EXISTENCE AND CONVERGENCE FOR QUASI–STATIC EVOLUTION IN BRITTLE FRACTURE

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Abstract

This paper investigates the mathematical well–posedness of the variational model of quasi–static growth for a brittle crack proposed by FRANCFORT & MARIGO in [14]. The starting point is a time discretized version of that evolution which results in a sequence of minimization problems of MUMFORD & SHAH type functionals. The natural weak setting is that of special functions of bounded variation, and the main difficulty in showing existence of the time–continuous quasi–static growth is to pass to the limit as the time–discretization step tends to 0. This is performed with the help of a jump transfer theorem which permits, under weak convergence assumptions for a sequence $\{u_n\}$ of SBV–functions to its BV–limit u, to transfer the part of the jump set of any test field that lies in the jump set of u onto that of the converging sequence $\{u_n\}$. In particular, it is shown that the notion of minimizer of a MUMFORD & SHAH type functional for its own jump set is stable under weak convergence assumptions. Furthermore, our analysis justifies numerical methods used for computing the time–continuous quasi–static evolution.

Keywords: fracture, functions of bounded variation, geometric measure theory, free discontinuity problems, MUMFORD & SHAH functional, quasi–static evolution.

1 Introduction

1.1 The mechanical environment

The foundation of the theory of brittle fracture stands unshaken since the work of A. GRIFFITH in the 1920's [16]. He viewed fracture in an elastic material as the result of the competition between surface and bulk energy. The modern theory of brittle fracture was formalized in his footsteps.

The basic ingredients are the toughness (surface energy density), denoted by G_c, and the energy release rate usually denoted by G and defined, in a two–dimensional setting, as

$$G := -\frac{dW}{dl},$$

where $W(l)$ is the bulk energy ([elastic] $-$ [work of external loads]) stored in the material for a crack of length l. Of course, this presupposes that the crack is unambiguously defined, once its length is known. The model is then extremely simple: Propagation will take place ($\dot{l} \neq 0$) if $G = G_c$, and will not if $G < G_c$.

That theory has been widely used and has demonstrated the breadth of its scope. It is however plagued by three major defects which we now briefly mention, referring the interested reader to e.g. [14] for a more in-depth analysis. Crack initiation is generically impossible, so that a previously uncracked specimen will remain such, independently of the magnitude of the load. The crack path has to be guessed *a priori* to lend any kind of meaning to the formulation. The crack length cannot experience jumps; in other words, crack evolution along its path is smooth.

Mechanicians have proposed various remedies – crack notching to promote initiation, branching criteria to vary the crack path, generalized energy release rate to palliate crack jumps (see e.g. [14] and references therein for more detail on this). The merits of additional ingredients notwithstanding, mechanics seems forced to import elements from outside the original framework in an *ad–hoc* manner.

The main purpose of [14] was to demonstrate how a slight departure from the classical theory could placate the aforementioned obstacles. The main idea, borrowed from D. Mumford & J. Shah's approach to image segmentation [21], is that the crack wants to quasi–statically minimize its total energy among all legal competitors. In other words, the crack $\Gamma(t)$ must minimize $W(\Gamma) +$ length of Γ among all $\Gamma \supset \Gamma(s),\ s < t$.

Specifically, in a N–dimensional setting, let $g(t)$ (a function of x) be the boundary condition on the sample Ω at time t, and, for any compact $\Gamma \subset \overline{\Omega}$, denote by $u(t, \Gamma)$ the solution to the elastic equilibrium on $\Omega \setminus \Gamma$ with boundary condition

$$\begin{cases} u(t,\Gamma) = g(t) \text{ on } \partial\Omega_f^c \setminus \Gamma \\ \dfrac{\partial W}{\partial \varepsilon}\Big(\varepsilon(u(t,\Gamma))\Big)n = 0 \text{ on } \partial\Omega_f \cup \Gamma, \end{cases}$$

where $\partial\Omega_f$ is the force–free part of the boundary $\partial\Omega$, while n denotes the unit normal to points on $\partial\Omega_f \cup \Gamma$. We remark that the competing crack Γ may choose to run alongside the part of the boundary $\partial\Omega_f^c$ where the displacement load is applied.

The crack at time t, $\Gamma(t)$, must minimize the total energy

$$\mathcal{E}(t) := \int_{\Omega\setminus\Gamma} W(\varepsilon(u(t,\Gamma)))\ dx + G_c \mathcal{H}^{N-1}(\Gamma \setminus \partial\Omega_f), \tag{1.1}$$

among all sets $\Gamma \supset \bigcup_{s<t} \Gamma(s)$. In the previous expression for the total energy, the elastic energy, a function of the symmetrized gradient $\varepsilon(u) := 1/2(\nabla u + \nabla^t u)$, is a well-behaved convex energy — in linearized elasticity, it is quadratic in ε, that is $W(\varepsilon) := 1/2 A\varepsilon \cdot \varepsilon$, A being the stiffness tensor— and the surface energy is assumed, following A. Griffith, to be proportional to the surface area of the crack.

Simple examples of indeterminacy [14] show that the evolution must be further constrained by imposition of a condition on the time evolution of $\mathcal{E}(t)$ if the propagation criterion of A. Griffith, that is $G = G_c$ if $\dot{l} \neq 0$ is to be preserved in the current setting. In [14], a criterion is proposed in the case where $g(t)(x) = tg(x)$, labelled that of *proportional loads*. The general form of that criterion, independent of the form of the loading process, was first explicitly stated in the study of G. Dal Maso & R. Toader [11]. The change in total energy is precisely the work of the external loads, or in other words,

$$\mathcal{E}(t) - \mathcal{E}(s) = \int_s^t \int_\Omega \frac{\partial W}{\partial \varepsilon}\Big(\varepsilon(u(\tau, \Gamma(\tau))\Big) \cdot \varepsilon(\dot{g}(\tau))\ dx\ d\tau. \tag{1.2}$$

The time–continuous model (1.1,1.2) is discussed at length in [14]. It is shown there to cure the three defects of "classical" brittle fracture: initiation always occurs in finite time for a proportional load; the crack path(s) is(are) picked by the minimization process, and the crack length may jump if it so wishes.

The proposed model only accommodates displacement loads because global minimization of the sum of bulk and surface energy will fail if, for example, surface traction loads are applied to part of the boundary of the sample, as is immediately seen by introducing a crack of finite surface area that will separate the loaded part of the boundary from the remaining part of the sample. Various remedies have been proposed, most notable among them, the search for local minimizers. In this direction, see [20]. Note however that the investigation of local minimality is severely limited, in dimension greater than 1, by the paucity of available mathematical tools.

Numerical implementation of the time evolution has been attempted by several authors (see e.g. [5], [4], [3]). In all cases, the continuous–time evolution is replaced by a finite time step approach: at each time step, a "static" problem is solved and time evolution is only evidenced through the time–discretized irreversibility constraints (for example, the admissible crack sites at a given time all contain the crack site at the prior time step). Further comments on numerics will be made in Subsection 1.3.

Until very recently, there was no framework for the existence of the continuous–time evolution. The first breakthrough in that direction is that of G. Dal Maso & R. Toader [11] who obtained an existence result in the antiplane setting (u scalar–valued), albeit with a method that requires a *two–dimensional setting* and the following restrictive assumption on possible cracks: the maximal number of connected components of the potential cracks is set *a priori*. This was immediately followed by the work of A. Chambolle [8] who tackled the planar elasticity setting under the same restrictive assumptions.

In this paper, we do away with all such limitations and establish existence for the time–continuous evolution in the *generalized antiplane case* (Theorem 1.1), this in *any dimension and without any restriction regarding connectedness.* For the sake of simplicity, we restrict our attention to the case of a homogeneous material with quadratic elastic energy, but there would be no obstacles to considering a heterogeneous material, each phase possessing a smooth (strictly) convex elastic energy with p–growth ($p > 1$). We also a–dimensionalize the problem and, in particular, drop the factor 1/2 in front of the elastic energy.

We remark that the elastic case remains unsolved at present for reasons that will be evoked in Subsection 1.4.

1.2 The mathematical environment

As mentioned above, the similarity between the total energy functional and the functional devised by D. Mumford & J. Shah in the context of image segmentation is striking. In essence, at fixed time, the problem is as follows: Minimize

$$\int_{\Omega\setminus\Gamma} |\nabla u|^2\, dx + \mathcal{H}^{N-1}(\Gamma), \tag{1.3}$$

among

$$\begin{cases} u \in H^1(\Omega'\setminus\Gamma);\ u \equiv g \text{ on } \Omega' \setminus \overline{\Omega} \\ \Gamma \subset \overline{\Omega};\ \Gamma \text{ closed}, \end{cases}$$

with $\Omega' \supset\supset \Omega$ and $g \in H^1(\Omega') \cap L^\infty(\Omega')$. We refer to this formulation as the *strong* formulation.

A host of obstacles have to be overcome, so as to successfully perform that minimization process. In particular, a Neumann sieve type phenomenon must be excluded. We recall that, typically, a Neumann sieve situation occurs when boundaries close up at a critical speed that creates channels in the domain of non–zero capacity [22]. As an example, consider the Neumann problem

$$-\Delta u_n = 0 \text{ on } \Omega_n := (-1,1)^2 \setminus \Big\{(0,y) : y \notin \bigcup_{p=0,\dots,n} (\pm p/n - \exp^{-n}, \pm p/n + \exp^{-n})\Big\},$$

with

$$\begin{cases} \dfrac{\partial u_n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_n \setminus \{x = \pm 1\} \\ u = 0 & \text{on} \quad \partial\Omega_n \cap \{x = -1\} \\ u = 1 & \text{on} \quad \partial\Omega_n \cap \{x = 1\}. \end{cases}$$

Then, $u_n \to u$ strongly in $L^2(\Omega)$, with $\Omega = [(-1,0) \cup (0,1)] \times (-1,1)$ and u the solution to

$$-\Delta u = 0 \text{ on } \Omega,$$

with

$$\begin{cases} \dfrac{\partial u}{\partial y} &=& 0 & \text{on} & \partial\Omega \cap \{y = \pm 1\} \\ u &=& 0 & \text{on} & \partial\Omega \cap \{x = -1\} \\ u &=& 1 & \text{on} & \partial\Omega \cap \{x = 1\} \\ \dfrac{\partial u}{\partial x} &=& \mu[u] & \text{on} & \{0\} \times (-1,1), \end{cases}$$

where $\mu[u] > 0$. Thus u_n does not converge to the solution

$$u' = \begin{cases} 0 \text{ on } (-1,0) \times (-1,1) \\ 1 \text{ on } (0,1) \times (-1,1) \end{cases}$$

of the Neumann problem for the complement Ω of the Hausdorff limit Ω^c of Ω_n^c.

Two methods have been devised to overcome such obstacles. The first tackles directly the strong formulation (1.3). If no assumptions of connectedness are placed on the test Γ's, a rather elaborate two-dimensional existence result is derived in [10], which was extended to any dimension in [19]. In the *two–dimensional* case however, if the number of connected components of the test Γ's is *a priori* uniformly bounded — which will in particular do away with the Neumann sieve issue — then existence is much easier: it results from the analysis of the limit of the solutions of Neumann problems on varying domains, that is

$$\begin{aligned} -\Delta u_n + u_n = f \quad &\text{on} \quad \Omega_n \\ \frac{\partial u_n}{\partial \nu} = 0 \quad &\text{on} \quad \partial\Omega_n \end{aligned}$$

where $\partial\Omega_n$ is connected, $\mathcal{H}^1(\partial\Omega_n) \le C < \infty$, and $\Omega_n^c \to \Omega^c$ in the sense of the Hausdorff metric. The solutions are then shown to satisfy $\nabla u_n \to \nabla u$, strongly in $L^2(\Omega)$. For this and related extensions, see [9], [6].

The second, devised by E. De Giorgi, performs the analysis of a weak formulation of (1.3) on the space $SBV(\Omega')$. It consists of minimizing

$$\int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \cap \overline{\Omega}) \tag{1.4}$$

among $\{v \in SBV(\Omega') : v \equiv g \text{ on } \Omega' \setminus \overline{\Omega}\}$. Use of L. Ambrosio's SBV–compactness theorem [1] easily leads to the existence of a minimum u. Furthermore, a regularity result — namely that $\mathcal{H}^{N-1}(\overline{S(u)} \setminus S(u)) = 0$ — obtained in [12], [7] establishes that the pair $\left(u, \Gamma := \overline{S(u)}\right)$ also minimizes (1.3).

