varepsilon, e_0, p_0^\varepsilon)$. By Proposition 3.3 the bounds (3.4) and (3.5) are satisfied.

Step 1 – Compactness and continuity. By application of the Helly Theorem (see [15]) to (3.5), there exists an element $p \in BV([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ and a subsequence of $\{p_\varepsilon\}$ (still indexed by ε) such that, for every $t \in [0,T]$,

$$p_\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly}^* \text{ in } \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}). \tag{4.5}$$

We extend $p_\varepsilon(t)$ to p_0^ε for $t<0$ and to $p_\varepsilon(T)$ for $t>T$, so that $\dot p_\varepsilon(t)=0$ for $t<0$ and for $t>T$. In view of the third bound in (3.4), we have that

$$\dot p_\varepsilon \rightharpoonup \dot p \quad \text{weakly}^* \text{ in } \mathcal{M}([0,T]\times\overline{\Omega};\mathbb{M}^{n\times n}_{sym}), \tag{4.6}$$

that is,

$$\int_{[0,T]\times\overline{\Omega}} \phi : d\dot p_\varepsilon \to \int_{[0,T]\times\overline{\Omega}} \phi : d\dot p$$

for every test function $\phi \in \mathcal{C}([0,T]\times\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$.

By the first two bounds in (3.4) and the Korn-Poincaré Inequality in $BD(\Omega)$ we deduce that, at any given time t, there exists a time-dependent subsequence $\{\varepsilon_t\}\subset\{\varepsilon\}$ such that

$$\begin{cases} e_{\varepsilon_t}(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),\\ \sigma_{\varepsilon_t}(t) \rightharpoonup \sigma(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \quad \text{with } \operatorname{div}\sigma(t)=0,\\ u_{\varepsilon_t}(t) \rightharpoonup u(t) \quad \text{weakly}^* \text{ in } BD(\Omega),\end{cases}$$

with $(u(t),e(t),p(t)) \in A(w(t))$. In fact, there is no need for subsequence extraction. Indeed, for any $\varphi \in H^1_0(\Omega;\mathbb{R}^n)$ we infer from $\operatorname{div}\sigma(t)=0$ that

$$\mathcal{Q}(e(t)) \le \mathcal{Q}(e(t)+E\varphi). \tag{4.7}$$

Thus, if another subsequence $\{\varepsilon'_t\}$ is such that $e_{\varepsilon'_t} \rightharpoonup e'(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $u_{\varepsilon_t} \rightharpoonup u'(t)$ weakly* in $BD(\Omega)$ with $(u'(t),e'(t),p(t)) \in A(w(t))$, then $u(t)-u'(t)$ is immediately seen to be in $H^1_0(\Omega;\mathbb{R}^n)$ with $E(u(t)-u'(t)) = e(t)-e'(t)$. We thus get from (4.7) that $\mathcal{Q}(e(t)) \le \mathcal{Q}(e'(t))$, hence switching the roles of e and e' that $\mathcal{Q}(e(t)) = \mathcal{Q}(e'(t))$. Using the strict convexity of $\mathcal{Q}$ we conclude that $e(t)=e'(t)$, from which we also have that $u(t)=u'(t)$.

We have therefore proved that for every $t\in[0,T]$

$$\begin{cases} e_\varepsilon(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),\\ \sigma_\varepsilon(t) \rightharpoonup \sigma(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \quad \text{with } \operatorname{div}\sigma(t)=0,\\ u_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly}^* \text{ in } BD(\Omega).\end{cases} \tag{4.8}$$

In particular, the initial condition is satisfied.

