Homogenization of monotone operators in divergence form with x–dependent multivalued graphs

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Abstract

In a previous paper [4], we proved the existence of solutions to $-div\ a(x, grad\, u) = f$, together with appropriate boundary conditions, whenever $a(x, e)$ belongs, for every fixed x, to a certain class of maximal monotone graphs in e. Here, we derive the corresponding homogenization result, letting $a(x, e)$ depend upon a parameter ε, and imposing adequate ε-uniform boundedness and coercivity properties. The resulting homogenized graphs belong to the same class of maximal monotone graphs. Our results do not assume any kind of periodicity.

Keywords: Homogenization, H-convergence, monotone graphs, multi-valued graphs, maximality.

1 Introduction

In a previous paper, we investigated equations of the form

$$\begin{cases} -div\ d = f & \text{in } W^{-1,p'}(\Omega) \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $(grad\, u, d) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$ and, for a.e. $x \in \Omega$, $(grad\, u(x), d(x)) \in \mathbf{a}(x)$, where $\mathbf{a}(x)$ is, for each x, a maximal monotone graph. We showed existence of a pair solution (u, d), provided that the graph $\mathbf{a}(x)$ belongs to a class defined in Section 2 below. For now it is enough to point out that the key to our existence result consists in investigating the 45° rotation of the graph. The class of graphs under consideration is then defined as those graphs $\mathbf{a}_\varphi(x)$ that are of the form

$$\mathbf{a}_\varphi(x) = \left\{(e,d) \in \mathbf{R}^N \times \mathbf{R}^N : d - e = \varphi(x, d+e)\right\}, \tag{1.1}$$

for a.e. x in Ω, where $\varphi(x,\lambda) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory contraction, i.e., is measurable in x for every λ and satisfies, for a.e. x in Ω,

$$|\varphi(x,\lambda) - \varphi(x,\lambda')| \le |\lambda - \lambda'|, \quad \lambda, \lambda' \in \mathbf{R}^N.$$

Note that such graphs are maximal monotone, as established in Lemma 2.1 of [4], which we now recall, for the reader's convenience.

Lemma 1.1 $\mathbf{a}_\varphi \subset \mathbf{R}^N \times \mathbf{R}^N$ *is a monotone graph if and only if* φ *is a contraction on its domain of definition. Furthermore,* $\mathbf{a}_\varphi$ *is maximal if and only if the domain of definition of* φ *is all of* $\mathbf{R}^N$*.*

Further, φ must satisfy some coercivity estimate detailed in Section 2 below.

We refer the reader to the introduction of [4] for a review of previously available results and recall that our existence result is exactly that derived in [3] with the additional assumption that $(0,0)$ is in the graph, for a.e. $x \in \Omega$. Our proof is however completely different from that in [3]; in particular, it does not use any measurable selection criteria, and all measurability issues are trivialized in our work.

Our first task in the present paper is to free ourselves from the (apparently innocuous) constraint that $(0,0)$ be in the graph. This is the object of Section 2 where we state and prove an extended existence theorem (Theorem 2.4). Section 2 also precisely defines the class of graphs under investigation through the associated class $\mathbf{M}^*(\alpha, m, p, \Omega)$ of Carathéodory contractions (defined in Definition 2.3).

We then derive a homogenization result for sequence of graphs associated to the class $\mathbf{M}^*(\alpha, m, p, \Omega)$. Specifically, we prove the following result:

Theorem 1.2 *Let* φ^ε *be an* ε*-indexed sequence of elements of* $\mathbf{M}^*(\alpha, m, p, \Omega)$ *(defined in Definition 2.3). Then, there exists* $\varphi^0 \in \mathbf{M}^*(\alpha, m, p, \Omega)$ *and a subsequence* $\{\varepsilon'\} \subset \{\varepsilon\}$ *such that, for any* $f \in W^{-1,p'}(\Omega)$*, the accumulation points* u, d *– for the weak topologies of* $W_0^{1,p}(\Omega)$ *and* $L^{p'}(\Omega; \mathbf{R}^N)$ *respectively – of any sequence* $(u^{\varepsilon'}, d^{\varepsilon'})$*, solution to*

$$\begin{cases} -div\ d^{\varepsilon'} = f \\ u^{\varepsilon'} = 0 \quad on\ \partial\Omega, \end{cases}$$

with $(grad\, u^{\varepsilon'}, d^{\varepsilon'}) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$ *and, for a.e.* $x \in \Omega$*,*

$$d^{\varepsilon'}(x) - grad\, u^{\varepsilon'}(x) = \varphi^{\varepsilon'}\left(x, d^{\varepsilon'}(x) + grad\, u^{\varepsilon'}(x)\right),$$

satisfy

$$\begin{cases} -div\ d = f \\ u = 0 \quad on\ \partial\Omega, \end{cases}$$

and, for a.e. $x \in \Omega$,

$$d(x) - grad\, u(x) = \varphi^0 \left(x, d(x) + grad\, u(x)\right).$$

Once again, the above result is identical to the corresponding homogenization (G-convergence) result in [3], but it is obtained with totally different techniques. We also emphasize that, although our result is established in a scalar context for simplicity sake, there is no obstacle to a generalization to vector-valued systems of higher order equations: at no point throughout this study do we use the maximum principle.

The plan of the paper is as follows. Section 2 has already been summarized above. Section 3 addresses, as is usual in a homogenization proof (see e.g. [5],[6],[7]), an abstract homogenization result for sequences of maximal monotone graphs in $W_0^{1,p}(\Omega) \times W^{-1,p'}(\Omega)$, To this end, it is actually convenient to add to the graph a compact single-valued graph. Then, Section 4 addresses the concrete homogenization setting, and proves Theorem 1.2, first with the addition of a (compact) 0-th order term (Theorem 4.1), then as stated in the theorem. We also show in Lemma 4.4 that the result is independent of the well chosen boundary conditions, provided that those deliver an adequate a priori bound. The proofs require a uniqueness theorem for maximal monotone extensions of monotone graphs which is new, to the best of our knowledge. That result is established in the last section (Section 5).

Note that, throughout the text, the term homogenization is synonymous with that of H-convergence (see [5]); in particular it does not entail any kind of periodicity assumptions.

As far as notation is concerned, $\{\varepsilon\}$, $\{\varepsilon'\}$ or $\{\varepsilon''\}$ will always denote decreasing sequences of positive real numbers that tend to 0, while $\{n\}$ or $\{n'\}$ will denote increasing sequences of integers that tend to ∞. If E and F are two sets, the domain of a subset $\mathcal{A}$ of $E \times F$ (also referred to as a graph) is the projection of $\mathcal{A}$ onto E and it is denoted by $dom(\mathcal{A})$. The euclidean scalar product on $\mathbf{R}^N$ will be denoted by $x \cdot y$, for $x, y \in \mathbf{R}^N$. Throughout $\overset{V}{\rightharpoonup}$ will denote weak convergence in V, while $\overset{V}{\longrightarrow}$ will denote strong convergence in V.

We loosely call "coercivity" any estimate that guarantees an a priori bound on the sequence under investigation. We call *contraction* any Lipschitz map defined on (a subset of) $\mathbf{R}^N$ with Lipschitz constant less than or equal to 1. Finally, since we are only investigating monotone graphs, we call *maximal extension of a graph* any maximal monotone extension of that graph.

2 Existence revisited

Consider Ω a bounded open domain in $\mathbf{R}^N$, $p \in (1, \infty)$, $p' = p/(p-1)$, $m(x)$ a fixed non-negative function in $L^1(\Omega)$ and α a positive real number.

Define $j(t) := |t|^{p-2}t,\ t \in \mathbf{R}$.

Definition 2.1 $\mathbf{M}(\alpha, m, p, \Omega)$ *is the set of functions* $\varphi(x, \lambda) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ *with the following properties:*

(i) φ *is Carathéodory;*

(ii) $\varphi(x, \cdot)$ *is a contraction for a.e.* x *in* Ω*;*

(iii) for any given λ *in* $\mathbf{R}^N$ *and a.e.* x *in* Ω*, define* $e(x)$ *and* $d(x)$ *as*

$$\begin{cases} d(x) + e(x) = \lambda \\ d(x) - e(x) = \varphi(x, \lambda); \end{cases}$$

then,

$$d(x).e(x) \geq -m(x) + \alpha(|e(x)|^p + |d(x)|^{p'}),$$

for a.e. $x \in \Omega$*;*

(iv) $\varphi(x, 0) = 0$*, for a.e.* $x \in \Omega$*.*

For a.e. x in Ω we denote by $\mathbf{a}_\varphi(x)$ the graph associated to $\varphi(x, \cdot)$ as in (1.1).

We observed in [4] – see comments just before Remark 2.2 in that paper – that $\mathbf{M}(\alpha, m, p, \Omega)$ is non empty whenever e.g. α is sufficiently small, and $m(x) \geq 0$, for a.e. x in Ω, and obtained the following existence result (Theorems 2.3 and 6.1 of that reference):

Theorem 2.2 *Consider* $\varphi \in \mathbf{M}(\alpha, m, p, \Omega)$*. For any* $f \in W^{-1,p'}(\Omega)$*, there exists a pair* (u, d) *such that*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in L^{p'}(\Omega; \mathbf{R}^N) \\ -div\ d = f \\ d(x) - grad\, u(x) = \varphi\left(x, d(x) + grad\, u(x)\right), \quad \text{for a.e. } x \text{ in } \Omega, \end{cases}$$

(or equivalently $(grad\, u(x), d(x)) \in \mathbf{a}_\varphi(x)$ *for a.e.* x *in* Ω*).*

Similarly, there exists a pair (u, d)*, with* u *uniquely defined, such that*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in L^{p'}(\Omega; \mathbf{R}^N) \\ -div\ d + j(u) = f \\ d(x) - grad\, u(x) = \varphi\left(x, d(x) + grad\, u(x)\right), \quad \text{for a.e. } x \text{ in } \Omega, \end{cases}$$

(or equivalently $(grad\, u(x), d(x)) \in \mathbf{a}_\varphi(x)$ *for a.e.* x *in* Ω*).*

In fact, condition (iv) in the definition of the admissible set $\mathbf{M}(\alpha, m, p, \Omega)$ is not required, as demonstrated by the extended existence theorem below. Note that, upon dropping the requirement that $(0, 0) \in \mathbf{a}_\varphi(x)$ for a.e. x in Ω, we exactly recover the class used for proving existence in [3] (see Remark 2.2 in [4]).