Now, in the context of fracture, time is not set and a natural idea is to start with a time–discretized evolution and to let the time step tend to 0. In the two–dimensional antiplane case and for an *a priori* bound on the number of connected components of the test cracks, G. Dal Maso & R. Toader perform that limit process in [11]. This is in turn what A. Chambolle achieves in plane elasticity, still with the same restriction on the test cracks.

Our goal in this paper is to perform the same limit process in the generalized antiplane case. This will be achieved through the use of the weak formulation *in any dimension and with no restrictions on the test cracks*. The results are briefly described in Subsection 1.4.

1.3 A note on numerics

When taking the time–continuous limit of a discrete time–step approximation, one cannot generally expect strong convergence of solutions or of energies. Our method produces precisely these strong convergences, thereby reducing the numerical issues for the time–continuous problem to those at a fixed time–step.

Now, to our knowledge, all justified numerical approximations are based upon Γ–convergence results to the weak, à la E. De Giorgi, formulation (see e.g. [3] and references therein), which is precisely the standpoint adopted in the present study. In particular, our result validates *a posteriori* the computations presented in [4].

1.4 Mathematical setting and outline

So as to describe our results more precisely, we need to introduce a few concepts and definitions. First, we refer to e.g. [13], [2], [23] for the definitions and basic properties of BV and SBV functions, which will not be recalled here.

For the remainder of this paper, $\to$ will denote strong convergence in the appropriate topology, $\rightharpoonup$, weak convergence in the appropriate topology, and $\overset{*}{\rightharpoonup}$, weak-* convergence in the appropriate topology. Further, Ω will be a Lipschitz bounded domain and Ω' a bounded domain such that $\overline{\Omega} \subset \Omega'$ (also denoted $\Omega \subset\subset \Omega'$). The introduction of Ω' facilitates the enforcement of Dirichlet boundary conditions, but the choice of a specific Ω' is unimportant.

Throughout the paper, we will denote by $[v(x)]$ the jump of v at the point x. We will say that

- A sequence $\{v_n\}$ (of elements of $SBV(\Omega)$) $SBV(\Omega)$*–converges to* $v \in SBV(\Omega)$ if

$$\begin{cases} \nabla v_n \rightharpoonup \nabla v \text{ in } L^1(\Omega); \\ [v_n]\nu_n \mathcal{H}^{N-1} \lfloor S(v_n) \overset{*}{\rightharpoonup} [v]\nu\mathcal{H}^{N-1}\lfloor S(v) \text{ as measures;} \\ v_n \to v \text{ in } L^1(\Omega);\ \text{and} \\ v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty(\Omega), \end{cases}$$

where ν denotes the normal to $S(v)$ at a given point. Note that, as a consequence of the above convergences (see [1]),

$$\mathcal{H}^{N-1}(S(v)) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(S(v_n)).$$

Also note that, if E is a Borel set such that $\mathcal{H}^{N-1}(E) < \infty$, then,

$$\mathcal{H}^{N-1}(S(v) \setminus E) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(S(v_n) \setminus E),$$

as can be immediately checked by using the fact that $\mathcal{H}^{N-1}\lfloor E$ being a Radon measure, $\mathcal{H}^{N-1}\lfloor(S(v))$ can be approximated from above by $\mathcal{H}^{N-1}\lfloor(U)$ for some open set $U \supset S(v)$.

- An element u of $SBV(\Omega')$ is *a minimizer for its own jump set* if and only if

$$\int_\Omega |\nabla u|^2\, dx \le \int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}(S(v) \setminus S(u)),$$

for all $v \in SBV(\Omega')$ with $v \equiv u$ on $\Omega' \setminus \overline{\Omega}$.

Our first result presented in Section 2 is a geometric measure theoretic result (Theorem 2.1); it potentially applies to many situations other than the specific setting of fracture, and will hopefully reveal useful. It is the cornerstone of our analysis because it enables us to show that minimizers for their own jump sets are stable under SBV–convergence (Corollary 2.10), and further, that the absolutely continuous part of the measure gradient of those minimizers converges strongly in L^2. Actually, a generalization of the transfer theorem to a more general setting proves useful (Theorem 2.8). The reader who has no interest in the more technical issues of geometric measure theory may take Theorems 2.1 and 2.8 at faith value without prejudice for the understanding of the remainder of the paper.

Section 3 addresses the issue of fracture evolution in the weak, à la E. De Giorgi, setting, at discrete times between 0 and 1. We assume that the displacement load $g(t)$ is applied to a part $\partial\Omega_f^c$ of the boundary $\partial\Omega$, while the remaining part $\partial\Omega_f$ of the boundary (a closed set) is load free. A crack will consequently pay no surface energy when sitting on $\partial\Omega_f$, a fact that we will transcribe by removing the contribution of that part of the crack to the surface energy. The load g is assumed to possess the following regularity:

$$g \in L^\infty((0,1) \times \Omega') \cap W^{1,1}((0,1); H^1(\Omega')).$$

Note that we assume an L^∞–bound on g, so as to be in a position to apply the maximum principle. This is a major hurdle in the fully elastic case —in lieu of the generalized antiplane case— because there is no maximum principle, so that even the existence of minimizers for the weak formulation at discrete times is not clear.

We pick the discretization times t in a countable dense set $I_\infty \subset [0,1]$ containing 0, and then pick a sequence of finite nested subsets $\{I_n\}$ and write $I_n = \{t_k^n : k = 0, ..., n\}$ with $t_0^n = 0$. At time t_k^n, we consider u_k^n, a minimizer for

$$\int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus \Big[\bigcup_{0\le j\le k-1} S(u_j^n) \cup \partial\Omega_f \Big]\Big)$$

in $\{v \in SBV(\Omega') : v \equiv g(t_k^n) \text{ on } \Omega' \setminus \overline{\Omega}\}$. In essence, the crack at time t_k^n is $\bigcup_{0\le j\le k} S(u_j^n)$. We want to let $n \nearrow \infty$. For a fixed $t = t_{k(n)}^n \in I_\infty$, we easily obtain estimates that permit us to conclude the existence of an $SBV(\Omega')$–converging subsequence; the limit is denoted by $u(t)$. Further, with the help of Lemma 3.1, we control the total jump set $\mathcal{H}^{N-1}\Big(\bigcup_{\substack{s\in I_\infty \\ s\le t}} S(u(s))\Big)$, independently of $t \in I_\infty$.

We then apply successively the Jump Transfer Theorems, Theorem 2.1 and its extension Theorem 2.8, to show that $\nabla u_n(t) \to \nabla u(t)$ in $L^2(\Omega')$, and that $u(t)$ minimizes

$$\int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus [\bigcup_{\substack{\tau\in I_\infty \\ \tau\le t}} S(u(\tau)) \cup \partial\Omega_f]);$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$ (see Lemma 3.4).

Defining the total energy to be

$$\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2 \, dx + \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\le t \\ \tau\in I_\infty}} S(u(\tau)) \setminus \partial\Omega_f \Big)$$

we obtain in Lemma 3.7 two inequalities that resemble what we would like to obtain, that is the integral version of (1.2),

$$\mathcal{E}(t) = \mathcal{E}(0) + 2 \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s) \, dx \, ds,$$

but $u(t)$ is not even defined yet for $t \notin I_\infty$.

We then proceed to extend u to all t's in $[0,1]$ by considering increasing sequences $t_n \in I_\infty \nearrow t$ and setting $u(t) := \lim_{n\to\infty} u(t_n)$. We show, using a consequence of the Jump Transfer Theorem (Lemma 3.8), that $u(t)$ minimizes

$$\int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t) \cup \partial\Omega_f)),$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$. The crack $\Gamma(t)$ is defined as

$$\Gamma(t) := \bigcup_{\substack{\tau\in I_\infty \\ \tau\le t}} S(u(\tau)),$$

and it is shown that $\Gamma(t) \supset S(u(t))$, up to a set of $\mathcal{H}^{N-1}$-measure zero.

It then only remains to show that we indeed obtain the integral version of (1.2) for the total energy at t. This is the object of Lemma 3.9; the argument is based on a somewhat delicate estimate of $\|\nabla u(t) - \nabla u(s)\|_{L^2(\Omega)}$ in terms of both $\int_s^t \|\nabla \dot{g}(\tau)\|_{L^2(\Omega)}\, d\tau$ and of the increase $\mathcal{H}^{N-1}(\Gamma(t) \setminus \Gamma(s))$ in crack area from s to t.

The final result, which establishes the existence of a continuous–time fracture evolution following the model proposed in [14] is given in the following

Theorem 1.1 *For any Lipschitz domain $\Omega \subset \mathbb{R}^N$, any closed $\partial\Omega_f \subset \partial\Omega$, and any element $g \in L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^N)) \cap W^{1,1}_{loc}([0,\infty); H^1(\mathbb{R}^N))$, there exists a crack $\Gamma(t) \subset \overline{\Omega}$ and a field $u(t) \in SBV(\mathbb{R}^N)$ such that*

- *$\Gamma(t)$ increases with t;*
- *$u(0)$ minimizes*
$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}(S(v) \setminus \partial\Omega_f)$$
among all v in $SBV(\mathbb{R}^N)$ with $v = g(0)$ on $\mathbb{R}^N \setminus \overline{\Omega}$;
- *$u(t)$, $t > 0$, minimizes*
$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t) \cup \partial\Omega_f))$$
among all v in $SBV(\mathbb{R}^N)$ with $v = g(t)$ on $\mathbb{R}^N \setminus \overline{\Omega}$; and
- *$S(u(0)) = \Gamma(0)$, and $S(u(t)) \subset \Gamma(t)$ up to a set of $\mathcal{H}^{N-1}$ – measure 0.*

Furthermore, the total energy
$$\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2 dx + \mathcal{H}^{N-1}(\Gamma(t) \setminus \partial\Omega_f)$$
is absolutely continuous and is given by
$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s)\, dx\, ds.$$

Finally, for any countable, dense set $I \subset [0,\infty)$ containing zero, the crack $\Gamma(t)$ and the field $u(t)$ can be chosen such that
$$\Gamma(t) = \bigcup_{\substack{\tau \in I \\ \tau \leq t}} S(u(\tau)).$$

Remark 1.2 *The extension to continuous times uses limits of sequences from below. Once Γ is defined on a countable dense set I, the limit from below is actually unique, be it for $\Gamma(t)$, or for $u(t)$, as can be easily deduced from the minimality of the limit $u(t)$, together with the strict convexity of the bulk energy.*

Remark 1.3 *With Theorem 1.1 in hand, we can easily recover in the present context the various properties of the crack set(s) evidenced in [11].*

2 Transfer of Jump Sets

The main tool of our analysis is a geometric measure theoretic result which will enable us to transfer most of the jump set of any function in SBV that lies inside the jump set of a function u onto that of u_n, if u_n SBV-converges to u.

We set $SBV_q(\Omega) := \{v \in SBV(\Omega) : \nabla v \in L^q(\Omega)\}$.

Theorem 2.1 *Let $\overline{\Omega} \subset \Omega'$, with $\partial\Omega$ Lipschitz, and let $\{u_n\} \subset SBV(\Omega')$ be such that*

- $S(u_n) \subset \overline{\Omega}$;
- $|\nabla u_n|$ *weakly converges in* $L^1(\Omega')$; *and*
- $u_n \to u$ *in* $L^1(\Omega')$,

where $u \in BV(\Omega')$ with $\mathcal{H}^{N-1}(S(u)) < \infty$. Then, for every $\phi \in SBV_q(\Omega')$, $1 \le q < \infty$, with $\mathcal{H}^{N-1}(S(\phi)) < \infty$, there exists a sequence $\{\phi_n\} \subset SBV_q(\Omega')$ with $\phi_n = \phi$ on $\Omega' \setminus \overline{\Omega}$ such that

i) $\phi_n \to \phi$ *strongly in* $L^1(\Omega')$;

ii) $\nabla\phi_n \to \nabla\phi$ *strongly in* $L^q(\Omega')$; *and*

iii) $\mathcal{H}^{N-1}\Big([S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]\Big) \to 0.$

Remark 2.2 *In particular, item iii) of the theorem implies that*

$$\limsup_{n\to\infty} \mathcal{H}^{N-1}(S(\phi_n) \setminus S(u_n)) \le \mathcal{H}^{N-1}(S(\phi) \setminus S(u)).$$

Remark 2.3 *Various extensions of Theorem 2.1 can be easily established. Firstly, the Lipschitz assumption on Ω of Theorem 2.1 can be replaced by Ω having finite perimeter with no alteration in the proof. Secondly, the assumption that $\mathcal{H}^{N-1}(S(u)) < \infty$ can also be dropped with minor alterations in the proof.*

In a different direction, a more delicate refinement of Theorem 2.1 is being pursued in [18], in order to study some more general settings, including that where other types of surface energies can be considered, besides the $\mathcal{H}^{N-1}$–measure of the jump set.