Further, because of the L^∞-bounds in (3.4) on e_ε and σ_ε and of (4.8), $t\mapsto e(t)$ and $t\mapsto\sigma(t)$ are weakly measurable (with values in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$), hence strongly measurable, so that

$$\begin{cases} e_\varepsilon \rightharpoonup e \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ \sigma_\varepsilon \rightharpoonup \sigma \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})).\end{cases} \tag{4.9}$$

Let N be the countable set of discontinuity points of $t\mapsto \operatorname{Var}(p;0,t)$. Then the map $t\mapsto p(t)$ is continuous in the strong topology of $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ at all times in $[0,T]\setminus N$. We now show that

$$t\mapsto (e(t),\sigma(t)) \text{ is weakly continuous in } [L^2(\Omega;\mathbb{M}^{n\times n}_{sym})]^2 \text{ at all points of } [0,T]\setminus N \tag{4.10}$$

and

$$t\mapsto u(t) \text{ is weakly}^* \text{ continuous in } BD(\Omega) \text{ at all points of } [0,T]\setminus N.$$

Indeed, let $t_k\to t$ with $t\notin N$. Since

$$\sup_{s\in[0,T]} \|e(s)\|_2 \le C_T, \tag{4.11}$$

for a subsequence $e(t_{k_j}) \rightharpoonup e^*$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, with $\operatorname{div} Ae^* = 0$ in Ω. As $t \notin N$, we have that $p(t_{k_j}) \to p(t)$ strongly in $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. Hence, $u(t_{k_j}) \rightharpoonup u^*$ weakly* in $BD(\Omega)$ with $(u^*, e^*, p(t)) \in A(w(t))$. From the previous minimality argument we deduce that $u^* = u(t)$ and $e^* = e(t)$. Hence, the whole sequences $\{e(t_k)\}$ and $\{u(t_k)\}$ converge to $e(t)$ and $u(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and weakly* in $BD(\Omega)$, respectively.

Step 2 – Stress constraint. Recall the modified stress constraint from Proposition 3.2, namely

$$\sigma_\varepsilon(t,x) - \varepsilon\dot{p}_\varepsilon(t,x) \in \partial_2 H((\sigma_\varepsilon * \rho)(t,x), 0) \quad \text{for a.e. } (t,x) \in [0,T]\times\Omega.$$

Then, for any $q \in \mathbb{M}^{n\times n}_{sym}$,

$$H((\sigma_\varepsilon * \rho)(t,x), q) \geq (\sigma_\varepsilon(t,x) - \varepsilon\dot{p}_\varepsilon(t,x)) : q \quad \text{for a.e. } (t,x) \in [0,T]\times\Omega,$$

hence, for any measurable set $E \subset \Omega$ and for a.e. t,

$$\int_E H((\sigma_\varepsilon * \rho)(t,x), q)\, dx \geq \int_E (\sigma_\varepsilon(t,x) - \varepsilon\dot{p}_\varepsilon(t,x)) : q\, dx.$$

Because of the second convergence in (4.9), $(\sigma_\varepsilon * \rho)(t) \to (\sigma * \rho)(t)$ uniformly in $\overline{\Omega}$ for every $t \in [0,T]$. Thanks to hypothesis (H3), (4.8) and the fourth bound in (3.4), we deduce that

$$\int_E H((\sigma * \rho)(t,x), q)\, dx \geq \int_E \sigma(t,x) : q\, dx$$

for a.e. $t \in [0,T]$. By the weak continuity (4.10) of $t \mapsto \sigma(t)$ in $[0,T]\setminus N$, we conclude that the previous relation actually holds for every $t \in [0,T]\setminus N$. Thus, for $t \in [0,T]\setminus N$,

$$H((\sigma * \rho)(t,x), q) \geq \sigma(t,x) : q \quad \text{for a.e. } x \in \Omega,$$

or, equivalently,

$$\sigma(t) \in \mathcal{K}((\sigma * \rho)(t)). \tag{4.12}$$

Now take $t_k \to t$ with $t_k, t \notin N$. Then, in view of (4.12), $\sigma(t_k) \in \mathcal{K}((\sigma * \rho)(t_k))$, while $\operatorname{div}\sigma(t_k) = 0$, so that, appealing to, *e.g.*, [10, Proposition 3.9, (6.5) and (6.20)] and using (2.2),

$$\begin{aligned}
&\beta_H\big|\mathrm{Var}(p;0,t) - \mathrm{Var}(p;0,t_k)\big| \geq \mathcal{H}((\sigma * \rho)(t_k), p(t) - p(t_k)) \\
&\geq \; -\int_\Omega \sigma(t_k) : (e(t) - e(t_k))\, dx + \int_\Omega \sigma(t_k) : (Ew(t) - Ew(t_k))\, dx \\
&\geq \; \frac{1}{2}\int_\Omega Ae(t_k) : e(t_k)\, dx - \frac{1}{2}\int_\Omega Ae(t) : e(t)\, dx + \int_\Omega \sigma(t_k) : (Ew(t) - Ew(t_k))\, dx,
\end{aligned}$$