We now define the class $\mathbf{M}^*(\alpha, m, p, \Omega)$ as

Definition 2.3 $\mathbf{M}^*(\alpha, m, p, \Omega)$ *is the set of functions* $\varphi(x, \lambda) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ *satisfying properties* $(i) - (iii)$ *of Definition 2.1.*

Elements in $\mathbf{M}^*(\alpha, m, p, \Omega)$ do not necessarily satisfy (iv), so that

$$\mathbf{M}(\alpha, m, p, \Omega) = \mathbf{M}^*(\alpha, m, p, \Omega) \cap \{\varphi : \varphi(x, 0) = 0, \text{ for a.e. } x \text{ in } \Omega\}.$$

We further assume, throughout the remainder of the paper, that

$$\mathbf{M}^*(\alpha, m, p, \Omega) \neq \emptyset, \tag{2.1}$$

an assumption which is satisfied whenever α is sufficiently small, and $m(x) \geq 0$, for a.e. x in Ω since, as already mentioned, $\mathbf{M}(\alpha, m, p, \Omega) \neq \emptyset$ under these conditions.

Then, the following existence theorem holds:

Theorem 2.4 *Consider* $\varphi \in \mathbf{M}^*(\alpha, m, p, \Omega)$*. For any* $f \in W^{-1,p'}(\Omega)$*, there exists a pair* (u, d) *such that*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in L^{p'}(\Omega; \mathbf{R}^N) \\ -div\ d = f \\ d(x) - grad\, u(x) = \varphi\left(x, d(x) + grad\, u(x)\right), \quad \text{for a.e. } x \text{ in } \Omega, \end{cases}$$

(or equivalently $(grad\, u(x), d(x)) \in \mathbf{a}_\varphi(x)$ *for a.e.* x *in* Ω*).*

Similarly, there exists a unique pair (u, d)*, with* u *uniquely defined, such that*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in L^{p'}(\Omega; \mathbf{R}^N) \\ -div\ d + j(u) = f \\ d(x) - grad\, u(x) = \varphi\left(x, d(x) + grad\, u(x)\right), \quad \text{for a.e. } x \text{ in } \Omega, \end{cases}$$

(or equivalently $(grad\, u(x), d(x)) \in \mathbf{a}_\varphi(x)$ *for a.e.* x *in* Ω*).*

Proof. First, we note that

$$d^*(x) := \frac{1}{2}\varphi(x, 0),\ e^*(x) := -d^*(x)$$

is a measurable pair satisfying

$$\begin{cases} d^*(x) + e^*(x) = 0 \\ d^*(x) - e^*(x) = \varphi(x, 0), \end{cases}$$

so that, according to (iii),

$$0 \geq -|e^*(x)|^2 \geq -m(x) + \alpha(|e^*(x)|^p + |d^*(x)|^{p'}),$$

and consequently $(e^*, d^*) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$.

We then define the function φ^* as

$$\varphi^*(x, \lambda) := \varphi\left(x, \lambda + d^*(x) + e^*(x)\right) - \left(d^*(x) - e^*(x), \right), \text{ for a.e. } x \text{ in } \Omega, \tag{2.2}$$

or still as

$$\varphi^*(x, \lambda) = \varphi(x, \lambda) - \varphi(x, 0), \text{ for a.e. } x \text{ in } \Omega,$$

so that $\varphi^*(x, 0) = 0$. It is immediate that φ^* satisfies conditions $(i), (ii), (iv)$; consequently, we can apply Theorem 5.1 in [4]. That theorem (applied to $-\varphi^*$, so that the roles of e and d are switched) states that

$$A^* := \{(e, d) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N) : \\ d(x) - e(x) = \varphi^*\left(x, d(x) + e(x)\right), \text{for a.e. } x \text{ in } \Omega\}$$

is a monotone graph in $L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$ that is such that:

$$\text{For any } \delta > 0 \text{ and any } d \in L^{p'}(\Omega; \mathbf{R}^N), \text{ there exists a (unique) element } (e', d') \in A^* \\ \text{satisfying } d' + \delta \widehat{\jmath}(e') = d, \text{ with } \widehat{\jmath}(e') := |e'|^{p-2} e'. \tag{2.3}$$

We will call this property $\widehat{\jmath}$-surjectivity in analogy with a similar definition in [4] (with, once again, the roles of e and d switched).

Further, in view of (2.2), for any $(e, d) \in A^*$, we can apply (iii) to the pair $(e + e^*, d + d^*)$. We then deduce the existence of some M and some $\beta > 0$ such that

$$\langle e, d \rangle \ \geq \beta(\|e\|^p_{L^p} + \|d\|^{p'}_{L^{p'}}) - M, \tag{2.4}$$

where $\langle , \rangle$ denotes the duality product between $L^p(\Omega; \mathbf{R}^N)$ and $L^{p'}(\Omega; \mathbf{R}^N)$, and M is a constant. Then, the domain of A^* is all of $L^p(\Omega; \mathbf{R}^N)$ (and, exchanging the roles of e and d, its range all of $L^{p'}(\Omega; \mathbf{R}^N)$). Indeed,

Lemma 2.5 *A monotone graph B in $L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$, which is $\widehat{\jmath}$-surjective in the sense of (2.3) and coercive in the sense of (2.4), has all of $L^p(\Omega; \mathbf{R}^N)$ for domain.*

Proof. Since B is $\widehat{\jmath}$-surjective, for any $e \in L^p(\Omega; \mathbf{R}^N)$, there exists a pair $(e_n, d_n) \in B$ with

$$d_n + n\widehat{\jmath}(e_n) = n\widehat{\jmath}(e).$$

Then

$$\langle d_n, e_n \rangle + n\langle \widehat{\jmath}(e_n) - \widehat{\jmath}(e), e_n - e \rangle = \langle d_n, e \rangle. \tag{2.5}$$

But, $\langle \widehat{\jmath}(e_n) - \widehat{\jmath}(e), e_n - e \rangle \ \geq 0$, so that coercivity implies that e_n and d_n are respectively bounded in $L^p(\Omega; \mathbf{R}^N)$ and $L^{p'}(\Omega; \mathbf{R}^N)$. A subsequence of (e_n, d_n) (still indexed by n) converges weakly in $L^p(\Omega;\mathbf{R}^N) \times L^{p'}(\Omega;\mathbf{R}^N)$ to (e_∞, d_∞). But, in view of (2.5),

$$\langle \widehat{\jmath}(e_n) - \widehat{\jmath}(e), e_n - e\rangle \to 0. \tag{2.6}$$

Either $e = 0$ and (2.6), together with the lower semi-continuous character of the L^p-norm, implies that $e_\infty = 0$, or $e \neq 0$, and (2.6), together with classical strong monotonicity properties of $\widehat{\jmath}$, implies that e_n converges to e, strongly in $L^p(\Omega;\mathbf{R}^N)$, hence that $e_\infty = e$.

But $(e_\infty = e, d_\infty) \in B$. Indeed, since $(e_n, d_n) \in B$, and since B is monotone,

$$\langle d_n - \overline{d}, e_n - \overline{e}\rangle \geq 0, \text{ for every } (\overline{e}, \overline{d}) \text{ in } B. \tag{2.7}$$

But, according to (2.5),

$$\limsup_n \langle d_n, e_n\rangle \leq \langle d_\infty, e\rangle, \tag{2.8}$$

so that, with (2.7)

$$\langle d_\infty - \overline{d}, e_\infty - \overline{e}\rangle \geq 0.$$

Now, it is easily checked that B, being $\widehat{\jmath}$-surjective, is maximal (see Remark 3.3 below for more on this). Thus $(e_\infty = e, d_\infty)$ belongs to B, which proves that the domain of B is all of $L^p(\Omega;\mathbf{R}^N)$. □

We now conclude the proof of Theorem 2.4. Since the domain of A^* is all of $L^p(\Omega;\mathbf{R}^N)$, it contains $-e^*$, that is that there exists $d_0 \in L^{p'}(\Omega;\mathbf{R}^N)$ with

$$d_0(x) + e^*(x) = \varphi^*\left(x, d_0(x) - e^*(x)\right), \text{for a.e. } x \text{ in } \Omega,$$

or still, with (2.2),

$$d_0(x) + d^*(x) = \varphi\left(x, d_0(x) + d^*(x)\right), \text{for a.e. } x \text{ in } \Omega.$$

Setting $\widehat{d} := d_0 + d^*$, we define, as already observed in Section 2 of [4],

$$\widehat{\varphi}(x,\lambda) = \varphi\left(x, \lambda + \widehat{d}(x)\right) - \widehat{d}(x), \text{for a.e. } x \text{ in } \Omega.$$