If, in the proof of Theorem 2.1 presented below, no attention is paid to the part of the jump sets that lie on $\partial\Omega$, Theorem 2.1 becomes

Theorem 2.4 *Let Ω be a bounded domain and let $\{u_n\} \subset SBV(\Omega)$ be such that*

- $|\nabla u_n|$ *weakly converges in* $L^1(\Omega)$;*and*
- $u_n \to u$ *in* $L^1(\Omega)$,

where $u \in BV(\Omega)$ with $\mathcal{H}^{N-1}(S(u)) < \infty$. Then for every $\phi \in SBV_q(\Omega)$, $1 \le q < \infty$, with $\mathcal{H}^{N-1}(S(\phi)) < \infty$, there exists a sequence $\{\phi_n\} \subset SBV_q(\Omega)$ such that

i) $\phi_n \to \phi$ *strongly in* $L^1(\Omega)$;

ii) $\nabla\phi_n \to \nabla\phi$ *strongly in* $L^q(\Omega)$; *and* *iii)* $\mathcal{H}^{N-1}\Big([S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]\Big) \to 0.$

Remark 2.5 *In Theorem 2.4, we actually have $\phi_n = \phi$ outside a compact subset of Ω.*

Proof of Theorem 2.1. As in [17], our strategy is to analyze $S(u)$ by considering a countable family of level sets for u, and noting that $S(u)$ equals the union of pairwise intersections of these sets. The coarea formula then allows us to control the parts of the corresponding level sets for u_n that lie outside $S(u_n)$. It is onto pieces of these level sets that we will transfer $S(\phi) \cap S(u)$.

The plan is as follows. First, we write $S(u)$ as a countable union of pairwise intersections of boundaries of super-level sets, $\partial_* E_s \cap \partial_* E_t$. We cover this union with cubes such that within each cube, $S(u) \cap \partial_* E_t$ is close to a hyperplane through the center of the cube. We then choose a finite disjoint subcollection that almost covers $S(u)$. By the L^1 convergence of u_n to u, we have that $\chi_{E_t^n} \to \chi_{E_t}$ in L^1, where E_t^n is the t level set for u_n. By the equiintegrability of ∇u_n, we know that within each cube Q_i, we can choose t_i such that $\mathcal{H}^{N-1}(\partial_* E_{t_i}^n \setminus S(u_n))$ goes to zero. Finally, within each cube, we define two functions, ϕ_- and ϕ_+. ϕ_- is obtained by reflecting ϕ from one side of a hyperplane close to the hyperplane through the center of the cube, and analogously for ϕ_+ from the other side of this hyperplane. We then take ϕ_n to be equal to ϕ_- on its side of $\partial_* E_{t_i}^n$ and equal to ϕ_+ on the other side, thus transferring $S(\phi) \cap S(u)$ from $S(u)$ to $\partial_* E_{t_i}^n$, except for a set of small $\mathcal{H}^{N-1}$ measure. As noted above, these boundaries lie almost entirely within $S(u_n)$, resulting in ϕ_n having all the properties stated in the theorem.

In the sequel of this proof, we set E_t to be the set of all Lebesgue–density 1 points for $\{x \in \Omega : u(x) > t\}$, and, similarly E_t^n to be that of all Lebesgue–density 1 points for $\{x \in \Omega : u_n(x) > t\}$.

Define $L := \{t \in \mathbb{R} : |\{x \in \Omega : u(x) = t\}| = 0\}$. Note that L^c is at most countable, because u is measurable. Now consider a dense countable set $D \subset L$ such that for every $t \in D$, E_t has finite perimeter. This is possible due to the coarea formula for BV functions.

Recall that

$$S(u) = \bigcup_{\substack{t_1, t_2 \in D \\ t_1 < t_2}} (\partial_* E_{t_1} \cap \partial_* E_{t_2}).$$

Indeed, if $x \in S(u)$, then $x \in \partial_* E_t$ for all $t \in (u_-(x), u_+(x))$ (see the proof of Theorem 1 in Section 5.9 of [13]). Therefore, up to a set of $\mathcal{H}^{N-1}$ measure zero, $S(u)$ equals

$$G := \bigcup_{\substack{t_1, t_2 \in D \\ t_1 < t_2}} (\partial^* E_{t_1} \cap \partial^* E_{t_2})$$

(the difference being that we have switched from measure theoretic boundaries to reduced boundaries). For each $x \in G$, we choose $t_1(x) < t_2(x)$ from D such that $x \in \partial^* E_{t_1(x)} \cap \partial^* E_{t_2(x)}$ and $t_2(x) - t_1(x) \geq \frac{1}{2}(u_+(x) - u_-(x))$. Note that, by Theorem 5.9.6 (iii) of [23], $\partial^* E_{t_1(x)}$ and $\partial^* E_{t_2(x)}$ share a common outward normal $\nu(x)$ at x.

Since for all $t \in D$, $\mathcal{H}^{N-1}$-a.e. $x \in S(u) \cap \partial^* E_t$ is a Lebesgue point for $\chi_{S(u)}$ with respect to the measure $\mathcal{H}^{N-1} \lfloor \partial^* E_t$,

$$\frac{\mathcal{H}^{N-1}(S(u) \cap \partial^* E_t \cap Q_r(x))}{\mathcal{H}^{N-1}(\partial^* E_t \cap Q_r(x))} \to 1$$

as $r \to 0$, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u) \cap \partial^* E_t$. Here, $Q_r(x)$ has the same normal as $\partial^* E_t$ at x. By Corollary 1 in Section 5.7.2 and Theorem 2 in Section 5.7.3 of [13], we have

$$\frac{\mathcal{H}^{N-1}(\partial^* E_t \cap Q_r(x))}{(2r)^{N-1}} \to 1$$

as $r \to 0$, and therefore,

$$\frac{\mathcal{H}^{N-1}(S(u) \cap \partial^* E_t \cap Q_r(x))}{(2r)^{N-1}} \to 1$$

as $r \to 0$, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u) \cap \partial^* E_t$.

Furthermore, according to Theorem 2.63 in [2], for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$,

$$\frac{\mathcal{H}^{N-1}(S(u) \cap Q_r(x))}{(2r)^{N-1}} \to 1$$

as $r \to 0$. Thus,

$$\frac{\mathcal{H}^{N-1}([S(u) \setminus \partial^* E_t] \cap Q_r(x))}{(2r)^{N-1}} \to 0 \tag{2.1}$$

as $r \to 0$, again for $\mathcal{H}^{N-1}$-a.e. $x \in S(u) \cap \partial^* E_t$. Since D is countable, (2.1) holds for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$ and all $t \in D$ such that $x \in \partial^* E_t$.

Set $C := \{x \in \partial\Omega : \partial\Omega \text{ is not differentiable at } x\}$ and set

$$G_j := \{x \in G \backslash C : (2.1) \text{ holds with } t = t_1(x) \text{ and } [u(x)] > \frac{1}{j}\},$$

and note that $G = \cup_{j \in \mathbb{N}} G_j$ up to a set of $\mathcal{H}^{N-1}$-measure zero. Now fix $\varepsilon > 0$. Choose j such that

$$\mathcal{H}^{N-1}(G \setminus G_j) < \varepsilon. \tag{2.2}$$

We now finely cover G_j up to a set of $\mathcal{H}^{N-1}$-measure zero, as follows. We first consider all closed cubes $Q_r(x)$ centered at $x \in G_j$ with radius r and with normal $\nu(x)$, such that $Q_r(x) \subset \Omega$ if $x \in \Omega$. We denote by $H_r(x)$ the intersection of $Q_r(x)$ with the hyperplane through x with normal $\nu(x)$. Our fine cover is then the family of all such cubes with the following additional properties:

$$\begin{aligned}
&1.\quad \mathcal{H}^{N-1}(S(u) \cap \partial Q_r(x)) = 0;\\
&2.\quad r < r(x), \text{ with } r(x) \text{ to be specified below;}\\
&3.\quad r^{N-1} \le c\mathcal{H}^{N-1}(S(u) \cap Q_r(x));\\
&4.\quad \mathcal{H}^{N-1}([S(u) \setminus \partial^* E_{t_1(x)}] \cap Q_r(x)) < \varepsilon r^{N-1};\\
&5.\quad \mathcal{H}^{N-1}(\{y \in \partial^* E_{t_1(x)} \cap Q_r(x) : \operatorname{dist}(y, H_r(x)) \ge \tfrac{\varepsilon}{2} r\}) < \varepsilon r^{N-1}; \text{ and}\\
&6.\quad \mathcal{H}^{N-1}\Big((S(\phi) \setminus S(u)) \cap Q_r(x)\Big) < \varepsilon r^{N-1},
\end{aligned} \tag{2.3}$$

where property 3 is possible due to Theorem 2 in Section 2.3 of [13], property 4 follows immediately from (2.1), property 5 from (5.6.11) in Theorem 5.6.5 of [23], and property 6 comes from the fact that $S(\phi)\setminus S(u)$ has $\mathcal{H}^{N-1}$–density 0, $\mathcal{H}^{N-1}$–a.e. in $S(u)$. In addition, for $x \in \partial\Omega$, we require that

$$\mathrm{dist}(Q_r(x) \cap \partial\Omega, H_r(x)) < \frac{\varepsilon}{2} r \tag{2.4}$$

as well as that

$$\mathcal{H}^{N-1}(Q_r(x) \cap \partial\Omega) - (2r)^{N-1} < \varepsilon r^{N-1},$$

which we can do since $x \notin C$.

Using Theorem 1 in Section 5.7.2 of [13], we can choose $r(x)$ so small that for $r < r(x)$,

$$\|\chi_{E_{t_1(x)} \cap Q_r(x)} - \chi_{Q_r^-(x)}\|_{L^1(Q_{r(x)})} < \varepsilon^2 |Q_r(x)|, \tag{2.5}$$

and similarly for $E_{t_2(x)}$. Here, $Q_r^-(x) := \{y \in Q_r(x) : (y - x) \cdot \nu(x) < 0\}$.

We now apply the Besicovitch covering theorem to this fine cover, with respect to the Radon measure $(1 + v)\mathcal{L}^N + \mathcal{H}^{N-1}\lfloor S(u)$, where v is the weak-L^1 limit of $|\nabla u_n|$. Consequently, there exists a *finite disjoint* subcollection $\{Q_{r_i}(x_i)\}$ such that

$$\begin{cases} |\cup_i Q_{r_i}(x_i)| & < \varepsilon \\ \mathcal{H}^{N-1}(G_j \setminus \cup_i Q_{r_i}(x_i)) & < \varepsilon \\ \displaystyle\int_{\cup_i Q_{r_i}(x_i)} v \, dx & < \frac{\varepsilon}{2j}. \end{cases} \tag{2.6}$$

The idea is to define ϕ_n so that $\phi_n = \phi$ outside this finite subcollection, and within each cube $Q_{r_i}(x_i)$, almost all of the jump set of ϕ that lies in G_j is transferred onto the jump set of u_n. This will enable us to prove statement 3 of the theorem.

Because of the weak convergence of $|\nabla u_n|$ to v in L^1 and the last inequality in (2.6), for n sufficiently large, we have

$$\int_{\cup_i Q_{r_i}(x_i)} |\nabla u_n| \, dx < \frac{\varepsilon}{2j},$$

which, by the coarea formula, implies that

$$\int_{\mathbb{R}} \mathcal{H}^{N-1}\Big((\partial_* E_t^n \cap (\cup_i Q_{r_i}(x_i))) \setminus S(u_n)\Big) \, dt < \frac{\varepsilon}{2j}.$$

Note that we use the fact that $u_n \in SBV$ which means that the singular part of the variation measure of u_n does not see the complement of $S(u_n)$. In turn, this implies that

$$\sum_i \int_{t_1(x_i)}^{t_2(x_i)} \mathcal{H}^{N-1}\Big((\partial_* E_t^n \cap Q_{r_i}(x_i)) \setminus S(u_n)\Big) \, dt < \frac{\varepsilon}{2j},$$

so that there exist $t_i \in [t_1(x_i), t_2(x_i)]$ such that

$$\sum_i (t_2(x_i) - t_1(x_i)) \mathcal{H}^{N-1}\Big((\partial_* E_{t_i}^n \cap Q_{r_i}(x_i)) \setminus S(u_n)\Big) < \frac{\varepsilon}{2j}.$$

Since $t_2(x_i) - t_1(x_i) > \frac{1}{2j}$, we conclude that

$$\sum_i \mathcal{H}^{N-1}\Big((\partial_* E_{t_i}^n \cap Q_{r_i}(x_i)) \setminus S(u_n)\Big) < \varepsilon. \tag{2.7}$$

Our reason for deriving (2.7) is that we will actually transfer the jump set of ϕ lying in G_j onto $\cup_i(\partial_* E^n_{t_i} \cap Q_{r_i}(x_i))$, which by (2.7) lies almost entirely inside $S(u_n)$.