or still

$$\begin{aligned}
\frac{1}{2}\int_\Omega Ae(t_k) : e(t_k)\, dx \leq \frac{1}{2}\int_\Omega Ae(t) : e(t)\, dx - \int_\Omega \sigma(t_k) : (Ew(t) - Ew(t_k))\, dx \\
+ \beta_H\big|\mathrm{Var}(p;0,t) - \mathrm{Var}(p;0,t_k)\big|.
\end{aligned}$$

Recalling (2.4) and the weak continuity (4.10) of $t \mapsto e(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ at $t \in [0,T]\setminus N$, we obtain that

$$e(t_k) \to e(t) \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \text{ for every } \{t_k\} \to t \text{ with } t_k, t \in [0,T]\setminus N. \tag{4.13}$$

Step 3 – Lower semi-continuity of the dissipated energy. We now show that

$$\liminf_{\varepsilon\to 0} \int_0^t \mathcal{H}((\sigma_\varepsilon * \rho)(s), \dot{p}_\varepsilon(s))\, ds \geq \int_{[0,t)\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot{p}}{|\dot{p}|}(s,x)\Big)\, d|\dot{p}| \tag{4.14}$$

for every $t \in [0,T)$, and

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{H}((\sigma_\varepsilon * \rho)(s), \dot p_\varepsilon(s))\, ds \ge \int_{[0,T]\times\overline{\Omega}} H\Big((\sigma*\rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p|. \tag{4.15}$$

Indeed, for $t \in [0,T]$ we have from (H3) that

$$\int_0^t \mathcal{H}((\sigma_\varepsilon * \rho)(s), \dot p_\varepsilon(s))\, ds \ge \int_0^t \mathcal{H}((\sigma * \rho)(s), \dot p_\varepsilon(s))\, ds - \sup_{s\in[0,t]} \|(\sigma_\varepsilon*\rho)(s) - (\sigma*\rho)(s)\|_\infty \int_0^t \|\dot p_\varepsilon(s)\|_1\, ds.$$

Now, in view of the second convergence in (4.9) and of (2.5), for any $\eta > 0$ there exists some $\bar\varepsilon_\eta > 0$ such that

$$\sup_{0<\varepsilon<\bar\varepsilon_\eta} \sup_{s\in[0,t]} \|(\sigma_\varepsilon*\rho)(s) - (\sigma*\rho)(s)\|_\infty \le \eta.$$

Consequently, using the third bound in (3.4), we have

$$\liminf_{\varepsilon\to 0} \int_0^t \mathcal{H}((\sigma_\varepsilon * \rho)(s), \dot p_\varepsilon(s))\, ds \ge \liminf_{\varepsilon\to 0} \int_0^t \mathcal{H}((\sigma * \rho)(s), \dot p_\varepsilon(s))\, ds - C_T\eta.$$

Now, for every $t \in [0,T]$, $[0,t)\times\overline{\Omega}$ is a locally compact set and (4.6) implies the weak* convergence of $\dot p_\varepsilon$ to $\dot p$ in $\mathcal{M}([0,t)\times\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, that is, against any test function in $C_0([0,t)\times\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$. Thus, in view of the assumptions (H0), (H1) on H, application of the Reshetnyak Theorem in the inequality above is licit; letting $\eta \searrow 0$ yields (4.14).

For $t = T$, we can apply the Reshetnyak Theorem on $[0,T]\times\overline{\Omega}$ and obtain (4.15) by virtue of the weak* convergence of $\dot p_\varepsilon$ to $\dot p$ in $\mathcal{M}([0,T]\times\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$.