Then, $\widehat{\varphi}$ is easily seen to satisfy properties $(i) - (iv)$, albeit for a different m and a different $\alpha > 0$, so that Theorem 2.2 applies, and allows to assert, for any $f \in W^{-1,p'}(\Omega)$, the existence of $u \in W^{1,p}_0(\Omega), d' \in L^{p'}(\Omega;\mathbf{R}^N)$ with $d'(x) - grad\, u(x) = \widehat{\varphi}\left(x, d'(x) + grad\, u(x)\right)$, for a.e. x in Ω, and

$$-div\ d' = f + div\ \widehat{d}. \tag{2.9}$$

But then, setting $d := d' + \widehat{d}$, we have

$$d(x) - grad\, u(x) = \varphi\left(x, d(x) + grad\, u(x)\right), \text{for a.e. } x \text{ in } \Omega,$$

while, in view of (2.9),

$$-div\ d = f.$$

An identical argument, using the second part of Theorem 2.2 would establish the second part of Theorem 2.4. □

3 Abstract homogenization

In this section, we develop a homogenization result in an abstract setting that is suited to the problem at hand. Specifically, we consider a reflexive separable Banach space V, and denote its dual by V'. We denote by $\langle\!\langle , \rangle\!\rangle$ the duality product between V and V', that is $\langle\!\langle u, g\rangle\!\rangle = g(u),\ u \in V, g \in V'$. We say that a subset $\mathcal{A}$ of $V \times V'$ is monotone if

$$\langle\!\langle u - u', g - g'\rangle\!\rangle \geq 0,\ \forall (u, g) \in \mathcal{A},\ \forall (u', g') \in \mathcal{A}.$$

Further, we consider J to be a compact, strictly monotone operator from V into V', that is

$$\begin{cases} J \text{ is compact} \\ \langle\!\langle u - v, J(u) - J(v)\rangle\!\rangle > 0, \forall u \neq v. \end{cases} \tag{3.1}$$

In view of (3.1), J is sequentially weakly continuous from V, with the weak topology, into V', with the strong topology.

We will mainly need a more specialized class of V's and J's. In that specialized setting, V is densely and compactly embedded in some reflexive Banach space W,

$$V \overset{i}{\hookrightarrow} W, \tag{3.2}$$

the duality product $\langle\!\langle u, g\rangle\!\rangle$ identifies with the duality product $\langle , \rangle_{W',W}$ between W and W' whenever $g \in W'$, that is

$$\langle\!\langle u, i^*(g)\rangle\!\rangle = \langle g, i(u)\rangle_{W',W}, \forall u \in V,\ \forall g \in W$$

and we assume that, for some positive β, some $1 < q < \infty$, $q' = \dfrac{q}{q-1}$, and some $b \in \mathbf{R}$,

$$\begin{cases} J : W \to W' \text{ continuous, strictly monotone} \\ \langle J(w), w\rangle_{W',W} \ \geq \beta(\|w\|_W^q + \|J(w)\|_{W'}^{q'}) - b.\ \forall w \in W. \end{cases} \tag{3.3}$$

In such a case, we will identify, for convenience, J with $i^* \circ J \circ i$, with $J \circ i$, or with $i^* \circ J$, whenever necessary.

Let $1 < p < \infty, p' = \dfrac{p}{p-1}, \alpha > 0, a \in \mathbf{R}$. With J satisfying (3.1), we define the following subset of $V \times V'$:

Definition 3.1 $\mathsf{M}(p, \alpha, a)$ *is the subset of those* $\mathcal{A} \in V \times V'$ *such that*

I. $\mathcal{A}$ *is monotone;*

II. $\mathcal{A}$ *is coercive: for any* $(u, g) \in \mathcal{A}$, $\langle\!\langle u, g\rangle\!\rangle \geq \alpha(\|u\|_V^p + \|g\|_{V'}^{p'}) - a$;

III. $\mathcal{A}$ *is* J*-surjective, that is*

$$\begin{cases} \text{for any } f \in V', \text{ there exists a unique } (u, g) \in \mathcal{A} \text{ with} \\ g + J(u) = f. \end{cases}$$

Remark 3.2 The uniqueness of $(u, g) \in \mathcal{A}$ is an immediate consequence of (3.1), since if $(u, g), (u', g') \in \mathcal{A}$ are such that

$$g + J(u) = f, \; g' + J(u') = f,$$

then, taking the difference and multiplying by $u - u'$ yields

$$\langle\!\langle u - u', J(u) - J(u') \rangle\!\rangle \leq 0,$$

hence the result.

Remark 3.3 Note that it is immediately checked that J-surjectivity implies maximality of $\mathcal{A}$. The converse, that is that maximality of a graph implies J-surjectivity is a difficult theorem that we will not use in this study; see [2], Proposition 2.2, in a Hilbert space setting, or [1], Theorem 1.2, in a reflexive Banach space setting. This justifies the fact that the set $\mathsf{M}(p, \alpha, a)$ has no explicit J-dependence, since maximality will imply J-surjectivity for all J's satisfying (3.1).

In the context of Remark 3.3, the independence of the J-surjective character upon the specific choice of J can be checked, for J's that also satisfy (3.2),(3.3), from a straightforward application of Shauder's fixed point theorem, as demonstrated in the following lemma:

Lemma 3.4 *Assume (3.2) and also that J and $\widehat{J}$ both satisfy (3.3). Then, all elements in $\mathsf{M}(p, \alpha, a)$ are both J and $\widehat{J}$-surjective. Further, the domain of any element in $\mathsf{M}(p, \alpha, a)$ is all of V.*

Proof. Take $\mathcal{A} \in \mathsf{M}(p, \alpha, a)$. Then assume that, for some $\theta \in [0, 1]$, and for all $f \in V'$, there exists (an obviously unique) $(u, g) \in \mathcal{A}$ with

$$g + (1 - \theta)J(u) + \theta\widehat{J}(u) = f. \tag{3.4}$$

(Note that this is true for $\theta = 0$ by hypothesis.) Then, consider, for $\delta > 0$ small enough, the mapping $T : W \to V$ defined by considering $(T(w), H) \in \mathcal{A}$ with

$$H + (1 - \theta)J(T(w)) + \theta\widehat{J}(T(w)) = f + \delta\left(J(w) - \widehat{J}(w)\right). \tag{3.5}$$

The mapping T is continuous, since, if $w_n \xrightarrow{W} w$, then $T(w_n)$ is bounded in V by coercivity of $\mathcal{A}$; for a subsequence $\{n'\} \subset \{n\}$, $T(w_{n'}) \overset{V}{\rightharpoonup} U$, and, by compactness,

$$G_n := f + \delta\left(J(w_{n'}) - \widehat{J}(w_{n'})\right) - (1 - \theta)J(T(w_{n'})) - \theta\widehat{J}(T(w_{n'}))$$

tends strongly in V' to

$$G := f + \delta\left(J(w) - \widehat{J}(w)\right) - (1 - \theta)J(U) - \theta\widehat{J}(U).$$

But, since $(T(w_{n'}), G_{n'}) \in \mathcal{A}$,

$$\langle\!\langle T(w_{n'}) - u, G_{n'} - g \rangle\!\rangle \geq 0 \xrightarrow{n'} \langle\!\langle U - u, G - g \rangle\!\rangle \geq 0, \ \forall (u, g) \in \mathcal{A},$$

so that the maximal character of $\mathcal{A}$ implies that $(U, G) \in \mathcal{A}$, hence, by uniqueness of the solution to (3.5), $U = T(w)$, $G = H$ and there is no need to extract subsequences.

The mapping $i \circ T$ is then compact, since i is compact. Finally, $i \circ T$ sends the ball $\{w \in W : \|w\|_W \leq M\}$ into itself, provided that M is carefully chosen. Indeed, from (3.5), plus coercivity of $\mathcal{A}$ and of $J, \hat{J}$,

$$\begin{aligned} \alpha\|T(w)\|_V^p - a + \beta\|i \circ T(w)\|_W^q - b \ \leq \ & \|f\|_{V'}\|T(w)\|_V \\ & + \delta \left(\|J(w)\|_{W'} + \|\widehat{J}(w)\|_{W'} \right) \|i \circ T(w)\|_W. \end{aligned}$$

But, by (3.3), $\|J(w)\|_{W'}, \|\widehat{J}(w)\|_{W'} \leq C(1 + \|w\|^{q-1})$ with a constant C that depends only on q, b, β. Thus, by virtue of Young's inequality,

$$\beta\|i \circ T(w)\|_W^q \leq C_1 + 2\delta C \left(1 + M^{q-1}\right) \|i \circ T(w)\|_W,$$

where C_1 is a constant depending only on α, a, b, p and $\|f\|_{V'}$. If $2\delta C < \beta$, and M is large enough, so that $\beta M^q \geq C_1 + 2\delta C(1 + M^{q-1})M$, then $\|i \circ T(w)\|_W \leq M$ as desired. Note that the upper bound on δ is independent of θ. Application of Schauder's fixed point theorem permits to conclude to the existence of a fixed point w for $i \circ T$. Hence, for any δ small enough (independently of θ), there exists a (unique) solution $(v := T(w), h) \in \mathcal{A}$ of

$$h + (1 - (\theta + \delta))\, J(v) + (\theta + \delta)\widehat{J}(v) = f.$$

Iterating as many times as necessary, we reach the value $\theta = 1$ in (3.4), which shows the first part of the lemma.