By the definition of L, we know that for each $t \in L$, $\chi_{E^n_t} \to \chi_{E_t}$ in $L^1(\Omega)$. In particular, this implies that

$$|(E^n_{t_1(x_i)} \triangle E_{t_1(x_i)}) \cap Q_{r_i}(x_i)| \to 0.$$

By (2.5), we thus conclude that, for n large enough,

$$|(E^n_{t_1(x_i)} \cap Q_{r_i}(x_i)) \triangle Q^-_{r_i}(x_i)| < \varepsilon^2 |Q_{r_i}(x_i)|$$

for all i, and the same result holds for $E^n_{t_2(x_i)}$. Since

$$E^n_{t_1(x_i)} \supset E^n_{t_i} \supset E^n_{t_2(x_i)} \text{ and } E_{t_1(x_i)} \supset E_{t_i} \supset E_{t_2(x_i)},$$

we have

$$|(E^n_{t_i} \cap Q_{r_i}(x_i)) \triangle Q^-_{r_i}(x_i)| < \varepsilon^2 |Q_{r_i}(x_i)| \tag{2.8}$$

for n sufficiently large. Note also that from (2.5) we get

$$|(E_{t_i} \cap Q_{r_i}(x_i)) \triangle Q^-_{r_i}(x_i)| < \varepsilon^2 |Q_{r_i}(x_i)|. \tag{2.9}$$

We now choose $N(\varepsilon)$ such that for $n \geq N(\varepsilon)$, (2.7) and (2.8) hold for all i.

Fix $Q_{r_i}(x_i)$. Inequalities (2.8) and (2.9) imply that

$$\int_{\frac{\varepsilon}{2} r_i}^{\varepsilon r_i} \Big[\mathcal{H}^{N-1}(H_i(s) \setminus E^n_{t_i}) + \mathcal{H}^{N-1}(H_i(s) \setminus E_{t_i}) \Big] ds < 2\varepsilon^2 (2r_i)^N,$$

where $H_i(s) := \{y \in Q_{r_i}(x_i) : (y - x_i) \cdot \nu(x_i) = -s\}$. We select an s_i so that $H^-_i := H_i(s_i)$ satisfies

$$\begin{cases} \mathcal{H}^{N-1}(H^-_i \setminus E^n_{t_i}) & < & 8\varepsilon (2r_i)^{N-1} \\ \mathcal{H}^{N-1}(H^-_i \setminus E_{t_i}) & < & 8\varepsilon (2r_i)^{N-1} \\ \operatorname{dist}(H^-_i, H_i) & \in & [\frac{\varepsilon}{2} r_i, \varepsilon r_i]. \end{cases} \tag{2.10}$$

In (2.10), $H_i := H_{r_i}(x_i)$. Analogously, we choose H^+_i on the other side of H_i so that

$$\begin{cases} \mathcal{H}^{N-1}(H^+_i \cap E^n_{t_i}) & < & 8\varepsilon (2r_i)^{N-1} \\ \mathcal{H}^{N-1}(H^+_i \cap E_{t_i}) & < & 8\varepsilon (2r_i)^{N-1} \\ \operatorname{dist}(H^+_i, H_i) & \in & [\frac{\varepsilon}{2} r_i, \varepsilon r_i]. \end{cases} \tag{2.11}$$

If $x \in \partial\Omega$, note that $\partial\Omega \cap Q_{r_i}(x_i)$ lies between H^-_i and H^+_i, due to (2.4).

We finally define R_i to be the region in $Q_{r_i}(x_i)$ between H^-_i and H^+_i, and note that

$$\mathcal{H}^{N-1}(G_j \setminus [\cup_i R_i]) < c\varepsilon. \tag{2.12}$$

Indeed, according to the second inequality in (2.6) together with property 4 of (2.3) which holds *a fortiori* true for G_j in lieu of $S(u)$, we have

$$\begin{aligned} \mathcal{H}^{N-1}\Big(G_j \setminus \cup_i [\partial^* E_{t_1}(x_i) \cap Q_{r_i}(x_i)]\Big) & < \varepsilon + \varepsilon \textstyle\sum_i r_i^{N-1} \\ & < \varepsilon(1 + c\mathcal{H}^{N-1}(S(u))), \end{aligned} \tag{2.13}$$

where we have further appealed to property 3 of (2.3) in deriving the second inequality.

Now, applying property 5 of (2.3) to the point x_i and the cube $Q_{r_i}(x_i)$, and summing the resulting inequalities over i, we obtain, still with the help of property 3 of (2.3) together with the last inequalities of both (2.10) and (2.11),

$$\begin{aligned}\mathcal{H}^{N-1}\Big(\bigcup_i[\partial^* E_{t_1(x_i)} \cap Q_{r_i}(x_i)] \setminus \bigcup_i R_i\Big) &\le \mathcal{H}^{N-1}\Big(\bigcup_i\{(\partial^* E_{t_1(x_i)} \cap Q_{r_i}(x_i)) \setminus R_i\}\Big) \\ &\le c\mathcal{H}^{N-1}(S(u))\varepsilon.\end{aligned} \tag{2.14}$$

Collecting (2.13) and (2.14), we obtain (2.12).

We now begin the transfer of the jump set $S(\phi)$ in $G_j \cap \cup_i Q_{r_i}(x_i)$ to $\cup_i(\partial_* E^n_{t_i} \cap Q_{r_i}(x_i))$. If $x_i \notin \partial\Omega$, we set $\phi_- := \phi\chi_{[Q^-_{r_i}(x_i)\setminus R_i]}$, extended by reflection to R_i (this is possible when ε is small enough because R_i is a small neighborhood of H_i). Similarly, we define ϕ_+ on $[Q_{r_i}(x_i) \setminus Q^-_{r_i}(x_i)] \cup R_i$. Define ϕ_n on $Q_{r_i}(x_i)$ as follows:

$$\phi_n := \begin{cases} \phi_- & \text{on} \quad Q^-_{r_i}(x_i) \setminus R_i \\ \phi_+ & \text{on} \quad Q_{r_i}(x_i) \setminus [Q^-_{r_i}(x_i) \cup R_i] \\ \phi_- & \text{on} \quad E^n_{t_i} \cap R_i \\ \phi_+ & \text{on} \quad R_i \setminus E^n_{t_i}. \end{cases}$$

For $x_i \in \partial\Omega$, note that either $Q^-_{r_i}(x_i) \setminus R_i \subset \Omega$, or $Q_{r_i}(x_i) \setminus (Q^-_{r_i}(x_i) \cup R_i) \subset \Omega$. Assuming the former, the only modification to ϕ_n above is that $\phi_n = \phi_-$ on $E^n_{t_i} \cap R_i \cap \Omega$, and $\phi_n = \phi_+$ on $R_i \setminus (E^n_{t_i} \cap \Omega)$. We repeat this construction for all i, so ϕ_n is defined on $\cup_i Q_{r_i}(x_i)$. Outside this union, we take $\phi_n \equiv \phi$.

Taking a sequence $\varepsilon_i \searrow 0$ generates a sequence ϕ_n, by choosing ϕ_n as above using ε_i for $n \in [N(\varepsilon_i), N(\varepsilon_{i+1}))$. Statements 1 and 2 of the theorem are easily verified for the sequence $\{\phi_n\}$. Indeed, for example,

$$\|\nabla\phi_n\|_{L^q(Q_{r_i}(x_i))} \le 2\|\nabla\phi\|_{L^q(Q_{r_i}(x_i))}$$

so that

$$\begin{aligned}\|\nabla\phi - \nabla\phi_n\|_{L^q(\Omega)} &\le \|\nabla\phi_n\|_{L^q(\cup_i Q_{r_i}(x_i))} + \|\nabla\phi\|_{L^q(\cup_i Q_{r_i}(x_i))} \\ &\le 3\|\nabla\phi\|_{L^q(\cup_i Q_{r_i}(x_i))} \to 0\end{aligned}$$

as $n \to 0$ since $|\nabla\phi|^q \in L^1(\Omega)$ together with the first inequality in (2.6) (recall that $\cup_i Q_{r_i}(x_i)$ depends on ε).

It remains to show that

$$\mathcal{H}^{N-1}\Big([S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]\Big) \le O(\varepsilon). \tag{2.15}$$

We now come to a bookkeeping argument on $S(\phi_n)$. First, let us look at the part of $S(\phi_n) \setminus S(u_n)$ that is outside of $\cup_i\overline{R_i}$. Since $\phi_n = \phi$ outside this union,

$$(S(\phi_n) \setminus S(u_n)) \setminus \cup_i\overline{R_i} \subset S(\phi) \setminus \cup_i\overline{R_i}.$$

But

$$[S(\phi) \setminus \cup_i \overline{R_i}] \cap S(u) \subset S(u) \setminus \cup_i \overline{R_i},$$

and this last set has $\mathcal{H}^{N-1}$ measure of order ε according to (2.12) and (2.2). Therefore,

$$\mathcal{H}^{N-1}\Big(\{[S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]\} \setminus \cup_i \overline{R_i}\Big) \leq O(\varepsilon). \tag{2.16}$$

We now show that

$$\mathcal{H}^{N-1}\Big(\{[S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]\} \cap \cup_i \overline{R_i}\Big) \leq O(\varepsilon) \tag{2.17}$$

by showing that

$$\mathcal{H}^{N-1}\Big([S(\phi_n) \setminus S(u_n)] \cap \cup_i \overline{R_i}\Big) \leq O(\varepsilon). \tag{2.18}$$

We consider $[S(\phi_n) \setminus S(u_n)] \cap \overline{R_i}$ for a fixed i, with $x_i \in \Omega$ (the case $x_i \in \partial\Omega$ is similar). We break $\overline{R_i}$ into four parts that we deal with one at a time:

$$\begin{aligned}\overline{R_i} = P_i^1 \cup P_i^2 \cup P_i^3 \cup P_i^4 \quad := \quad & \Big(\partial_* E^n_{t_i} \cap \overline{R_i}\Big) \cup \Big(R_i \setminus \partial_* E^n_{t_i}\Big) \cup \Big([H_i^+ \cup H_i^-] \setminus \partial_* E^n_{t_i}\Big) \\ & \cup \Big(\partial R_i \setminus [H_i^+ \cup H_i^- \cup \partial_* E^n_{t_i}]\Big),\end{aligned}$$

with obvious notation. By (2.7), we *a fortiori* have

$$\sum_i \mathcal{H}^{N-1}(P_i^1 \setminus S(u_n)) < \varepsilon. \tag{2.19}$$

We now turn to P_i^2. The part of $S(\phi_n) \setminus \partial_* E^n_{t_i}$ that lies in $E^n_{t_i} \cap R_i$ is a reflected version of $S(\phi)$ lying in $Q^-_{r_i}(x_i) \setminus R_i$, according to the definition of ϕ_n and ϕ_-. An analogous statement holds for the part of $S(\phi_n) \setminus \partial_* E^n_{t_i}$ in $R_i \setminus E^n_{t_i}$. But according to (2.12) and (2.2)

$$\mathcal{H}^{N-1}\Big((S(\phi) \cap S(u)) \setminus \cup_i R_i\Big) \leq O(\varepsilon).$$

Furthermore, by properties 3 and 6 of (2.3),

$$\mathcal{H}^{N-1}\Big((S(\phi) \setminus S(u)) \setminus \cup_i R_i\Big) \leq O(\varepsilon).$$

Consequently,

$$\sum_i \mathcal{H}^{N-1}(P_i^2 \cap S(\phi_n)) \leq O(\varepsilon). \tag{2.20}$$