Step 4 – Energy equality. We first pass to the $\liminf_\varepsilon$ in the energy equality of Proposition 3.2. Using the convergences in (4.8), together with the second bound in (3.4) and inequality (4.14), we immediately get

$$\mathcal{Q}(e(t)) + \int_{[0,t)\times\overline{\Omega}} H\Big((\sigma*\rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p| + \liminf_{\varepsilon} \varepsilon \int_0^t \|\dot p_\varepsilon(s)\|_2^2\, ds \le \mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma(s) : E\dot w(s)\, dx\, ds \tag{4.16}$$

for every $t \in [0,T)$. Similarly, using (4.15), we obtain

$$\mathcal{Q}(e(T)) + \int_{[0,T]\times\overline{\Omega}} H\Big((\sigma*\rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p| + \liminf_{\varepsilon} \varepsilon \int_0^T \|\dot p_\varepsilon(s)\|_2^2\, ds \le \mathcal{Q}(e_0) + \int_0^T \int_\Omega \sigma(s) : E\dot w(s)\, dx\, ds. \tag{4.17}$$

Conversely, let λ be the weak* limit of $|\dot p_\varepsilon|$ in $\mathcal{M}([0,T]\times\overline{\Omega})$. We introduce the set

$$M := N \cup \big\{t \in [0,T] :\ \lambda(\{t\}\times\overline{\Omega}) \ne 0\big\}$$

and we note that M is a countable set. We also remark that, since $\operatorname{div}\sigma(t) = 0$ for every t, while (4.12) holds at $t \notin N$ and also at $t = 0$ in view of (4.1), minimality also holds at all $t \notin N$ and at $t = 0$, that is, for any $(v,\eta,q) \in A(w(t))$

$$\mathcal{Q}(e(t)) \le \mathcal{Q}(\eta) + \mathcal{H}((\sigma*\rho)(t), q - p(t)), \tag{4.18}$$

see, *e.g.*, [3, Theorem 3.6].

Let $t \in (0,T]$. Consider then a (k-indexed) sequence of partitions $\{s_i^k\}_{i=0,\dots,n_k}$ of $[0,t]$ with $s_0^k = 0$ and $s_{n_k}^k = t$ such that $s_i^k \notin M$ for every $i \neq 0, n_k$ and $\lim_{k\to\infty} \max_i (s_i^k - s_{i-1}^k) = 0$. In view of (4.18), the regularity of w implies that, at s_i^k, $i \neq n_k$,

$$\begin{aligned}\mathcal{Q}(e(s_i^k)) &\le \mathcal{Q}(e(s_{i+1}^k) + Ew(s_i^k) - Ew(s_{i+1}^k)) + \mathcal{H}((\sigma * \rho)(s_i^k), p(s_{i+1}^k) - p(s_i^k)) \\ &\le \mathcal{Q}(e(s_{i+1}^k)) - \int_{s_i^k}^{s_{i+1}^k} \int_\Omega \sigma(s_{i+1}^k) : E\dot{w}(s)\, dx\, ds \\ &\quad + \mathcal{H}((\sigma * \rho)(s_i^k), p(s_{i+1}^k) - p(s_i^k)) + o(1) \int_{s_i^k}^{s_{i+1}^k} \int_\Omega |E\dot{w}(s)|^2\, dx\, ds.\end{aligned}$$

Iterating yields, with $\sigma^k(s) := \sigma(s_{i+1}^k)$ for $s \in [s_i^k, s_{i+1}^k)$,

$$\begin{aligned}&\mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma^k(s) : E\dot{w}(s)\, dx\, ds \\ &\qquad \le \mathcal{Q}(e(t)) + \sum_i \mathcal{H}((\sigma * \rho)(s_i^k), p(s_{i+1}^k) - p(s_i^k)) + o(1) \int_0^t \int_\Omega |E\dot{w}(s)|^2\, dx\, ds.\end{aligned}$$

By (4.10), (4.11), and the Dominated Convergence Theorem we obtain that, as $k \to \infty$,

$$\int_0^t \int_\Omega \sigma^k(s) : E\dot{w}(s)\, dx\, ds \to \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s)\, dx\, ds.$$