Since, whenever J satisfies (3.3), so does nJ, $n \in \mathbf{N}$, there exists, for any $u \in V$, a unique $(u_n, g_n) \in \mathcal{A}$ with

$$g_n + nJ(u_n) = nJ(u). \tag{3.6}$$

By coercivity of $\mathcal{A}$, multiplication of (3.6) by $u_n - u$ implies that u_n is bounded in V, so that, for a subsequence $\{n'\} \subset \{n\}$,

$$\begin{cases} u_{n'} \overset{V}{\rightharpoonup} \overline{u} \\ g_{n'} \overset{V'}{\rightharpoonup} \overline{g}. \end{cases}$$

But, exactly as in the proof of (2.8) in Lemma 2.5, we get

$$\limsup_{n'} \langle\!\langle u_{n'}, g_{n'} \rangle\!\rangle \leq \langle\!\langle \overline{u}, \overline{g} \rangle\!\rangle,$$

so that, for any $(v,h)\in\mathcal{A}$, monotonicity is preserved, i.e.,

$$\langle\!\langle \overline{u}-v,\overline{g}-h\rangle\!\rangle \geq 0.$$

By maximality, we conclude that

$$(\overline{u},\overline{g})\in\mathcal{A}. \tag{3.7}$$

Now, multiplication of (3.6) – at n' – by $u_{n'}-u$ and compactness of the identity mapping from V into W imply that

$$\langle J(\overline{u})-J(u),\overline{u}-u\rangle_{W',W}=0,$$

hence that $\overline{u}=u$, which, together with (3.7) yields the desired result. □

We are now ready to prove the main result of this section.

Theorem 3.5 *Assume that $\mathcal{A}^\varepsilon\in\mathsf{M}(p,\alpha,a)$. Then, there exists a subsequence $\{\varepsilon'\}\subset\{\varepsilon\}$ and an element $\mathcal{A}^0\in\mathsf{M}(p,\alpha,a)$ such that, for any $f^{\varepsilon'}\stackrel{V'}{\longrightarrow}f$, the solution of*

$$\begin{cases}(u^{\varepsilon'},g^{\varepsilon'})\in\mathcal{A}^{\varepsilon'}\\ g^{\varepsilon'}+J(u^{\varepsilon'})=f^{\varepsilon'}\end{cases}$$

satisfies

$$u^{\varepsilon'}\stackrel{V}{\rightharpoonup}u$$

$$g^{\varepsilon'}\stackrel{V'}{\longrightarrow}g,$$

with

$$\begin{cases}(u,g)\in\mathcal{A}^0\\ g+J(u)=f.\end{cases}$$

We will say that $\mathcal{A}^{\varepsilon'}$ H-converges to $\mathcal{A}^0$.

Assume that (3.2) holds. Then, the set $\mathcal{A}^0$ does not depend upon the specific choice of J satisfying (3.3). Finally, in that case, $dom(\mathcal{A}^0)=V$.

Proof. Step 1 – Definition of $\mathcal{A}^0$: Let $X=\{f_n: n\in\mathbf{N}\}$ be a countable dense subset of V' and consider $(u_n^\varepsilon,g_n^\varepsilon)\in\mathcal{A}^\varepsilon$, the solution of

$$g_n^\varepsilon+J(u_n^\varepsilon)=f_n. \tag{3.8}$$

Multiplication by u_n^ε immediately implies, in view of the coercivity hypothesis (II) in Definition 3.1, that

$$\alpha\|u_n^\varepsilon\|_V^p\leq\left(\|f_n\|_{V'}+\|J(0)\|_{V'}\right)\|u_n^\varepsilon\|_V+a,$$

so that, for some continuous function $\Phi:\mathbf{R}\to\mathbf{R}$,

$$\|u_n^\varepsilon\|_V\leq\Phi(\|f_n\|_{V'}). \tag{3.9}$$

Thus a diagonalization process allows us to extract from $\{\varepsilon\}$ a subsequence $\{\varepsilon'\}$ with

$$u_n^{\varepsilon'} \overset{V}{\rightharpoonup} u_{f_n}, \forall n \in \mathbf{N}.$$

Then, by virtue of (3.8), and since J is compact,

$$g_n^{\varepsilon'} \overset{V'}{\longrightarrow} g_{f_n}, \forall n \in \mathbf{N},$$

with

$$g_{f_n} + J(u_{f_n}) = f_n. \tag{3.10}$$

We can then pass to the limit in $\langle\!\langle u_n^{\varepsilon'} - u_m^{\varepsilon'}, g_n^{\varepsilon'} - g_m^{\varepsilon'} \rangle\!\rangle \geq 0$. We get

$$\langle\!\langle u_{f_n} - u_{f_m}, g_{f_n} - g_{f_m} \rangle\!\rangle \geq 0. \tag{3.11}$$

Further, by weak lower semicontinuity of the norm, (II) becomes

$$\langle\!\langle u_{f_n}, g_{f_n} \rangle\!\rangle \geq \alpha(\|u_{f_n}\|_V^p + \|g_{f_n}\|_{V'}^{p'}) - a, \tag{3.12}$$

while (3.9) yields

$$\|u_{f_n}\|_V^p \leq \Phi(\|f_n\|_{V'}). \tag{3.13}$$

Now, let $f \in V'$ be arbitrary. Since X is dense in V', there exists $f_n \in X$ with $f_n \overset{V'}{\longrightarrow} f$. From (3.13) and the continuity of Φ, a subsequence $u_{f_{n'}}$ of u_{f_n} converges weakly to some $u \in V$. Say a different subsequence $u_{f_{n''}}$ converges weakly to some $\widehat{u} \in V$. Then, by virtue of (3.10) and (3.11),

$$\langle\!\langle u_{f_{n'}} - u_{f_{n''}}, J(u_{f_{n'}}) - J(u_{f_{n''}}) \rangle\!\rangle \leq \langle\!\langle u_{f_{n'}} - u_{f_{n''}}, f_{n'} - f_{n''} \rangle\!\rangle.$$

But the right hand-side of the previous inequality tends to 0 as $n', n'' \nearrow \infty$, while the left hand-side tends to $\langle\!\langle u - \widehat{u}, J(u) - J(\widehat{u}) \rangle\!\rangle$, which is thus non-positive. The strictly monotone character of J implies that $u = \widehat{u}$ and the whole sequence u_{f_n} tends to u, weakly in V.

Thus $u_f := u$ depends only on f, not on the approximating sequence for f in X, and g_f, defined by

$$g_f + J(u_f) = f \tag{3.14}$$

as well.

Define

$$\mathcal{A}^0 := \{(u_f, g_f) : f \in V\}.$$

In view of (3.11) and (3.12) which pass to the limit, and recalling (3.14), we immediately conclude that $\mathcal{A}^0 \in \mathsf{M}(p, \alpha, a)$.

Step 2 – Proof of the convergence result: Consider, $f^{\varepsilon'} \overset{V'}{\longrightarrow} f$, and the associated $(u^{\varepsilon'}, g^{\varepsilon'}) \in \mathcal{A}^{\varepsilon'}$ with

$$g^{\varepsilon'} + J(u^{\varepsilon'}) = f^{\varepsilon'}.$$

In view of the uniform coercivity property, there exists $u \in V$ such that, for a subsequence $\{\varepsilon''\} \subset \{\varepsilon'\}$,

$$u^{\varepsilon''} \overset{V}{\rightharpoonup} u,$$

and, by the compactness of J,

$$g^{\varepsilon''} \xrightarrow{V'} g$$

with

$$g = f - J(u).$$

We claim that $(u, g) \in \mathcal{A}^0$. Indeed, consider $f_n \in X \xrightarrow{V'} f$, and $(u_n^{\varepsilon''}, g_n^{\varepsilon''}) \in \mathcal{A}^{\varepsilon''}$ the pair that satisfies

$$g_n^{\varepsilon''} + J(u_n^{\varepsilon''}) = f_n.$$

Since $A^{\varepsilon''}$ is monotone

$$\langle\!\langle u^{\varepsilon''} - u_n^{\varepsilon''}, J(u^{\varepsilon''}) - J(u_n^{\varepsilon''})\rangle\!\rangle \leq \langle\!\langle u^{\varepsilon''} - u_n^{\varepsilon''}, f^{\varepsilon''} - f_n\rangle\!\rangle.$$

But, according to Step 1,

$$u_n^{\varepsilon''} \overset{V}{\rightharpoonup} u_{f_n},$$

with $(u_{f_n}, g_{f_n} = f_n - J(u_{f_n})) \in \mathcal{A}^0$, and, passing to the limit in ε'' in the previous inequality,

$$\langle\!\langle u - u_{f_n}, J(u) - J(u_{f_n})\rangle\!\rangle \leq \langle\!\langle u - u_{f_n}, f - f_n\rangle\!\rangle.$$

But, still with Step 1, $u_{f_n} \overset{V}{\rightharpoonup} u_f$, with $(u_f, g_f = f - J(u_f)) \in \mathcal{A}^0$, so that, passing to the limit in n in the previous inequality,

$$\langle\!\langle u - u_f, J(u) - J(u_f)\rangle\!\rangle \leq 0.$$

Hence $u = u_f$, $(u, g) \in \mathcal{A}^0$, and the whole sequence $u^{\varepsilon'}$ converges weakly in V to u_f.

Assume from now on that (3.2) and (3.3) are satisfied.