Now for P_i^3. Because of the reflection in defining ϕ_n and of our choice of H_i^+ and H_i^- for (2.10) and (2.11), the only possible jumps of ϕ_n lying in H_i^+ and H_i^- are $H_i^+ \cap E^n_{t_i}$ and $H_i^- \setminus E^n_{t_i}$ (up to a set of $\mathcal{H}^{N-1}$ measure zero). Appealing to the first inequalities of (2.10) and (2.11), we see that these two sets have $\mathcal{H}^{N-1}$ measure less than $C\varepsilon r_i^{N-1}$, so that, with property 3 of (2.3),

$$\sum_i \mathcal{H}^{N-1}(S(\phi_n) \cap P_i^3) \leq O(\varepsilon). \tag{2.21}$$

Finally, according to the third relations in (2.10) and (2.11),

$$\sum_i \mathcal{H}^{N-1}(P_i^4) \leq c \sum_i \Big[\mathrm{dist}(H_i^-, H_i) + \mathrm{dist}(H_i^+, H_i)\Big] r_i^{N-2} \leq c\varepsilon \sum_i r_i^{N-1} = O(\varepsilon), \tag{2.22}$$

where we have appealed once again to property 3 of (2.3). Collecting (2.19)-(2.22) establishes (2.18). The third part of the theorem follows from (2.16) and (2.17). □

Remark 2.6 *Since we take $\phi_n \equiv \phi$ outside a vanishing sequence of sets, we also have*

$$|\{\phi_n \neq \phi\}| \to 0$$

and

$$|\{\nabla\phi_n \neq \nabla\phi\}| \to 0.$$

Remark 2.7 *Note that, if Theorem 2.1 produces a sequence $\{\phi_n\}$, and if $\{w_n\} \subset H^1(\Omega)$ is a sequence converging strongly to zero, then $\{w_n + \phi_n\}$ has properties i to iii of Theorem 2.1, while having the same traces as $\{w_n + \phi\}$, that is $w_n + \phi_n = w_n + \phi$ on $\Omega' \setminus \overline{\Omega}$.*

The following generalization of Theorem 2.1 holds true:

Theorem 2.8 *Assume $\overline{\Omega} \subset \Omega'$ with $\partial\Omega$ Lipschitz. Let $k \in \{1, ..., p\}$, $\{u_{k,n}\} \subset SBV(\Omega')$ with*

- *$S(u_{k,n}) \subset \overline{\Omega}$;*
- *$|\nabla u_{k,n}|$ weakly converge in $L^1(\Omega')$; and*
- *$u_{k,n} \to u_k$ in $L^1(\Omega')$,*

where $u_k \in BV(\Omega')$ with $\mathcal{H}^{N-1}(S(u_k)) < \infty$. Then for every $\phi \in SBV_q(\Omega')$, $1 \leq q < \infty$, with $\mathcal{H}^{N-1}(S(\phi)) < \infty$, there exists a sequence $\{\phi_n\} \subset SBV_q(\Omega')$ with $\phi_n = \phi$ on $\Omega' \setminus \overline{\Omega}$ such that

i) $\phi_n \to \phi$ strongly in $L^1(\Omega')$;

ii) $\nabla\phi_n \to \nabla\phi$ strongly in $L^q(\Omega')$; and

iii) $\mathcal{H}^{N-1}\Big([S(\phi_n) \setminus \bigcup_{k=1,\dots,p} S(u_{k,n})] \setminus [S(\phi) \setminus \bigcup_{k=1,\dots,p} S(u_k)]\Big) \to 0.$

Remark 2.9 *An analogue of Theorem 2.4 would hold in the generalized setting of Theorem 2.8. We will not detail it here.*

Proof of Theorem 2.8. The proof is a straightforward adaptation of the proof of Theorem 2.1. It suffices to reproduce that proof with a covering, up to a set of $\mathcal{H}^{N-1}$-measure zero, of

$$G_j := \Big\{x \in \bigcup_{k=1,\dots,p} \Big(\bigcup_{\substack{t_1,t_2 \in D_k \\ t_1 < t_2}} [\partial^* E^k_{t_1} \cap \partial^* E^k_{t_2}]\Big) : \min_{k=1,\dots,p} [u_k(x)] > \frac{1}{j}\Big\},$$

where E^k_t and D_k are defined in the natural way. In particular, the covering has to be such that, for every cube $Q_r(x)$ in the cover, there exists an index $k \in \{1, ..., p\}$ such that $x \in S(u_k)$ and

$$\mathcal{H}^{N-1}\Big(\Big[\bigcup_{\substack{l=1,\dots,p \\ l \neq k}} S(u_l) \setminus S(u_k)\Big] \cap Q_r(x)\Big) < \varepsilon r^{N-1},$$

which is always possible since $\Big[\bigcup_{\substack{l=1,\dots,p \\ l \neq k}} S(u_l) \setminus S(u_k)\Big]$ has $\mathcal{H}^{N-1}$–density 0, $\mathcal{H}^{N-1}$–a.e. in $S(u_k)$. $\square$

We now present a useful corollary of Theorem 2.1 which essentially states that the minimizers, with respect to their own jump sets, of a bulk + surface energy functional are stable under very weak convergence hypotheses.

Corollary 2.10 *Let A be a Borel set with $\mathcal{H}^{N-1}(A^c) < \infty$, $u_n \to u$ in $L^1(\Omega')$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega')$, where $u_n, u \in SBV(\Omega')$, $u_n = U_n$ on $\Omega' \setminus \overline{\Omega}$, with $U_n \in W^{1,2}(\Omega')$, $\mathcal{H}^{N-1}(S(u)) < \infty$, and the u_n minimize*

$$\int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1} \lfloor A \Big(S(v) \setminus S(u_n) \Big),$$

with respect to functions with the same Dirichlet data, that is U_n, in $\Omega' \setminus \overline{\Omega}$ (in the terminology introduced in the introduction, u_n is a minimizer for its own jump set). Then u minimizes the same functional with respect to its own jump set. If further $U_n = U \in W^{1,2}(\Omega')$, then $\nabla u_n \to \nabla u$ in $L^2(\Omega)$.

Proof. Take ϕ to be an arbitrary element in $\{v \in SBV(\Omega') : v = 0 \text{ on } \Omega' \setminus \overline{\Omega}\}$ with $\mathcal{H}^{N-1}(S(\phi)) < \infty$. According to Theorem 2.1 there exists a sequence $\phi_n \in SBV(\Omega')$ with $\phi_n = 0$ on $\Omega' \setminus \overline{\Omega}$, such that

$$\begin{cases} \nabla \phi_n \to \nabla \phi \text{ in } L^2(\Omega') \\ \mathcal{H}^{N-1}\Big([S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)] \Big) \to 0. \end{cases}$$

Invoking the minimality of u_n, we get

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 \, dx &\leq \int_\Omega |\nabla u_n + \nabla \phi_n|^2 \, dx + \mathcal{H}^{N-1} \lfloor A \Big(S(u_n + \phi_n) \setminus S(u_n) \Big) \\ &= \int_\Omega |\nabla u_n|^2 \, dx + 2 \int_\Omega \nabla u_n \cdot \nabla \phi_n \, dx + \int_\Omega |\nabla \phi_n|^2 \, dx \\ &\quad + \mathcal{H}^{N-1} \lfloor A \Big(S(\phi_n) \setminus S(u_n) \Big), \end{aligned}$$

or equivalently

$$0 \leq 2 \int_\Omega \nabla u_n \cdot \nabla \phi_n \, dx + \int_\Omega |\nabla \phi_n|^2 \, dx + \mathcal{H}^{N-1} \lfloor A \Big(S(\phi_n) \setminus S(u_n) \Big).$$

But, in view of Theorem 2.1, we may pass to the limit in each term of the previous inequality and we obtain, upon adding $\|\nabla u\|^2_{L^2(\Omega)}$,

$$\begin{aligned} \int_\Omega |\nabla u|^2 \, dx &\leq \int_\Omega |\nabla u + \nabla \phi|^2 \, dx + \mathcal{H}^{N-1} \lfloor A \Big(S(\phi) \setminus S(u) \Big) \\ &= \int_\Omega |\nabla u + \nabla \phi|^2 \, dx + \mathcal{H}^{N-1} \lfloor A \Big(S(u + \phi) \setminus S(u) \Big), \end{aligned}$$

which was the sought result, whenever $\mathcal{H}^{N-1}(S(\phi)) < \infty$. But the result is obvious if $\mathcal{H}^{N-1}(S(\phi)) = \infty$ because, since $\mathcal{H}^{N-1}(A^c \cap S(\phi)) < \infty$, $\mathcal{H}^{N-1} \lfloor A(S(\phi + u) \setminus S(u)) = \infty$.

If $U_n = U$, take $\phi := u - U$, so that $\phi = 0$ on $\Omega' \setminus \overline{\Omega}$ and $S(\phi) = S(u)$. By Theorem 2.1, there exists a sequence $\{\phi_n\} \subset SBV(\Omega')$ such that $\phi_n = 0$ on $\Omega' \setminus \overline{\Omega}$, $\nabla \phi_n \to \nabla \phi$ in

$L^2(\Omega')$, and $\mathcal{H}^{N-1}(S(\phi_n) \setminus S(u_n)) \to 0$. From the minimality of u_n and since $U_n = U$, $U + \phi_n$ is an admissible test function in the minimization problem for u_n, so that

$$\int_\Omega |\nabla u_n|^2 dx \le \int_\Omega |\nabla U + \nabla \phi_n|^2 dx + \mathcal{H}^{N-1} \lfloor A\Big(S(\phi_n) \setminus S(u_n)\Big).$$

The above properties of $\{\phi_n\}$ then yield

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \le \int_\Omega |\nabla U + \nabla \phi|^2 dx = \int_\Omega |\nabla u|^2 dx.$$

But, by lower semicontinuity, we also have

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \ge \int_\Omega |\nabla u|^2 dx,$$

which gives $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$. Together with the weak L^2 convergence of ∇u_n to ∇u, we get strong L^2 convergence. □

3 Analysis of Crack Growth

In this section, we propose to pass to the time–continuous limit in the time–discretized formulation of the model discussed in the introduction as the discretization step tends to zero.

In all that follows, we will not relabel converging subsequences of a given sequence, unless confusion might ensue.

3.1 The discrete formulation

We consider a countable and dense subset I_∞ in $[0, 1]$, and, for each $n \in \mathbb{N}$, a subset $I_n = \{t_0^n = 0 < t_1^n < ... < t_n^n\}$, such that $\{I_n\}$ form an increasing sequence of nested sets whose union is I_∞. We set $\Delta_n := \sup_{k\in\{1,...,n\}}(t_k^n - t_{k-1}^n)$. Note that $\Delta_n \searrow 0$. As seen in the introduction we are given boundary data $g \in W^{1,1}((0, 1); H^1(\Omega')) \cap L^\infty((0, 1) \times \Omega')$, so that at time t, the kinematically admissible fields v should satisfy $v \equiv g(t)$ in $\Omega' \setminus \overline{\Omega}$. For each $n \in \mathbb{N}$ and $t \in [0, 1]$ we define $g_n(t) := g_k^n := g(t_k^n)$ for $t \in [t_k^n, t_{k+1}^n)$, and immediately note that, for each $t \in I_\infty$, $g(t) = \lim_{n\to\infty} g_n(t)$, strongly in $H^1(\Omega')$. Actually, $g(t) = g_n(t)$, if n is large enough.

At time t_k^n, we consider u_k^n, a minimizer for

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus \Big[\bigcup_{0\le j\le k-1} S(u_j^n) \cup \partial\Omega_f\Big]\Big) \tag{3.1}$$

in $\{v \in SBV(\Omega') : v \equiv g_k^n \text{ on } \Omega' \setminus \overline{\Omega}\}$. The existence of a minimizer for (3.1) is a straightforward iterated application of the SBV–compactness theorem (see e.g. [1]) and of its consequences mentioned in Subsection 1.4. We then define $u^n(t) := u_k^n$ in $[t_k^n, t_{k+1}^n)$.

The *a priori* estimates on $u^n(t)$ below are very close to those obtained in [11]. Their derivation is nearly identical but, since the setting is different — weak formulation in a SBV setting versus strong formulation in a two-dimensional setting with "connected" compact crack sites —, we establish them anew, while inviting the reader familiar with [11] to skip the derivation and go to (3.7). In any case, for some constant $C > 0$ the following holds true:

$$\begin{array}{lrcl} 1. & \|\nabla u^n(t)\|_{L^2(\Omega)} & \le & C; \\ 2. & \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty \\ \tau\le t}} S(u^n(\tau))\Big) & \le & C; \text{ and} \\ 3. & \|u^n(t)\|_{L^\infty(\Omega)} & \le & C. \end{array} \tag{3.2}$$

Indeed, at time t^n_k, take g^n_k as test function for the minimality of u^n_k in (3.1). We obtain

$$\|\nabla u^n_k\|^2_{L^2(\Omega)} + \mathcal{H}^{N-1}\Big(S(u^n_k) \setminus \Big[\bigcup_{0\le j\le k-1} S(u^n_j) \cup \partial\Omega_f\Big]\Big) \le \|\nabla g^n_k\|^2_{L^2(\Omega)},$$

which implies, since $u^n_k \equiv g^n_k$ on $\Omega' \setminus \overline{\Omega}$, that

$$\|\nabla u^n(t)\|_{L^2(\Omega)} \le \|\nabla g^n(t)\|_{L^2(\Omega)} \le \int_0^1 \|\nabla \dot g(s)\|_{L^2(\Omega)} ds + \|\nabla g(0)\|_{L^2(\Omega)} \le c.$$

This proves the first bound.