We then get

$$\mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s)\, dx\, ds \le \mathcal{Q}(e(t)) + \widetilde{\mathrm{Var}}_H(p; 0, t), \tag{4.19}$$

where

$$\widetilde{\mathrm{Var}}_H(p; 0, t) := \liminf_{k\to\infty} \sum_i \mathcal{H}\big((\sigma * \rho)(s_i^k), p(s_{i+1}^k) - p(s_i^k)\big).$$

We will show below that for $t \notin M$ and for $t = T$,

$$\widetilde{\mathrm{Var}}_H(p; 0, t) \le \int_{[0,t]\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot{p}}{|\dot{p}|}(s,x)\Big)\, d|\dot{p}|. \tag{4.20}$$

Assuming for now that (4.20) holds true, we get from (4.19) that

$$\mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s)\, dx\, ds \le \mathcal{Q}(e(t)) + \int_{[0,t]\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot{p}}{|\dot{p}|}(s,x)\Big)\, d|\dot{p}| \tag{4.21}$$

for $t \notin M$ and for $t = T$. Comparing (4.17) with (4.21) at $t = T$ immediately leads to

$$\liminf_\varepsilon \varepsilon \int_0^T \|\dot{p}_\varepsilon(s)\|_2^2\, ds = 0.$$

Combining now (4.16) and (4.21) yields the following energy equality for $t \notin M$:

$$\mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s)\, dx\, ds = \mathcal{Q}(e(t)) + \int_{[0,t]\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot{p}}{|\dot{p}|}(s,x)\Big)\, d|\dot{p}|, \tag{4.22}$$

where we used that $\lambda \ge |\dot{p}|$ by [1, Proposition 1.62], hence $|\dot{p}|(\{t\} \times \overline{\Omega}) = 0$ for $t \notin M$. This completes the proof of the energy equality, except for those $t \in M \setminus N$. For such t's we consider a decreasing sequence $\{t_k\} \subset [0,T] \setminus M$ that converges to t. The energy equality (4.22) holds at such t_k for every k. In view of (4.13) we immediately conclude that the same equality holds at t.

Moreover,

$$t \mapsto e(t) \text{ is strongly continuous in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \text{ at all points of } [0,T] \setminus N \tag{4.23}$$

and

$$t \mapsto u(t) \text{ is strongly continuous in } BD(\Omega) \text{ at all points of } [0,T]\setminus N. \tag{4.24}$$

Indeed, let $t \in [0,T]\setminus N$ and let $\{t_k\}$ be any sequence converging to t. We can always find a sequence $\{s_k\} \to t$ such that $(s_k) \subset [0,T]\setminus N$ and $s_k < t_k$ for every k. Applying (4.16) at t_k and (4.22) at s_k we obtain

$$\mathcal{Q}(e(t_k)) + \int_{(s_k,t_k)\times\overline{\Omega}} H\Big((\sigma*\rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p| \le \mathcal{Q}(e(s_k)) + \int_{s_k}^{t_k}\int_\Omega \sigma(s) : E\dot w(s)\, dx\, ds.$$

Since the second term on the left-handside is nonnegative, we deduce that

$$\limsup_{k\to\infty} \mathcal{Q}(e(t_k)) \le \limsup_{k\to\infty} \mathcal{Q}(e(s_k)) = \mathcal{Q}(e(t)),$$

where the last equality is a consequence of (4.13) and of the assumption $\{s_k\} \subset [0,T]\setminus N$. This proves (4.23). Since $t \mapsto p(t)$ is continuous in the strong topology of $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ at all times in $[0,T]\setminus N$, the Korn-Poincaré Inequality in $BD(\Omega)$ and (4.23) yield (4.24).

Equality (4.22), together with (4.23), also implies that $|\dot p|(\{t\}\times\overline{\Omega}) = 0$ for $t \notin N$. Indeed, let $t \notin N$ and let $\{s_k\} \to t^-$ and $\{t_k\} \to t^+$, with $\{s_k\},\{t_k\} \subset [0,T]\setminus N$. By (4.22) at s_k and at t_k we have that

$$\mathcal{Q}(e(t_k)) + \int_{(s_k,t_k]\times\overline{\Omega}} H\Big((\sigma*\rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p| = \mathcal{Q}(e(s_k)) + \int_{s_k}^{t_k}\int_\Omega \sigma(s) : E\dot w(s)\, dx\, ds. \tag{4.25}$$

Passing to the limit as $k \to \infty$ and using (4.23), we deduce that

$$\int_{\{t\}\times\overline{\Omega}} H\Big((\sigma*\rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p| = 0,$$

hence $|\dot p|(\{t\}\times\overline{\Omega}) = 0$ by (2.2).