Step 3 – Independence of $\mathcal{A}^0$ upon the choice of J: Consider $\widehat{J}$ satisfying (3.3). Then, for $f^{\varepsilon'} \xrightarrow{V'} f$, let $(u^{\varepsilon'}, g^{\varepsilon'}) \in \mathcal{A}^{\varepsilon'}$ solve

$$g^{\varepsilon'} + \widehat{J}(u^{\varepsilon'}) = f^{\varepsilon'},$$

which is always possible according to Lemma 3.4. Equivalently, $(u^{\varepsilon'}, g^{\varepsilon'})$ solves

$$g^{\varepsilon'} + J(u^{\varepsilon'}) = f^{\varepsilon'} + J(u^{\varepsilon'}) - \widehat{J}(u^{\varepsilon'}).$$

But the coercivity of $\mathcal{A}^{\varepsilon'}$ implies that, for a subsequence $\{\varepsilon''\} \subset \{\varepsilon'\}$,

$$u^{\varepsilon''} \overset{V}{\rightharpoonup} u,$$

so that, by virtue of the compactness of J and $\hat{J}$,

$$f^{\varepsilon''} + J(u^{\varepsilon''}) - \widehat{J}(u^{\varepsilon''}) \xrightarrow{V'} f + J(u) - \widehat{J}(u).$$

We can apply Step 2 and conclude that $\left(u, g = f + J(u) - \widehat{J}(u) - J(u)\right) \in \mathcal{A}^0$, while

$$g + \widehat{J}(u) = f.$$

Since (u, g) is unique (see Remark 3.2), there is no need to extract the subsequence $\{\varepsilon''\}$ and the result is proved.

Step 4 – $dom(\mathcal{A}^0) = V$: This is a direct application of Lemma 3.4 to $\mathcal{A}^0$, as element of $\mathsf{M}(p, \alpha, a)$. □

Remark 3.6 In the spirit of the sought result (Theorem 1.2), we can modify Theorem 3.5 by dropping the compact term $J(u)$. Indeed, assume that $\mathcal{A}^\varepsilon$ H-converges to $\mathcal{A}^0$, then, let $(u^\varepsilon, g^\varepsilon)$ be in $\mathcal{A}^\varepsilon \in \mathsf{M}(p, \alpha, a)$. Consider an accumulation point (u, g) of $(u^\varepsilon, g^\varepsilon)$ for the (sequentially) weak $\times$ strong topology on $V \times V'$, that is the weak $\times$ strong limit in $V \times V'$ of a subsequence $(u^{\varepsilon'}, g^{\varepsilon'})$ of $(u^\varepsilon, g^\varepsilon)$ ($\{\varepsilon'\} \subset \{\varepsilon\}$). Then, by compactness

$$g^{\varepsilon'} + J(u^{\varepsilon'}) \xrightarrow{V'} g + J(u),$$

thus, by H-convergence, $(u, g) \in \mathcal{A}^0$.

In other words, if $\mathcal{A}^\varepsilon$ H-converges to $\mathcal{A}^0$, then all weak $\times$ strong accumulation points of pair-sequences in $\mathcal{A}^\varepsilon$ belong to $\mathcal{A}^0$. Note that the existence of weak $\times$ weak accumulation points is guaranteed by uniform coercivity of $\mathcal{A}^\varepsilon$, but not that of weak $\times$ strong accumulation points.

4 Concrete homogenization

In this section, we prove Theorem 1.2. As before, we add a compact perturbation to the graph of the form $j(u)$ with, as in Section 2,

$$j(t) := |t|^{p-2}t, \ t \in \mathbf{R},$$

and first prove the following

Theorem 4.1 *Let φ^ε be an ε-indexed sequence of elements of $\mathbf{M}^*(\alpha, m, p, \Omega)$ (defined in Definition 2.3). Then, there exist $\varphi^0 \in \mathbf{M}^*(\alpha, m, p, \Omega)$ and a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$ such that*

- *for any $f \in W^{-1,p'}(\Omega)$,*
- *for any $(u^{\varepsilon'}, d^{\varepsilon'})$ (with $u^{\varepsilon'}$ uniquely defined) solution to*

$$-div\ d^{\varepsilon'} + j(u^{\varepsilon'}) = f \tag{4.1}$$

$$u^{\varepsilon'} = 0 \quad on\ \partial\Omega,$$

with $(grad\, u^{\varepsilon'}, d^{\varepsilon'}) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$ and, for a.e. $x \in \Omega$,

$$d^{\varepsilon'}(x) - grad\, u^{\varepsilon'}(x) = \varphi^{\varepsilon'}\left(x, d^{\varepsilon'}(x) + grad\, u^{\varepsilon'}(x)\right),$$

- *and, for any subsequence $\{\varepsilon''\}$ of $\{\varepsilon'\}$, with $d^{\varepsilon''} \rightharpoonup d$, weakly in $L^{p'}(\Omega;\mathbf{R}^N)$,*

we have

$$u^{\varepsilon'} \overset{V}{\rightharpoonup} u,$$

where (u,d) satisfy

$$\begin{cases} -div\ d + j(u) = f \\ u = 0 \quad on\ \partial\Omega, \end{cases}$$

and, for a.e. x in Ω,

$$d(x) - grad\, u(x) = \varphi^0\left(x, d(x) + grad\, u(x)\right).$$

Remark 4.2 As previously stated in Remark 3.6, the sequence $d^{\varepsilon'}$ always admits accumulation points because coercivity (item (iii) in Definition 2.1 for the admissible class $\mathbf{M}^*(\alpha,m,p,\Omega)$) immediately yields, upon multiplication of (4.1) by $u^{\varepsilon'}$ and integration by parts, that $d^{\varepsilon'}$ is bounded in $L^{p'}(\Omega;\mathbf{R}^N)$.

Remark 4.3 Note that (3.2) is satisfied with $V := W_0^{1,p}(\Omega)$, $W := L^p(\Omega)$, $W' := L^{p'}(\Omega)$, $V' := W^{-1,p'}(\Omega)$. The mapping $J : L^p(\Omega) \to L^{p'}(\Omega)$ defined as $J(w) := j(w)$ obviously satisfies (3.3) with $p = q$. Further, if we define

$$\mathcal{A}^\varepsilon := \{(u, -div\, d) : d(x) - grad\, u(x) = \varphi^\varepsilon\left(x, d(x) + grad\, u(x)\right), \text{ for a.e. } x \text{ in } \Omega\}, \tag{4.2}$$

it belongs to $\mathsf{M}(p,\alpha,a)$, with $a := \int_\Omega m(x)\ dx$;. Indeed, items $(I),(II)$ in the definition of the elements of $\mathsf{M}(p,\alpha,a)$ are trivially verified, while item (III) stems directly from the second part of Theorem 2.4.

In other words, there is a natural way to embed all elements of $\mathbf{M}^*(\alpha,m,p,\Omega)$ into $\mathsf{M}(p,\alpha,a)$, with $a := \int_\Omega m(x)\ dx$.

Proof of Theorem 4.1.

Step 1 – Construction of an adequate Carathéodory contraction: First, we consider an open cube C of sidelength c containing Ω and a still larger open domain $\widehat{\Omega}$ compactly containing C. Recalling (2.1), we assume that, for an adequate extension of m to $L^1(\widehat{\Omega})$, there exist an element $\widehat{\varphi} \in \mathbf{M}^*(\alpha,m,p,\widehat{\Omega})$ and extend φ^ε by $\widehat{\varphi}$ to $\widehat{\Omega}\setminus\Omega$, thus producing a new sequence, still denoted by φ^ε, which belongs now to $\mathbf{M}^*(\alpha,m,p,\widehat{\Omega})$ and coincides with φ^ε on Ω.

By virtue of Remark 4.3 above, Theorem 3.5 in its entirety applies to $\mathcal{A}^\varepsilon$ defined through (4.2), as well as to $\hat{\mathcal{A}}^\varepsilon$ correspondingly extending $\mathcal{A}^\varepsilon$ to $\hat{\Omega}$. In particular, upon setting

$$V := W_0^{1,p}(\widehat{\Omega}),\ W := L^p(\widehat{\Omega}),\ W' := L^{p'}(\widehat{\Omega}),\ V' := W^{-1,p'}(\widehat{\Omega}),$$

there exists a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$, and $\widehat{\mathcal{A}}^0 \in \mathsf{M}(p,\alpha,a)$ such that, for any $f \in V'$, $(u^{\varepsilon'}, -div\, d^{\varepsilon'})$, unique solution to

$$\begin{cases} -div\ d^{\varepsilon'} + j(u^{\varepsilon'}) = f \\ \\ u^{\varepsilon'} = 0 \quad \text{on } \partial\widehat{\Omega}, \end{cases} \tag{4.3}$$

with $(grad\, u^{\varepsilon'}, d^{\varepsilon'}) \in W \times W'$ and

$$d^{\varepsilon'}(x) - grad\, u^{\varepsilon'}(x) = \varphi^{\varepsilon'}\left(x, d^{\varepsilon'}(x) + grad\, u^{\varepsilon'}(x)\right), \text{for a.e. } x \text{ in } \widehat{\Omega}, \qquad (4.4)$$

satisfies

$$\begin{cases} u^{\varepsilon'} \overset{V}{\rightharpoonup} u \\ -div\ d^{\varepsilon'} \overset{V'}{\longrightarrow} g, \end{cases} \qquad (4.5)$$

with

$$\begin{cases} g + j(u) = f \\ (u, g) \in \widehat{\mathcal{A}}^0. \end{cases} \qquad (4.6)$$

The sequence $\{\varepsilon'\}$ can also be chosen such that, upon setting this time

$$V := W_0^{1,p}(\Omega),\ W := L^p(\Omega),\ W' := L^{p'}(\Omega),\ V' := W^{-1,p'}(\Omega),$$

there also exists $\mathcal{A}^0$ such that (4.5), (4.6) hold true upon replacing $\widehat{\Omega}$ by Ω in (4.3), (4.4). Note that we will not use $\mathcal{A}^0$ per se, but only the fact that the convergences in (4.5) hold for a subsequence $\{\varepsilon'\}$ which is independent of $f \in W^{-1,p'}(\Omega)$; see Step 2 below.