Now, at time t^n_{k+1}, take $u^n_k + g^n_{k+1} - g^n_k$ as a test function for the minimality of u^n_{k+1} in (3.1). We obtain

$$\begin{aligned} \|\nabla u^n_{k+1}\|^2_{L^2(\Omega)} \;&+\; \mathcal{H}^{N-1}\Big(S(u^n_{k+1}) \setminus \Big[\bigcup_{0\le j\le k} S(u^n_j) \cup \partial\Omega_f\Big]\Big) \\ &\le \|\nabla u^n_k\|^2_{L^2(\Omega)} + 2\int_\Omega \nabla u^n_k \cdot (\nabla g^n_{k+1} - \nabla g^n_k)\, dx + \|\nabla g^n_{k+1} - \nabla g^n_k\|^2_{L^2(\Omega)} \\ &\le \|\nabla u^n_k\|^2_{L^2(\Omega)} + 2\int_{t^n_k}^{t^n_{k+1}} \int_\Omega \nabla u^n(s) \cdot \nabla \dot g(s)\, dxds \\ &\quad + \Big(\int_{t^n_k}^{t^n_{k+1}} \|\nabla \dot g(s)\|_{L^2(\Omega)}\, ds\Big)^2. \end{aligned}$$

In view of the integrability of $\|\nabla \dot g\|_{L^2(\Omega)}$, there exists $\omega(\delta) \searrow 0+$ as $\delta \searrow 0+$ such that

$$\int_{t^n_k}^{t^n_{k+1}} \|\nabla \dot g(s)\|_{L^2(\Omega)}\, ds \le \omega(\Delta_n),$$

so that the previous string of inequalities reduces to

$$\begin{aligned} &\|\nabla u^n_{k+1}\|^2_{L^2(\Omega)} + \mathcal{H}^{N-1}\Big(S(u^n_{k+1}) \setminus \Big[\bigcup_{0\le j\le k} S(u^n_j) \cup \partial\Omega_f\Big]\Big) \\ &\le \|\nabla u^n_k\|^2_{L^2(\Omega)} + 2\int_{t^n_k}^{t^n_{k+1}} \int_\Omega \nabla u^n(s) \cdot \nabla \dot g(s)\, dxds + \omega(\Delta_n) \cdot \int_{t^n_k}^{t^n_{k+1}} \|\nabla \dot g(s)\|_{L^2(\Omega)}\, ds. \end{aligned} \tag{3.3}$$

Summing up (3.3) for $k = 0, ..., i$, we obtain

$$\begin{aligned} \|\nabla u^n_{i+1}\|^2_{L^2(\Omega)} &+ \mathcal{H}^{N-1}\Big(\bigcup_{k=0}^{i} S(u^n_{k+1}) \setminus \partial\Omega_f\Big) \\ &\le \|\nabla u(0)\|^2_{L^2(\Omega)} + \mathcal{H}^{N-1}(S(u(0))) + 2\int_0^{t^n_{i+1}} \int_\Omega \nabla u^n(s) \cdot \nabla \dot{g}(s)\, dxds \\ &\quad + \omega(\Delta_n) \cdot \int_0^1 \|\nabla \dot{g}(s)\|_{L^2(\Omega)}\, ds, \end{aligned} \tag{3.4}$$

which, in view of the already established *a priori* bound (3.2-1), together with the Cauchy-Schwartz inequality, yields

$$\mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S(u^n(\tau)) \setminus \partial\Omega_f\Big) \le \mathcal{H}^{N-1}\Big(\bigcup_{k=0}^{i} S(u^n_{k+1}) \setminus \partial\Omega_f\Big) \le c, \tag{3.5}$$

or still, since $\mathcal{H}^{N-1}(\partial\Omega_f)$ is finite, for any $t \in [0, 1]$,

$$\mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S(u^n(\tau))\Big) \le c.$$

This proves the second bound.

The third bound is an immediate consequence of the L^∞-character of g, hence of g_n, as can be seen by truncation.

Thanks to estimates (3.2), we are now in a position — through a straightforward diagonalization process — to apply AMBROSIO's SBV-compactness theorem (see e.g. [1]) to $\{u_n(t) : n \in \mathbb{N}, t \in I_\infty\}$, and to conclude the existence of $\{u(t) \in SBV(\Omega') : t \in I_\infty\}$ such that, for all $t \in I_\infty$,

$$\begin{cases} \nabla u_n(t) \rightharpoonup \nabla u(t) \text{ in } L^2(\Omega'); \\ (u^+_n(t) - u^-_n(t))\nu_n \mathcal{H}^{N-1}\lfloor S(u_n(t)) \overset{*}{\rightharpoonup} (u^+(t) - u^-(t))\nu \mathcal{H}^{N-1}\lfloor S(u(t)) \text{ as measures;} \\ u_n(t) \to u(t) \text{ in } L^p(\Omega')(p < \infty); \text{ and} \\ u_n(t) \overset{*}{\rightharpoonup} u(t) \text{ in } L^\infty(\Omega'). \end{cases} \tag{3.6}$$

In view of convergences (3.6), we conclude, following the terminology of the introduction, that u^n $SBV(\Omega')$–converges to u.

Further, (3.2) implies the existence of a constant $C < \infty$ such that, for $t \in I_\infty$,

$$\begin{aligned} &1. \quad \|\nabla u(t)\|_{L^2(\Omega')} \le C; \\ &2. \quad \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S(u(\tau))\Big) \le C; \text{ and} \\ &3. \quad \|u(t)\|_{L^\infty(\Omega')} \le C. \end{aligned} \tag{3.7}$$

The bounds 1 and 3 above are obvious by lower semicontinuity. At the expense of a numbering of $\{s \in I_\infty;\ s \le t\}$, the bound 2 immediately results from an application of the following

Lemma 3.1 *For each $p \in \mathbb{N}$, let $\{v_p^n\}$ be a sequence in $SBV(\Omega)$ such that v_p^n $SBV(\Omega)$–converges to v_p. Then, for any open subset A of Ω,*

$$\mathcal{H}^{N-1}\lfloor A\Big(\bigcup_p S(v_p)\Big) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}\lfloor A\Big(\bigcup_p S(v_p^n)\Big).$$

Proof. Define, for $r \in (0,1]$,

$$A_r := \Big\{x \in S(v_1) : [v_1(x)] + r[v_2(x)] = 0\Big\}.$$

These are pairwise disjoint with $\mathcal{H}^{N-1}$–σ–finite union, and thus $\mathcal{H}^{N-1}(A_r) = 0$, except maybe for a countable number of r. Choose r so that $\mathcal{H}^{N-1}(A_r) = 0$. Since $v_1^n + rv_2^n$ $SBV(\Omega)$–converges to $v_1 + rv_2$,

$$\mathcal{H}^{N-1}\lfloor A\Big(S(v_1 + rv_2)\Big) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}\lfloor A\Big(S(v_1^n + rv_2^n)\Big).$$

But

$$S(v_1 + rv_2) \cup A_r = S(v_1) \cup S(v_2)$$

and

$$S(v_1^n + rv_2^n) \subset S(v_1^n) \cup S(v_2^n)$$

for each $n \in \mathbb{N}$. We thus get

$$\mathcal{H}^{N-1}\lfloor A\Big(S(v_1) \cup S(v_2)\Big) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}\lfloor A\Big(S(v_1^n) \cup S(v_2^n)\Big). \tag{3.8}$$

An iteration of the argument above would yield the same inequality (3.8) with the union from 1 to $k \in \mathbb{N}$ of the jumps $S(v_p)$. We thus get

$$\mathcal{H}^{N-1}\lfloor A\Big(\bigcup_{p=1,\dots,k} S(v_p)\Big) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}\lfloor A\Big(\bigcup_{p=1,\dots,k} S(v_p^n)\Big)$$

which implies that

$$\mathcal{H}^{N-1}\lfloor A\Big(\bigcup_{p=1,\dots,k} S(v_p)\Big) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}\lfloor A\Big(\bigcup_{p\in\mathbb{N}} S(v_p^n)\Big).$$

Letting k tend to ∞ in the previous inequality yields the desired result. □

We now investigate the minimality properties of $u(t)$. This is the object of the following lemmas which are easy consequences of Theorem 2.1. We will make repeated use of the following remark:

Remark 3.2 *For $t \in I_n$, $u^n(t)$ minimizes*

$$\int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}\Big(S(v) \setminus \Big[\bigcup_{\substack{\tau\in I_n\\ \tau\le t}} S(u^n(\tau)) \cup \partial\Omega_f\Big]\Big)$$

on $\{v \in SBV(\Omega') : v = g(t)$ on $\Omega' \setminus \overline{\Omega}\}$.

This minimality is an immediate consequence of the minimality in the choice of $u^n(t)$. In particular, $u^n(t)$ is a minimizer for its own jump set.

Lemma 3.3 *For each $t \in I_\infty$, $u(t)$ is a minimizer for its own jump set, or, in other words, it minimizes*

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus [S(u(t)) \cup \partial\Omega_f]\Big)$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$. Furthermore, $\nabla u^n(t) \to \nabla u(t)$ in $L^2(\Omega)$.

Proof. Since each $t \in I_\infty$ belongs to I_n for n large enough, and $g(t) = g_n(t)$ for n large enough, Remark 3.2 actually implies that $u^n(t),\ t \in I_\infty$, minimizes

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus [S(u^n(t)) \cup \partial\Omega_f]\Big)$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$. Now, in view of (3.6), (3.7) and since $g_n(t) = g(t)$ for n large enough, Corollary 2.10 with $A = (\partial\Omega_f)^c$ implies the sought result. □

Now actually, $u(t)$ is not only a minimizer for its own jump set, but it also satisfies the following minimality property

Lemma 3.4 *For each $t \in I_\infty$, $u(t)$ minimizes*

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus \Big[\bigcup_{\substack{\tau \in I_\infty \\ \tau \le t}} S(u(\tau)) \cup \partial\Omega_f\Big]\Big) \tag{3.9}$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$. (Note that the times τ are taken up to and including $\tau = t$ in the minimality property.)