Step 5 – Proof of (4.20). Let $0 \le t_1 < t_2 \le T$ with $t_1, t_2 \notin M$, unless $t_1 = 0$ or $t_2 = T$. Since $p_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, for every $\psi \in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ we have

$$\int_{\overline{\Omega}} (p_\varepsilon(t_2) - p_\varepsilon(t_1)) : \psi\, dx = \int_{t_1}^{t_2}\int_{\overline{\Omega}} \dot p_\varepsilon(s) : \psi\, dx\, ds$$

In view of (4.5), (4.6), and the assumption on t_1, t_2, this implies that

$$\int_{\overline{\Omega}} \psi : d(p(t_2) - p(t_1)) = \int_{[t_1,t_2]\times\overline{\Omega}} \psi : d\dot p \tag{4.26}$$

(see, *e.g.*, [1, Proposition 1.62]). We now recall that, by [21, Theorem 3.6],

$$\mathcal{H}((\sigma*\rho)(t_1), p(t_2) - p(t_1)) = \sup\Big\{ \int_{\overline{\Omega}} \zeta : d(p(t_2) - p(t_1)) :\ \zeta \in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) \cap \mathcal{K}((\sigma*\rho)(t_1)) \Big\}.$$

Therefore, for every $\eta > 0$ there exists $\zeta \in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) \cap \mathcal{K}((\sigma*\rho)(t_1))$ such that

$$\mathcal{H}((\sigma*\rho)(t_1), p(t_2) - p(t_1)) \le \int_{\overline{\Omega}} \zeta : d(p(t_2) - p(t_1)) + \eta.$$

On the other hand, by (4.26) we have

$$\int_{\overline{\Omega}} \zeta(x) : d(p(t_2) - p(t_1)) = \int_{[t_1,t_2]\times\overline{\Omega}} \zeta : d\dot p \le \int_{[t_1,t_2]\times\overline{\Omega}} H\Big((\sigma*\rho)(t_1,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p|(s,x),$$

where the last inequality follows from the fact that $\zeta \in \mathcal{K}((\sigma*\rho)(t_1))$ and the definition of H. Combining together the two previous inequalities and by the arbitrariness of η, we conclude that

$$\mathcal{H}((\sigma*\rho)(t_1), p(t_2) - p(t_1)) \le \int_{[t_1,t_2]\times\overline{\Omega}} H\Big((\sigma*\rho)(t_1,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p|(s,x).$$

Let now $t \in (0,T]$ with $t \notin M$ or $t = T$. The inequality above immediately implies that, in the notation of the previous step,

$$\widetilde{\mathrm{Var}}_H(p;0,t) \le \liminf_{k\to\infty} \sum_i \int_{[s_i^k, s_{i+1}^k]\times\overline{\Omega}} H\Big((\sigma*\rho)(s_i^k,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p|(s,x).$$

In turn, recalling (H3) and the uniform continuity of $\sigma*\rho$ over $[0,T]\times\overline{\Omega}$, we can replace $(\sigma*\rho)(s_i^k,x)$ by $(\sigma*\rho)(s,x)$ in the right-hand side of the above inequality, since $(\sigma*\rho)(s_i^k,x)-(\sigma*\rho)(s,x)$ can be chosen as small as desired, provided that $s-s_i^k$ is small enough, that is, that k is small enough. But then the right-hand side identifies with

$$\int_{[0,t]\times\overline{\Omega}} H\Big((\sigma*\rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p|(s,x) + \liminf_{k\to\infty} \sum_{i=1}^{n_k-1} \int_{\overline{\Omega}} H\Big((\sigma*\rho)(s_{i-1}^k,x), \frac{\dot p}{|\dot p|}(s_i^k,x)\Big)\, d|\dot p|(s_i^k,x).$$

Now, since $s_i^k \notin M$, for $i = 1,...,n_k-1$, we have that $|\dot p|(\{s_i^k\}\times\overline{\Omega}) = 0$, so that the second term in the expression above is identically 0, and we get the desired inequality.