Now, according to the second part of Theorem 3.5, the domain of $\widehat{\mathcal{A}}^0$ is all of V. Thus, fix $\psi \in \mathcal{C}_0^\infty(\widehat{\Omega})$ with $\psi \equiv 1$ on C; for any $\lambda \in \mathbf{Q}^N$, $u_\lambda := \psi(x)\lambda \cdot x \in dom(\widehat{\mathcal{A}}^0)$. Consider $g_\lambda \in V'$ such that $(u_\lambda, g_\lambda) \in \widehat{\mathcal{A}}^0$ and set

$$f_\lambda := g_\lambda + j(u_\lambda).$$

If $(u_\lambda^{\varepsilon'}, d_\lambda^{\varepsilon'})$ is a solution of (4.3),(4.4) with $f = f_\lambda$, we get that

$$u_\lambda^{\varepsilon'} \overset{V}{\rightharpoonup} u_\lambda, \forall \lambda \in \mathbf{Q}^N,$$

and, since $\|d_\lambda^{\varepsilon'}\|_{W'}$ is uniformly bounded in ε' by the uniform coercivity of $\varphi^{\varepsilon'}$ (item (iii) in Definition 2.3 of the class $\mathbf{M}^*(\alpha, m, p, \widehat{\Omega})$), a diagonalization process also yields, for a subsequence of $\{\varepsilon'\}$ that we will not relabel,

$$d_\lambda^{\varepsilon'} \overset{W'}{\rightharpoonup} d_\lambda, \forall \lambda \in \mathbf{Q}^N,$$

where, by (4.6),

$$-div\, d_\lambda = g_\lambda.$$

According to the above, we conclude that, for this new subsequence, indexed by $\{\varepsilon'\}$, both (4.5) and

$$\begin{cases} u_\lambda^{\varepsilon'} \overset{V}{\rightharpoonup} u_\lambda \\ d_\lambda^{\varepsilon'} \overset{W'}{\rightharpoonup} d_\lambda \end{cases} \forall \lambda \in \mathbf{Q}^N, \qquad (4.7)$$

hold true.

We set

$$B := \left\{(\lambda, d_\lambda) : \lambda \in \mathbf{Q}^N\right\} \subset L^p(C;\mathbf{R}^N) \times L^{p'}(C;\mathbf{R}^N). \tag{4.8}$$

That set is pointwise monotone, that is, for a.e. $x \in C$, and any $(\lambda, \mu) \in \mathbf{Q}^N$,

$$(d_\lambda(x) - d_\mu(x)) \cdot (\lambda - \mu) \geq 0. \tag{4.9}$$

Indeed, for any $\zeta \in \mathcal{C}_0^\infty(C)$, $\zeta \geq 0$, an elementary application of compensated compactness (here the div-curl lemma), together with the fact that $grad\, u_\lambda \equiv \lambda$ on C, yields

$$0 \leq \lim_{\varepsilon''} \int_C \zeta(x)(d_\lambda^{\varepsilon''} - d_\mu^{\varepsilon''})(x) \cdot (grad\, u_\lambda^{\varepsilon''} - grad\, u_\mu^{\varepsilon''})(x) dx = \int_C \zeta(x)(d_\lambda - d_\mu)(x) \cdot (\lambda - \mu) dx,$$

hence (4.9).

Further, a similar argument would show coercivity, that is that, for a.e. $x \in C$ and any $\lambda \in \mathbf{Q}^N$,

$$d_\lambda(x) \cdot \lambda \geq -m(x) + \alpha(|\lambda|^p + |d_\lambda(x)|^{p'}). \tag{4.10}$$

Split C into n^N cubes C_n^i of sidelength $\dfrac{c}{n}$ and denote by χ_n^i the characteristic function of C_n^i. Define, for any $h \in L^1(C)$,

$$[h]_n^i := \frac{1}{|C_n^i|} \int_{C_n^i} h(x)\, dx.$$

The set

$$B_n := \left\{ \left(\lambda, \sum_i [d_\lambda]_n^i \chi_n^i(x) \right) : \lambda \in \mathbf{Q}^N, x \in C \right\}$$

is obviously a pointwise monotone graph in the sense of (4.9) and, thanks to Jensen's inequality, it is coercive in the sense of (4.10) (with $\sum_i [d_\lambda(\text{ resp. } d_\mu)]_n^i \chi_n^i(x)$ replacing $d_\lambda(\text{ resp. } d_\mu)(x)$ and $\sum_i [m]_n^i \chi_n^i(x)$ replacing $m(x)$).

Consider the subset of $\mathbf{R}^N \times \mathbf{R}^N$ defined as

$$\mathbf{b}_n^i := \left\{(\lambda, [d_\lambda]_n^i) : \lambda \in \mathbf{Q}^N\right\}.$$

In view of the above, it is a monotone, coercive graph with dense domain, that is, for any $\lambda, \mu \in \mathbf{Q}^N$,

$$\begin{cases} ([d_\lambda]_n^i - [d_\mu]_n^i) \cdot (\lambda - \mu) \geq 0 \\ [d_\lambda]_n^i \cdot \lambda \geq -[m]_n^i + \alpha(|\lambda|^p + |[d_\lambda]_n^i|^{p'}) \\ dom(\mathbf{b}_n^i) \text{ is dense in } \mathbf{R}^N. \end{cases} \tag{4.11}$$

Note that, since $(0, d_0) \in B$,

$$(0, [d_0]_n^i) \in \mathbf{b}_n^i. \tag{4.12}$$

In view of (4.11), we are in a position to apply Theorem 5.2 from Section 5 below and we conclude to the existence of a (unique) maximal extension $\overline{\mathbf{b}}_n^i$ of $\mathbf{b}_n^i$. Appealing to Lemma 2.1 in [4] (Lemma 1.1 of the introduction), there exists a contraction $\overline{\varphi}_n^i : \mathbf{R}^N \to \mathbf{R}^N$ such that, for any $(\lambda, \mu) \in \mathbf{R}^N \times \mathbf{R}^N$,

$$(\lambda, \mu) \in \overline{\mathbf{b}}_n^i \quad \text{iff} \quad \mu - \lambda = \overline{\varphi}_n^i(\mu + \lambda),$$

so that, in particular,

$$[d_\lambda]_n^i - \lambda = \overline{\varphi}_n^i([d_\lambda]_n^i + \lambda), \ \forall \lambda \in \mathbf{Q}^N. \tag{4.13}$$

Let us emphasize that $\overline{\varphi}_n^i$ is defined on all of $\mathbf{R}^N$. Also, in view of (4.12),

$$[d_0]_n^i = \overline{\varphi}_n^i([d_0]_n^i). \tag{4.14}$$

We define

$$\varphi_n(x, \lambda) := \sum_i \overline{\varphi}_n^i(\lambda)\chi_n^i(x). \tag{4.15}$$

It is a Carathéodory contraction, which, in view of (4.14), satisfies

$$d_n^*(x) = \varphi_n\left(x, d_n^*(x)\right), \quad \text{with} \quad d_n^*(x) := \sum_i [d_0]_n^i \chi_n^i(x). \tag{4.16}$$

(Note that, as $n \nearrow \infty$, d_n^* tend to d_0 strongly in $L^{p'}(C; \mathbf{R}^N)$.) Consequently, for any $\lambda \in \mathbf{R}^N$ and a.e. $x \in C$,

$$|\varphi_n(x, \lambda)| \leq |d_n^*(x)| + |\lambda - d_n^*(x)|.$$

Thus, for every $\lambda \in \mathbf{Q}^N$, there exists $\varphi(x, \lambda) \in L^{p'}(C; \mathbf{R}^N)$, such that, for a subsequence (still labeled by n),

$$\varphi_n(., \lambda) \overset{L^{p'}(C;\mathbf{R}^N)}{\rightharpoonup} \varphi(., \lambda). \tag{4.17}$$

Further, $\varphi(x, .)$ is easily checked to be a contraction on $\mathbf{Q}^N$. Indeed, for a.e. $x \in C$, every $\nu \in \mathbf{R}^N$ and every $\lambda, \mu \in \mathbf{Q}$,

$$(\varphi_n(x, \lambda) - \varphi_n(x, \mu)) \cdot \nu \leq |\lambda - \mu||\nu|,$$

and this inequality is preserved upon passing to the weak limit in n. We then extend $\varphi(x, .)$ to all of $\mathbf{R}^N$ as the (strong) limit in $L^{p'}(C; \mathbf{R}^N)$ of the Cauchy sequences $\varphi(x, \lambda_n)$, with $\lambda_n \in \mathbf{Q}^N \to \lambda$. We thus obtain a Carathéodory contraction φ^0 defined on $C \times \mathbf{R}^N$.

We need to show that, if $\lambda \in \mathbf{Q}^N$, then, for a.e. $x \in C$,

$$d_\lambda(x) - \lambda = \varphi^0(x, d_\lambda(x) + \lambda), \tag{4.18}$$

where d_λ is such that $(\lambda, d_\lambda) \in B$, or, in other words that,

$$\lambda \in \mathbf{Q}^N \Rightarrow (\lambda, d_\lambda(x)) \in \mathbf{a}^0_\varphi(x), \quad \text{for a.e. } x \in C.$$

Assuming (4.18) to hold true for a moment, we show in the next step that the theorem is proved upon using φ^0 defined through convergence (4.17) (and its extension to $C \times \mathbf{R}^N$) and the subsequence $\{\varepsilon'\}$ defined at the onset of Step 1. Finally, Step 3 establishes (4.18).