Remark 3.5 *As in Remark 3.2, the minimality property in Lemma 3.4 implies* in particular *that $u(t)$ is a minimizer for its own jump set.*

Proof of Lemma 3.4. In view of (3.7),

$$\mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau \in I_\infty \\ \tau \le t}} S(u(\tau))\Big) \le c < \infty,$$

so that, because the union is over a countable set of times (in I_∞), for any $\eta > 0$, there exist $t_1 < t_2 < ... < t_p \le t$, such that

$$\mathcal{H}^{N-1}\Big(\bigcup_{k=1,\dots,p} S(u(t_k))\Big) \ge \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau \in I_\infty \\ \tau \le t}} S(u(\tau))\Big) - \eta. \tag{3.10}$$

According to Remark 3.2, for each $t \in I_\infty$, $u^n(t)$ minimizes

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus \Big[\bigcup_{\substack{\tau \in I_n \\ \tau \le t}} S(u^n(\tau)) \cup \partial\Omega_f\Big]\Big)$$

on $\{v \in SBV(\Omega') : v = g_n(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$; *a fortiori*, for n large enough so that $t_1, ..., t_p \in I_n$, we have

$$\int_\Omega |\nabla u^n(t)|^2\, dx \le \int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus \Big[\bigcup_{k=1,\ldots,p} S(u^n(t_k)) \cup \partial\Omega_f\Big]\Big). \tag{3.11}$$

The previous lemma implies in particular that, for each $t \in I_\infty$,

$$\nabla u^n(t) \rightharpoonup \nabla u(t). \tag{3.12}$$

Take ϕ to be an arbitrary element in $\{v \in SBV(\Omega') : v = 0 \text{ on } \Omega' \setminus \overline{\Omega}\}$ with $\mathcal{H}^{N-1}(S(\phi)) < \infty$. Theorem 2.8 implies the existence of a sequence $\phi_n \in SBV(\Omega')$ with $\phi_n \equiv 0$ on $\Omega' \setminus \overline{\Omega}$, such that

$$\begin{cases} \nabla\phi_n \to \nabla\phi \text{ in } L^2(\Omega') \\ \mathcal{H}^{N-1}\lfloor\partial\Omega_f^c([S(\phi_n) \setminus (\bigcup_{k=1,\ldots,p} S(u^n(t_k))] \setminus [S(\phi) \setminus (\bigcup_{k=1,\ldots,p} S(u(t_k))]) \to 0. \end{cases}$$

By virtue of (3.11), we get

$$\begin{aligned} \int_\Omega |\nabla u^n(t)|^2\, dx &\le \int_\Omega |\nabla u^n(t) + \nabla\phi_n|^2\, dx \\ &\quad + \mathcal{H}^{N-1}\lfloor\partial\Omega_f^c\Big(S(u^n(t) + \phi_n) \setminus \Big[\bigcup_{k=1,\ldots,p} S(u^n(t_k))\Big]\Big) \\ &= \int_\Omega |\nabla u^n(t)|^2\, dx + 2\int_\Omega \nabla u^n(t)\cdot\nabla\phi_n\, dx + \int_\Omega |\nabla\phi_n|^2\, dx \\ &\quad + \mathcal{H}^{N-1}\lfloor\partial\Omega_f^c\Big(S(u^n(t) + \phi_n) \setminus \Big[\bigcup_{k=1,\ldots,p} S(u^n(t_k))\Big]\Big), \end{aligned}$$

or equivalently

$$0 \le 2\int_\Omega \nabla u^n(t)\cdot\nabla\phi_n\, dx + \int_\Omega |\nabla\phi_n|^2\, dx + \mathcal{H}^{N-1}\lfloor\partial\Omega_f^c\Big(S(u^n(t)+\phi_n) \setminus \Big[\bigcup_{k=1,\ldots,p} S(u^n(t_k))\Big]\Big).$$

But, in view of Theorem 2.8 together with (3.12), we may pass to the limit in each term of the previous inequality and we obtain, upon adding $\|\nabla u(t)\|^2_{L^2(\Omega)}$,

$$\int_\Omega |\nabla u(t)|^2\, dx \le \int_\Omega |\nabla u(t) + \phi|^2\, dx + \mathcal{H}^{N-1}\lfloor\partial\Omega_f^c\Big(S(u(t) + \phi) \setminus \Big[\cup_{k=1,\ldots,p} S(u(t_k))\Big]\Big).$$

Appealing to (3.10), we obtain the sought result, whenever $\mathcal{H}^{N-1}(S(\phi)) < \infty$. But the result is obvious if $\mathcal{H}^{N-1}(S(\phi)) = \infty$. □

Remark 3.6 *Fix $t \notin I_\infty$. In view of the definition of $u^n(t) = u^n(t^n_k)$ for $t \in [t^n_k, t^n_{k+1})$, $u^n(t)$ satisfies the minimality property of Remark 3.2 with $u^n(t) = g^n(t)$ on $\Omega' \setminus \overline{\Omega}$. Then, if we denote by $\hat{u}(t)$ the $SBV(\Omega')$–limit of a subsequence of $\{u^n(t)\}$, $\hat{u}(t)$ will satisfy the conclusions of both Lemma 3.3 and 3.4. This is immediately seen upon reproducing the proofs of those lemmas in the setting of Remark 2.7 (note that $g^n(t) \to g(t)$ in $H^1(\Omega')$).*

Actually, the result of Lemma 3.4 can be strengthened in the case of $\hat{u}(t)$ (with the same proof) to conclude that $\hat{u}(t)$ minimizes

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus \big[\bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S(u(\tau)) \cup S(\hat{u}(t)) \cup \partial\Omega_f\big]\Big) \tag{3.13}$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$. We have added in (3.13) the jump set of $\hat{u}(t)$ to the union of all jump sets up to time t.

Of course, the converging subsequence potentially depends upon t, so that this may not be a convenient definition of an extension of $u(t)$ for $t \notin I_\infty$, although we shall see below that such is not the case.

We now derive elementary estimates on the total energy at time t, that is

$$\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2\, dx + \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S(u(\tau)) \setminus \partial\Omega_f\Big). \tag{3.14}$$

We also define the corresponding total energy for $u^n(t)$, namely

$$\mathcal{E}^n(t) := \int_\Omega |\nabla u^n(t)|^2\, dx + \mathcal{H}^{N-1}\Big(\bigcup_{\substack{\tau\in I_\infty\\ \tau\le t}} S(u^n(\tau)) \setminus \partial\Omega_f\Big).$$

The following lemma then holds:

Lemma 3.7 *For any $t \in I_\infty$,*

$$\mathcal{E}(t) \le \mathcal{E}(0) + 2\liminf_{n\to\infty} \Big[\int_0^t \Big(\int_\Omega \nabla u^n(s)\cdot\nabla\dot{g}(s)\, dx\Big)\, ds\Big] \tag{3.15}$$

and

$$\mathcal{E}(t) \ge \mathcal{E}(0) + 2\limsup_{n\to\infty} \Big[\sum_{i=0}^{k(n)} \int_\Omega \nabla u(t^n_{i+1})\cdot\Big(\int_{t^n_i}^{t^n_{i+1}} \nabla\dot{g}(s)\, ds\Big)\, dx\Big], \tag{3.16}$$

where $k(n)$ is such that $t^n_{k(n)+1} = t$. (Recall that, for each n, $t^n_i \nearrow$ with i.) Furthermore, $\mathcal{E}(t)$ has no negative jumps.

Proof. We recall (3.4,3.5) with $t = t^n_{i+1}$, namely

$$\begin{aligned}\mathcal{E}_n(t) &= \|\nabla u^n(t)\|^2_{L^2(\Omega)} + \mathcal{H}^{N-1}\Big(\Big[\bigcup_{\substack{\tau\le t\\ \tau\in I_\infty}} S(u^n(\tau))\Big] \setminus \partial\Omega_f\Big)\\ &\le \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u^n(s)\cdot\nabla\dot{g}(s)\, dxds + \omega(\Delta_n)\cdot\int_0^1 \|\nabla\dot{g}(s)\|_{L^2(\Omega)}\, ds,\end{aligned} \tag{3.17}$$

and pass to the limit in n. The $SBV-$ compactness theorem permits us to pass to the lim–inf in the first term and to the limit in each additional term, except for the term

$$\mathcal{H}^{N-1}\Big(\Big[\bigcup_{\substack{\tau\le t\\ \tau\in I_\infty}} S(u^n(\tau))\Big] \setminus \partial\Omega_f\Big)$$

for which we appeal to Lemma 3.1.

For the second inequality, we take $v \equiv u(t) + g(s) - g(t)$ as a competitor for $u(s)$ in Lemma 3.4, and get

$$\begin{aligned}\mathcal{E}(s) &\leq \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{N-1}\Big(S(u(t)) \setminus \Big(\Big[\bigcup_{\substack{\tau\leq s\\ \tau\in I_\infty}} S(u(\tau))\Big] \cup \partial\Omega_f\Big)\Big) \\ &\quad + \mathcal{H}^{N-1}\Big(\Big[\bigcup_{\substack{\tau\leq s\\ \tau\in I_\infty}} S(u(\tau))\Big] \setminus \partial\Omega_f\Big) \\ &\leq \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{N-1}\Big(\Big[\bigcup_{\substack{\tau\leq t\\ \tau\in I_\infty}} S(u(\tau))\Big] \setminus \partial\Omega_f\Big)\end{aligned}$$

so that

$$\begin{aligned}\mathcal{E}(t) - \mathcal{E}(s) &\geq \int_\Omega (|\nabla u(t)|^2 - |\nabla v|^2)\, dx \\ &= -2\int_\Omega \nabla u(t)\cdot\nabla(g(s)-g(t))\, dx - \int_\Omega |\nabla(g(s)-g(t))|^2\, dx.\end{aligned}$$

Since $g \in W^{1,1}((0,1); H^1(\Omega))$, we obtain the following lower bound:

$$\mathcal{E}(t) - \mathcal{E}(s) \geq 2\int_\Omega \nabla u(t)\cdot\nabla\Big(\int_s^t \dot{g}(\sigma)\, d\sigma\Big)\, dx - o(s-t) \tag{3.18}$$

from which we immediately infer, since $\|\nabla u(t)\|_{L^2(\Omega)} \leq C$ (see (3.7)) and $\nabla\dot{g}$ belongs to $L^1((0,1); H^1(\Omega))$, that $\mathcal{E}(t)$ cannot experience a negative jump.

Summing (3.18) for $s = t_i^n$, $t = t_{i+1}^n$, $t_i^n, t_{i+1}^n \in I_n, i \in \{0, ..., k(n)\}$ with $k(n)$ such that $t_{k(n)+1}^n = t$, and letting $n \nearrow \infty$, we get (3.16). □

3.2 The continuum extension

We now proceed to extend u to all $t \in [0,1]$. For each $t \in [0,1] \setminus I_\infty$, we consider an increasing sequence $t_n \in I_\infty \nearrow t$. Recalling (3.7), and invoking once again the SBV–compactness theorem, we obtain that a subsequence of $\{u(t_n)\}$, still denoted by $\{u(t_n)\}$, $SBV(\Omega')$–converges to some function labelled $u(t) \in SBV(\Omega')$.

Lemma 3.8 *Define the crack*

$$\Gamma(t) := \bigcup_{\substack{\tau\in I_\infty\\ \tau\leq t}} S(u(\tau)).$$

The function ∇u belongs to $L^\infty((0,1); L^2(\Omega'))$ and $\mathcal{H}^{N-1}(\Gamma(t)) < \infty$. Further, for all $t \in [0,1]$, we have

$$\int_\Omega |\nabla u(t)|^2\, dx \leq \int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t) \cup \partial\Omega_f)), \tag{3.19}$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$, while

$$S(u(t)) \subset \Gamma(t), \text{ up to a set of } \mathcal{H}^{N-1} - \text{measure } 0. \tag{3.20}$$

Finally, for a.e. $t \in [0,1]$, $u^n(t)$ $SBV(\Omega')$–converges to $u(t)$ and $\nabla u^n(t) \to \nabla u(t)$ in $L^2(\Omega)$.

Proof. That $\mathcal{H}^{N-1}(\Gamma(t)) < \infty$ comes from the fact that, for each $t \notin I_\infty$, there exists $t' \in I_\infty$ with $t' > t$, together with (3.7).

We argue as in the proof of Lemma 3.3: invoking Remark 2.7, the absolutely continuity of $g(t)$ in $H^1(\Omega)$, and Corollary 2.10, we conclude that $\nabla u(t_n) \to \nabla u(t)$ in $L^2(\Omega)$ and that $u(t)$ is a minimizer for its own jump set.

Further, according to Lemma 3.4,

$$\int_\Omega |\nabla u(t_n)|^2 \, dx \le \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t_n) \cup \partial\Omega_f)) \tag{3.21}$$

among all v in $\{v \in SBV(\Omega') : v = g(t_n) \text{ on } \Omega' \setminus \overline{\Omega}\}$.

For any $v \in SBV(\Omega')$ with $v = g(t)$ on $\Omega' \setminus \overline{\Omega}$, take $v_n = v - g(t) + g(t_n)$, so that $v_n = g(t_n)$ on $\Omega' \setminus \overline{\Omega}$, and $\nabla v_n \to \nabla v$ in $L^2(\Omega')$ while $S(v_n) \subset S(v)$. Inserting v_n as a test function in (3.21), we a fortiori get

$$\int_\Omega |\nabla u(t_n)|^2 \, dx \le \int_\Omega |\nabla v_n|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t_n) \cup \partial\Omega_f)),$$

and passing to the limit in n yields

$$\int_\Omega |\nabla u(t)|^2 \, dx \le \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t) \cup \partial\Omega_f)), \tag{3.22}$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$, since $\Gamma(t) = \bigcup_{t_n \le t} \Gamma(t_n)$.