Step 6 – Proof of (4.4). Since $p \in BV([0,T];\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$, the map $t \mapsto p(t)$ has a left and a right limit with respect to the strong topology of $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ at every $t \in N$. By the same argument as in (4.10) we deduce that that the map $t \mapsto e(t)$ has a left and a right limit with respect to the weak topology of $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ at every $t \in N$.

Let now $t \in N$ and $\{t_k\} \to t^-$ with $\{t_k\} \subset [0,T]\setminus N$. Arguing as in the proof of (4.13), we obtain that

$$\frac12\int_\Omega Ae(t_k):e(t_k)\,dx \le \frac12\int_\Omega Ae(t-):e(t-)\,dx - \int_\Omega \sigma(t_k):(Ew(t)-Ew(t_k))\,dx + \beta_H\|p(t-)-p(t_k)\|_1,$$

hence

$$e(t_k) \to e(t-) \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \text{ for every } \{t_k\}\to t^- \text{ with } t_k \in [0,T]\setminus N.$$

Analogously, for every $t \in N$ we have

$$e(t_k) \to e(t+) \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \text{ for every } \{t_k\}\to t^+ \text{ with } t_k \in [0,T]\setminus N.$$

Finally, arguing as in the proof of (4.23), we conclude that the map $t \mapsto e(t)$ has a left and a right limit with respect to the strong topology of $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ at every $t \in N$.

Let now $t \in N$. Let $\{s_k\}\to t^-$ and $\{t_k\}\to t^+$, with $\{s_k\},\{t_k\} \subset [0,T]\setminus N$. The energy equality (4.25) holds and by passing to the limit we deduce (4.4).

The proof of Theorem 4.1 is complete. □

Remark 4.2. Theorem 4.1 does not directly apply to the setting of the Armstrong-Frederick model described in [11]. However the interested reader will be quickly convinced that a straightforward adaptation of the theorem will yield the relevant result in that setting as well. The reader will also note that the analogue of [11, Proposition 4.14] will equally apply in "real" time. In other words a mere repetition of the proof of that proposition would yield the removal of the cap, at least for times $t \in [0,s_1)$, where $s_1 := \sup\{s : [0,s] \subset [0,T]\setminus N\}$, with N defined in (4.2). ¶

Remark 4.3. Our main result, Theorem 4.1, produces a notion of weak solution to the regularized evolution, which is a zero-viscosity limit of some visco-plastic evolution described in Theorem 3.1. In [19] a similar notion of weak solution is proposed through a limit process for a viscous approximation as the viscosity parameter vanishes. It is called a Balanced Viscosity solution by the authors and it is characterized by an energy identity at all times $t \in [0,T]$ involving a variation term, called $\mathrm{Var}_{\mathfrak{f}}$ there, that encompasses the so-called Finsler cost induced by the viscous term at the points of jump for the state field under consideration (see [19, (3.9)]).

Thanks to Theorem 4.1, we are at liberty to assert that the energy equality (4.3) holds true at all times, provided that we decide to set $e(t)$ at the value $e(t+)$ when $t \in N$.

But then, we do conclude that the solution that we have evidenced can be viewed as a Balanced Viscosity solution, provided that

$$\int_{[0,t]\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p|$$

is taken to be the definition of the variation $\mathrm{Var}_{\mathfrak{f}}(0,t)$. Of course, it would remain to reconcile our definition of the variation with that introduced in [19] which does not, in our opinion, so easily extend to the present setting. In particular, one would have to show that

$$\int_{\{t\}\times\overline{\Omega}} H\Big((\sigma * \rho)(s,x), \frac{\dot p}{|\dot p|}(s,x)\Big)\, d|\dot p|$$

can also be expressed as the Finsler cost defined in [19, (1.11)] at the jump points of the variation of $t \mapsto \mathrm{Var}(p;0,t)$, or, in other words, that there is indeed existence of a Balanced Viscosity solution in the problem at hand.

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