Step 2 – φ^0 defined in Step 1 meets the requirements of the theorem: Consider an arbitrary $f \in W^{-1,p'}(\Omega)$ and $(u^{\varepsilon'}, -div\, d^{\varepsilon'})$, unique solution to

$$\begin{cases} -div\ d^{\varepsilon'} + j(u^{\varepsilon'}) = f \\ u^{\varepsilon'} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $(grad\, u^{\varepsilon'}, d^{\varepsilon'}) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$ and, for a.e. $x \in \Omega$,

$$d^{\varepsilon'}(x) - grad\, u^{\varepsilon'}(x) = \varphi^{\varepsilon'}\left(x, d^{\varepsilon'}(x) + grad\, u^{\varepsilon'}(x)\right).$$

As already noted, (4.5), (4.6) also hold true with the appropriate replacement of $\widehat{\Omega}$ by Ω in the definitions of V, V'. Thus $(u^{\varepsilon'}, -div\, d^{\varepsilon'})$ satisfies

$$\begin{cases} u^{\varepsilon'} \overset{W_0^{1,p}(\Omega)}{\rightharpoonup} u \\ -div\ d^{\varepsilon'} \overset{W^{-1,p'}(\Omega)}{\longrightarrow} g, \end{cases}$$

with

$$\begin{cases} g + j(u) = f \\ (u, g) \in \mathcal{A}^0. \end{cases}$$

Consider a weak accumulation point $d \in L^{p'}(\Omega; \mathbf{R}^N)$ for $d^{\varepsilon'}$. An elementary application of compensated compactness (here the div-curl lemma) ensures, in view of (4.7) and the fact that $u_\lambda \equiv \lambda$ on Ω, that, for a.e. $x \in \Omega$ and for all $\lambda \in \mathbf{Q}^N$,

$$(d_\lambda(x) - d(x)) \cdot (\lambda - grad\, u(x)) \geq 0. \tag{4.19}$$

Fix $x \in \Omega$, so that (4.10), (4.18), (4.19) hold true for all $\lambda \in \mathbf{Q}^N$. Then

$$\mathbf{b}(x) := \left\{(\lambda, d_\lambda(x)) : \ \lambda \in \mathbf{Q}^N\right\}$$

is a monotone subset of $\mathbf{R}^N \times \mathbf{R}^N$, coercive in the sense of (4.10), with dense domain $\mathbf{Q}^N$. Applying once again Theorem 5.2 from Section 5 below, there exists a unique maximal extension, $\overline{\mathbf{b}}(x)$, coercive in the sense of (4.10), that is such that for a.e. $x \in C$ and any $(\lambda, \mu) \in \overline{\mathbf{b}}(x)$,

$$\mu \cdot \lambda \geq -m(x) + \alpha(|\lambda|^p + |\mu|^{p'}). \tag{4.20}$$

Equivalently, appealing to Lemma 2.1 in [4] (Lemma 1.1 of the introduction), there exists a unique contraction $\overline{\varphi}_x : \mathbf{R}^N \to \mathbf{R}^N$ such that, for any $(\lambda, \mu) \in \mathbf{R}^N \times \mathbf{R}^N$,

$$(\lambda, \mu) \in \overline{\mathbf{b}}(x) \quad \text{iff} \quad \mu - \lambda = \overline{\varphi}_x(\mu + \lambda).$$

Thus, in particular, since $\mathbf{b}(x) \subset \overline{\mathbf{b}}(x)$,

$$d_\lambda(x) - \lambda = \overline{\varphi}_x(d_\lambda(x) + \lambda),\ \lambda \in \mathbf{Q}^N.$$

But, by virtue of (4.18), the contraction $\varphi^0(x, \cdot) : \mathbf{R}^N \to \mathbf{R}^N$ also satisfies

$$d_\lambda(x) - \lambda = \varphi^0(x, d_\lambda(x) + \lambda),\ \lambda \in \mathbf{Q}^N.$$

The uniqueness of the contraction $\overline{\varphi}_x$ – which is equivalent, through Lemma 2.1 in [4] (Lemma 1.1 of the introduction) to the uniqueness of the maximal extension of $\mathbf{b}(x)$ – yields

$$\overline{\varphi}_x \equiv \varphi^0(x, \cdot), \tag{4.21}$$

while, by (4.19), $(grad\, u(x), d(x))$ belongs to some maximal extension of $\mathbf{b}(x)$, hence, by uniqueness, to $\overline{\mathbf{b}}(x)$, or still

$$d(x) - grad\, u(x) = \overline{\varphi}_x\left(d(x) + grad\, u(x)\right). \tag{4.22}$$

Since (4.20), (4.21), (4.22) hold true for a.e. $x \in \Omega$, $\varphi^0 \in \mathbf{M}^*(\alpha, m, p, \Omega)$ and Theorem 4.1 is proved.

Step 3 – Proof of (4.18): Recalling (4.13), together with the definition (4.15) of φ_n, and setting

$$d_\lambda^n := \sum_i [d_\lambda]_n^i \chi_n^i,$$

we have

$$d_\lambda^n(x) - \lambda = \varphi_n(x, d_\lambda^n(x) + \lambda),\ \forall \lambda \in \mathbf{Q}^N,\ \text{for a.e. } x \text{ in } C. \tag{4.23}$$

But

$$d_\lambda^n \overset{L^{p'}(C;\mathbf{R}^N)}{\longrightarrow} d_\lambda. \tag{4.24}$$

Now, for all $\lambda \in \mathbf{Q}^N$ and for a.e. $x \in C$,

$$\varphi_n(x, d_\lambda^n(x) + \lambda) = \varphi_n(x, d_\lambda(x) + \lambda) + \left(\varphi_n(x, d_\lambda^n(x) + \lambda) - \varphi_n\left(x, d_\lambda(x) + \lambda\right)\right),$$

so that, since

$$|\varphi_n(x, d_\lambda^n(x) + \lambda) - \varphi_n(x, d_\lambda(x) + \lambda)| \le |d_\lambda^n(x) - d_\lambda(x)|$$

because φ_n is a contraction, (4.24) yields

$$\varphi_n(\cdot, d_\lambda^n(\cdot) + \lambda) - \varphi_n(\cdot, d_\lambda(\cdot) + \lambda) \overset{L^{p'}(C;\mathbf{R}^N)}{\longrightarrow} 0. \tag{4.25}$$

Approximate d_λ, strongly in $L^{p'}(C;\mathbf{R}^N)$, by a sequence of piecewise constant functions with rational values $\{\rho_p\}_p$. Then, exactly as above,

$$\varphi_n(\cdot,d_\lambda(\cdot)+\lambda)-\varphi_n(\cdot,\rho_p(\cdot)+\lambda)\stackrel{L^{p'}(C;\mathbf{R}^N)}{\longrightarrow}0,\ p\nearrow\infty,\ \text{ uniformly in } n. \tag{4.26}$$

But, recalling (4.17) (where φ can be replaced by its extension φ^0), and observing that ρ_p takes its values in $\mathbf{Q}^N$,

$$\varphi_n(\cdot,\rho_p(\cdot)+\lambda)\stackrel{L^{p'}(C;\mathbf{R}^N)}{\rightharpoonup}\varphi^0(\cdot,\rho_p(\cdot)+\lambda). \tag{4.27}$$

Finally,

$$\varphi^0(\cdot,\rho_p(\cdot)+\lambda)-\varphi^0(\cdot,d_\lambda(.)+\lambda)\stackrel{L^{p'}(C;\mathbf{R}^N)}{\longrightarrow}0, \tag{4.28}$$

since φ^0 is a contraction.

Collecting (4.25)–(4.28), we conclude that the right hand side of (4.23) goes to $\varphi^0(\cdot,d_\lambda(\cdot)+\lambda)$, weakly in $L^{p'}(C;\mathbf{R}^N)$, which, together with (4.24) establishes (4.18).

The proof of Theorem 4.1 is complete. □

Theorem 1.2 is a straightforward consequence of Theorem 4.1.

Proof of Theorem 1.2. Consider the subsequence $\{\varepsilon'\}$ for which Theorem 4.1 holds true. If (u,d) is an accumulation point, for the weak topologies of $W_0^{1,p}(\Omega)$ and $L^{p'}(\Omega;\mathbf{R}^N)$ respectively, of a sequence $(u^{\varepsilon'},d^{\varepsilon'})$, solution to

$$\begin{cases}-div\ d^{\varepsilon'}=f\\ u^{\varepsilon'}=0\quad\text{on }\partial\Omega,\end{cases}$$

with $(grad\,u^{\varepsilon'},d^{\varepsilon'})\in L^p(\Omega;\mathbf{R}^N)\times L^{p'}(\Omega;\mathbf{R}^N)$ and, for a.e. $x\in\Omega$,

$$d^{\varepsilon'}(x)-grad\,u^{\varepsilon'}(x)=\varphi^{\varepsilon'}\left(x,d^{\varepsilon'}(x)+grad\,u^{\varepsilon'}(x)\right),$$

then, for a subsequence $\{\varepsilon''\}\subset\{\varepsilon'\}$,

$$\begin{cases}u^{\varepsilon''}\stackrel{W_0^{1,p}(\Omega)}{\rightharpoonup}u\\ d^{\varepsilon''}\stackrel{L^{p'}(\Omega;\mathbf{R}^N)}{\rightharpoonup}d.\end{cases}$$

But, the compactness of the injection from $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ implies that

$$j(u^{\varepsilon''})\stackrel{L^{p'}(\Omega)}{\longrightarrow}j(u),$$

so that

$$-div\ d^{\varepsilon''}+j(u^{\varepsilon''})=f+j(u^{\varepsilon''})\stackrel{W^{-1,p'}(\Omega)}{\longrightarrow}f+j(u).$$

Application of Theorem 4.1 yields that $(grad\,u(x),d(x))\in\mathbf{a}_{\varphi^0}(x)$, for a.e. $x\in\Omega$, while, of course, $-div\ d=f$. □

Now, the result of Theorem 1.2 is actually independent of the boundary conditions on $\partial\Omega$, provided that they guarantee an a priori bound on the solution-sequences. This is the object of the following

Lemma 4.4 *If, in the context of Theorem 1.2,* $(u^{\varepsilon'}, d^{\varepsilon'})$*, solution to*

$$\begin{cases} -div\ d^{\varepsilon'} = f \\ u^{\varepsilon'} \in W^{1,p}(\Omega), \end{cases}$$

with $(grad\, u^{\varepsilon'}, d^{\varepsilon'}) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$ *and, for a.e.* $x \in \Omega$,

$$d^{\varepsilon'}(x) - grad\, u^{\varepsilon'}(x) = \varphi^{\varepsilon'}\left(x, d^{\varepsilon'}(x) + grad\, u^{\varepsilon'}(x)\right),$$

is such that

$$u^{\varepsilon'} \text{ is bounded in } W^{1,p}(\Omega),$$

Then, all accumulation points (u, d) *– for the weak topologies of* $W^{1,p}(\Omega)$ *and* $L^{p'}(\Omega; \mathbf{R}^N)$ *respectively – of that sequence satisfy*

$$-div\ d = f,$$

with $(grad\, u, d) \in L^p(\Omega; \mathbf{R}^N) \times L^{p'}(\Omega; \mathbf{R}^N)$ *and, for a.e.* $x \in \Omega$,

$$d(x) - grad\, u(x) = \varphi^0\left(x, d(x) + grad\, u(x)\right).$$

Proof. First, observe that, by coercivity, $d^{\varepsilon'}$ is bounded in $L^{p'}(\Omega; \mathbf{R}^N)$, hence accumulation points (u, d) for $(u^{\varepsilon'}, d^{\varepsilon'})$ in the weak topology of $W^{1,p}(\Omega) \times L^{p'}(\Omega; \mathbf{R}^N)$ do exist. Then, consider, for any $\lambda \in \mathbf{Q}^N$, the sequence $(u_\lambda^{\varepsilon'}, d_\lambda^{\varepsilon'})$ defined in the proof of Theorem 4.1 (see (4.7)) and recall that, according to (4.18), its weak $\left(W^{1,p}(\Omega) \times L^{p'}\left(\Omega; \mathbf{R}^N\right)\right)$-limit (λ, d_λ) verifies, for a.e. $x \in \Omega$,

$$(\lambda, d_\lambda(x)) \in \mathbf{a}_{\varphi^0}(x).$$

Now, by monotonicity of $\mathbf{a}_{\varphi^{\varepsilon'}}(x)$, for a.e. $x \in \Omega$,

$$(d_\lambda^{\varepsilon'} - d^{\varepsilon'})(x) \cdot (grad\, u_\lambda^{\varepsilon'} - grad\, u^{\varepsilon'})(x) \geq 0.$$

But the inequality passes to the limit through an elementary application of compensated compactness (here the div-curl lemma). Thus, for a.e. $x \in \Omega$,

$$(d_\lambda(x) - d(x)) \cdot (\lambda - grad\, u(x)) \geq 0. \tag{4.29}$$

Since $\varphi^0(x, \cdot)$ is the contraction associated to the unique maximal extension $\overline{\mathbf{b}}(x)$ of $\mathbf{b}(x) = \left\{(\lambda, d_\lambda(x)) :\ \lambda \in \mathbf{Q}^N\right\}$, we conclude, in view of (4.29), that, for a.e. $x \in \Omega$, $(grad\, u(x), d(x)) \in \overline{\mathbf{b}}(x)$, or still that

$$d(x) - grad\, u(x) = \varphi^0\left(x, d(x) + grad\, u(x)\right),$$

and the result is proved. □

5 About maximal extensions

In this last section, we investigate maximal extensions of monotone graphs in a reflexive Banach space. This is a well studied topic and we refer the reader to e.g. [1], [2] for a compendium of results. To our own surprise however, the results we present below seem new. Of course, we do not pretend to have exhaustively surveyed the literature and we will happily renege on the novelty claim, should any reader point to an already available result in that direction.

In any case, let E be, for the remainder of this section, a reflexive Banach space, and let E' denote its topological dual; $\langle , \rangle$ denotes the duality product, that is $\langle \eta, x\rangle = \eta(x),\ x \in E, \eta \in E'$.

Let A be a graph on $E \times E'$ (a subset of $E \times E'$). We recall that $dom(A) := \{x \in E : \exists \eta \in E', (x,\eta) \in A\}$ and call section of A at x the set $A(x) := \{\eta \in E' : (x,\eta) \in A\}$. We say that $A \subset E \times E'$ is locally bounded if

$$\forall x \in dom(A), \exists r > 0 \text{ such that } \sup\{\|\eta\|_{E'} : (y,\eta) \in A \text{ and } \|y-x\| < r\} < \infty.$$

The following theorem holds true:

Theorem 5.1 *If $A \subset E \times E'$ is monotone, $dom(A) = E$, A is locally bounded, and A is sequentially closed in $E \times E'$ for the product of the strong topology on E and of the weak topology on E', then there exists a unique maximal extension $\widehat{A}$ of A. Moreover, for every $x \in E$, the sections of $\widehat{A}$ are the closed convex envelopes of the sections of A, i.e.,*

$$\widehat{A}(x) = \overline{conv A(x)},\ x \in E.$$

Proof. Let B be a maximal extension of A, Then it is straightforward that

$$\overline{conv A(x)} \subset B(x).$$

Assume that $\zeta \notin \overline{conv A(x)}$. Hahn–Banach's theorem implies the existence of $c > d$ and of $f \in E$ such that

$$\langle \zeta, f\rangle \geq c > d \geq \langle \eta, f\rangle,\ \forall \eta \in A(x).$$

Hence

$$\langle \zeta, f\rangle \quad > \sup_{\eta \in A(x)} \langle \eta, f\rangle. \tag{5.1}$$

Now, assume that $\zeta \in B(x)$. Take $x + \frac{1}{n} f \in E$; it belongs to $dom(A) = E$. Since A is locally bounded, $\left\| A\left(x + \frac{1}{n} f\right)\right\|_{E'} \leq C$ for n large enough. Consider $\xi_n \in A\left(x + \frac{1}{n} f\right)$. By Banach–Alaoglu's theorem, there exists $\{n'\} \subset \{n\}$ and $\xi \in E'$ such that

$$\xi_{n'} \overset{E'}{\rightharpoonup} \xi.$$

But A is sequentially strong $\times$ weak closed, thus $(x,\xi) \in A$. By monotonicity,

$$\left\langle \xi_{n'} - \zeta, x + \frac{1}{n'} f - x \right\rangle \quad \geq 0,$$

since $(x,\zeta) \in B$, $(x + \dfrac{1}{n'} f, \xi_{n'}) \in A \subset B$. Consequently,

$$\langle \xi_{n'} - \zeta, f \rangle \quad \geq 0,$$

hence, $\xi \in A(x)$ verifies

$$\langle \xi, f \rangle \quad \geq \langle \zeta, f \rangle,$$

in contradiction to (5.1). Thus $\overline{conv A(x)} \supset B(x)$. □

For any $A \subset E \times E'$, we define the graph

$$\overline{A} := \textit{sequential strong} \times \textit{weak closure of } A, \tag{5.2}$$

and $\widehat{\overline{A}}$, the graph with sections

$$\widehat{\overline{A}} := \{(x,\eta) \in E \times E' : x \in dom(\overline{A}), \eta \in \overline{conv \overline{A}(x)}\}, \tag{5.3}$$

We now obtain the following corollary to Theorem 5.1

Theorem 5.2 *If $A \subset E \times E'$ is monotone, $dom(A)$ is dense in E, and A is coercive in the sense that, for some $\alpha > 0$, $p \geq 0$, $q > 1$, $m \in \mathbf{R}$,*

$$\langle \zeta, x \rangle \quad \geq -m + \alpha(\|x\|_E^p + \|\zeta\|_{E'}^q),\ \forall (x,\zeta) \in A,$$

then there exists a unique maximal extension $\widehat{\overline{A}}$ of A, given by (5.2) and (5.3). Its domain is E and it is also coercive with the same constants.

Proof. First, A is obviously locally bounded (and even bounded on bounded sets). Further, sequential weak lower semi-continuity and convexity of the norm imply that both $\overline{A}$ and $\widehat{\overline{A}}$ are also coercive in that sense, hence (locally) bounded.

Now, $dom(\overline{A}) = E$. Indeed, if $x \in E$, there exists $x_n \in dom(A)$ converging to x. Take $\zeta_n \in A(x_n)$; since A is locally bounded $\|\zeta_n\|_{E'} \leq C$, provided n is large enough. Banach–Alaoglu's theorem implies the existence of $\{n'\} \subset \{n\}$ and $\zeta \in E'$ such that

$$\zeta_{n'} \overset{E'}{\rightharpoonup} \zeta.$$

But, by definition, $(x,\zeta) \in \overline{A}$, hence the result.

We now apply Theorem 5.1 to $\overline{A}$ and get $\widehat{\overline{A}}$ as the unique maximal extension of $\overline{A}$. But if B is a maximal extension of A, $\overline{B}$ is an extension of B, hence $B = \overline{B}$. Since $B \supset A$, $\overline{B} = B \supset \overline{A}$, thus B is a maximal extension of $\overline{A}$, hence, by uniqueness $B = \widehat{\overline{A}}$. □

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