Finally,

$$S(u(t)) \subset \Gamma(t) \cup N(t), \text{ with } \mathcal{H}^{N-1}(N(t)) = 0. \tag{3.23}$$

Otherwise, there exists an $x \in S(u(t)) \setminus \Gamma(t)$ such that $S(u(t))$ has $\mathcal{H}^{N-1}$ density bounded from below by, say $\alpha > 0$, at x, while $\Gamma(t)$ has $\mathcal{H}^{N-1}$ density 0 at x (see [13], Theorems 1 and 2 in Section 2.3), so that there exists a small open ball $B(x,r)$ such that

$$\mathcal{H}^{N-1}(B(x,r) \cap S(u(t)) = \mathcal{H}^{N-1}(\overline{B(x,r)} \cap S(u(t)) \ge \alpha r^{N-1}, \tag{3.24}$$

while

$$\mathcal{H}^{N-1}(B(x,r) \cap \Gamma(t)) = \mathcal{H}^{N-1}(\overline{B(x,r)} \cap \Gamma(t)) < 1/2\alpha r^{N-1}.$$

Since $S(u(t_n)) \subset \Gamma(t)$,

$$\mathcal{H}^{N-1}(B(x,r) \cap S(u(t_n)) < 1/2\alpha r^{N-1}.$$

But, by virtue of the SBV–compactness theorem,

$$\mathcal{H}^{N-1} \lfloor S(u(t)) \le \text{weak*–}\lim \mathcal{H}^{N-1} \lfloor S(u(t_n)),$$

so that, since $B(x,r)$ is open,

$$\mathcal{H}^{N-1}(B(x,r) \cap S(u(t)) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(B(x,r) \cap S(u(t_n)) \le 1/2\alpha r^{N-1},$$

which contradicts (3.24) and establishes (3.23).

As noted in (3.13) in Remark 3.6 above, for each $t \notin I_\infty$, a (potentially) t–dependent subsequence $\{u^{n_t}(t)\}$ of $\{u^n(t)\}$ $SBV(\Omega')$–converges to $\hat{u}(t)$ satisfying

$$\int_\Omega |\nabla \hat{u}(t)|^2 \, dx \le \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus (\Gamma(t) \cup S(\hat{u}(t)) \cup \partial\Omega_f)),$$

among all v in $\{v \in SBV(\Omega') : v = g(t) \text{ on } \Omega' \setminus \overline{\Omega}\}$, hence in particular, $\hat{u}(t)$ minimizes

$$\int_\Omega |\nabla v|^2 \, dx$$

among all $v \in SBV(\Omega')$ with $v = g(t)$ on $\Omega' \setminus \overline{\Omega}$ and $S(v) \subset \Gamma(t) \cup S(\hat{u}(t)) \cup \partial\Omega_f$.

But then $u(t)$ is an admissible test function, and since $\int_\Omega |\nabla v|^2 \, dx$ is strictly convex in v, we conclude that $u(t) \equiv \hat{u}(t)$, provided that

$$\int_\Omega |\nabla u(t)|^2 \, dx \leq \int_\Omega |\nabla \hat{u}(t)|^2 \, dx. \tag{3.25}$$

The proof of (3.25) is that used in [15]. We reproduce it here for the reader's convenience.

Define, for $t \in [0, 1]$,

$$\Gamma^n(t) := \bigcup_{\substack{\tau \in I_n \\ \tau \leq t}} S(u^n(\tau)).$$

and note that, by virtue of the second bound in (3.2),

$$l^n(t) := \mathcal{H}^{N-1}(\Gamma^n(t))$$

are uniformly bounded increasing function of t on $[0, 1]$, so that by Helly's theorem, there exists a monotone increasing function $\lambda(t)$ on $[0, 1]$ such that a subsequence of $l^n(t)$ (still indexed by n) converges pointwise to $\lambda(t)$ on $[0, 1]$; we also take the (potentially) t–dependent subsequence $\{n_t\}$ as a subsequence of that subsequence. Denote by H the (at most countable) set of discontinuity points of $\lambda(t)$. Then, for $t \notin H$, consider $t_p \in I_\infty \nearrow t$. We test the minimality of $u^n(t_p)$ with $u^{n_t}(t) + g^{n_t}(t_p) - g^{n_t}(t)$ and obtain

$$\int_\Omega |\nabla u^{n_t}(t_p)|^2 dx \leq \int_\Omega |\nabla(u^{n_t}(t) + g^{n_t}(t_p) - g^{n_t}(t))|^2 dx + \mathcal{H}^{N-1}(S(u^{n_t}(t)) \setminus (\Gamma^{n_t}(t_p) \cup \partial\Omega_f)),$$

or still,

$$\begin{aligned}
\int_\Omega |\nabla u^{n_t}(t_p)|^2 dx \quad &\leq \quad \int_\Omega |\nabla(u^{n_t}(t) + g^{n_t}(t_p) - g^{n_t}(t))|^2 \, dx \\
&\qquad\qquad + \mathcal{H}^{N-1}(\Gamma^{n_t}(t)) \setminus (\Gamma^{n_t}(t_p) \cup \partial\Omega_f)) \\
&= \quad \int_\Omega |\nabla(u^{n_t}(t) + g^{n_t}(t_p) - g^{n_t}(t))|^2 \, dx \\
&\qquad\qquad + \mathcal{H}^{N-1}(\Gamma^{n_t}(t) \cup \partial\Omega_f) - \mathcal{H}^{N-1}(\Gamma^{n_t}(t_p) \cup \partial\Omega_f),
\end{aligned}$$

where the last equality holds since $\Gamma^{n_t}(t) \supset \Gamma^{n_t}(t_p)$.

We now pass to the limit in n_t and obtain, since $\nabla u^{n_t}(t_p) \to \nabla u(t_p)$ strongly in $L^2(\Omega')$ by Lemma 3.3 while, as in Lemma 3.3 (see Remark 3.6), $\nabla u^{n_t}(t) \to \nabla \hat{u}(t)$ strongly in $L^2(\Omega')$,

$$\int_\Omega |\nabla u(t_p)|^2 dx \leq \int_\Omega |\nabla(\hat{u}(t) + g(t_p) - g(t))|^2 dx + \lambda(t) - \lambda(t_p).$$

Lettin $t_p \nearrow t$ and using the definition of $u(t)$ as well as the continuity of λ at t yields (3.25).

Thus $\hat{u}(t) \equiv u(t), t \notin H$, and the limit $\hat{u}(t)$ does not depend upon the choice of a specific t–dependent subsequence and the whole (sub)sequence of $\{u^n\}$ determined by the convergence of the $u^n(t)$ for $t \in I_\infty$ (and by that of l^n to λ) is such that, for all $t \in [0, 1] \setminus H$,

$u^n(t)$ $SBV(\Omega')$–converges to $u(t)$, while, as in Lemma 3.3, $\nabla u^n(t) \to \nabla u(t)$ strongly in $L^2(\Omega')$.

That ∇u belongs to $L^\infty((0,1); L^2(\Omega'))$ is now an immediate consequence of the lower semicontinuity of the relevant norms with respect to SBV–convergence.

This, together with the minimality property (3.22) and (3.23), completes the proof of the lemma.

□

It remains to investigate the energy $\mathcal{E}(t)$ associated to $u(t)$, defined as in (3.14).

Lemma 3.9 *$\mathcal{E}(t)$ is absolutely continuous on $[0,1]$ and is given by*

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s)\, dx\, ds.$$

Proof. In view of the strong convergence of $\nabla u(t_n)$ to $\nabla u(t)$, and of the monotonicity of $\Gamma(t)$, it is immediate that

$$\mathcal{E}(t_n) \to \mathcal{E}(t)$$

whenever $t_n (\in I_\infty) \nearrow t$. By virtue of Lemma 3.7, and since $\int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s)\, dx\, ds$ is continuous in t, it suffices to show that, for any $t \in I_\infty$,

$$\liminf_{p\to\infty} \Big[\int_0^t \Big(\int_\Omega \nabla u^p(s) \cdot \nabla \dot{g}(s)\, dx \Big)\, ds \Big] = \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s)\, dx\, ds \tag{3.26}$$

as well as that

$$\limsup_{p\to\infty} \Big[\sum_{i=0}^{k(p)} \int_\Omega \nabla u(t_{i+1}^p) \cdot \Big(\int_{t_i^p}^{t_{i+1}^p} \nabla \dot{g}(s)\, ds \Big)\, dx \Big] = \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s)\, dx\, ds, \tag{3.27}$$

with $t_{k(p)+1}^p = t$.

We first establish (3.26). In view of the convergence of $\nabla u^p(t)$ to $\nabla u(t)$ in $L^2(\Omega)$ for each t established in Lemma 3.8, together with the uniform bound on $\|\nabla u^p(t)\|_{L^2(\Omega')}$(see (3.7)), we immediately conclude that

$$\int_0^t \Big(\int_\Omega \nabla u^p(s) \cdot \nabla \dot{g}(s)\, dx \Big)\, ds \to \int_0^t \Big(\int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s)\, dx \Big)\, ds,$$

hence (3.26).

In order to establish (3.27), we will prove that, for $s \le t,\ s, t \in [0,1]$,

$$\|\nabla u(s) - \nabla u(t)\|_{L^2(\Omega)} \le \sqrt{\gamma(s,t)} + \int_s^t \|\nabla \dot{g}(s)\|_{L^2(\Omega)}\, ds, \tag{3.28}$$

where

$$\gamma(s,t) := \mathcal{H}^{N-1}(\Gamma(t) \setminus [\Gamma(s) \cup \partial\Omega_f]).$$

Postponing for a moment the proof of (3.28), we now prove (3.27). We define the crack length

$$l(t) := \mathcal{H}^{N-1}(\Gamma(t) \setminus \partial\Omega_f),$$

and observe that it is a bounded and increasing function of t. We now take $\varepsilon > 0$, and choose $\delta > 0$ such that $\int_s^t \|\nabla \dot{g}(\sigma)\|_{L^2(\Omega)} \, d\sigma < \varepsilon$ for $|t - s| < \delta$. We then choose a finite set A such that

$$\sum_{t \in I_\infty \setminus A} [l(t)] < \varepsilon,$$

which is always possible because $l(t)$ has bounded total variation and at most a countable number of jumps.

For a fixed $t \in I_\infty$, we now choose n large enough so that $[0, t] = \bigcup_{i=0}^{k(n)} [t_i^n, t_{i+1}^n]$ with $t_i^n \in I_n$, $t_{k(n)+1}^n = t$, and also such that

i) $|t_{i+1} - t_i| < \delta$;

ii) the integral of $\|\nabla \dot{g}\|_{L^2(\Omega)}$ over the union of intervals $[t_i, t_{i+1}]$ that intersect A is less than ε.

So, if $[t_i, t_{i+1}] \cap A = \emptyset$, (3.28), together with the first property of our partition $[t_i, t_{i+1}]$, implies that, for all $s \le t \in [t_i, t_{i+1}]$,

$$\|\nabla u(s) - \nabla u(t)\|_{L^2(\Omega)} \le \varepsilon + \sqrt{\varepsilon}. \tag{3.29}$$

We now aim to show (3.27). Summing over i, we obtain

$$\begin{aligned}
&\Big| \sum_{i=0}^{k(p)} \int_\Omega \nabla u(t_{i+1}^p) \cdot \Big(\int_{t_i^p}^{t_{i+1}^p} \nabla \dot{g}(s) \, ds \Big) \, dx - \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s) \, dx \, ds \Big| \\
\le\; & \sum_{i=0}^{k(p)} \int_{t_i^p}^{t_{i+1}^p} \Big| \int_\Omega (\nabla u(t_{i+1}^p) - \nabla u(s)) \cdot \nabla \dot{g}(s) \, dx \Big| \, ds \\
\le\; & \sum_{i=0}^{k(p)} \int_{t_i^p}^{t_{i+1}^p} \|\nabla u(t_{i+1}^p) - \nabla u(s)\|_{L^2(\Omega)} \|\nabla \dot{g}(s)\|_{L^2(\Omega)} \, ds \\
=\; & \sum_{\substack{i=0 \\ [t_i^p, t_{i+1}^p] \cap A = \emptyset}}^{k(p)} \int_{t_i^p}^{t_{i+1}^p} \|\nabla u(t_{i+1}^p) - \nabla u(s)\|_{L^2(\Omega)} \|\nabla \dot{g}(s)\|_{L^2(\Omega)} \, ds \\
&+ \sum_{\substack{i=0 \\ [t_i^p, t_{i+1}^p] \cap A \neq \emptyset}}^{k(p)} \int_{t_i^p}^{t_{i+1}^p} \|\nabla u(t_{i+1}^p) - \nabla u(s)\|_{L^2(\Omega)} \|\nabla \dot{g}(s)\|_{L^2(\Omega)} \, ds \\
\le\; & (\varepsilon + \sqrt{\varepsilon}) \sum_{\substack{i=0 \\ [t_i^p, t_{i+1}^p] \cap A = \emptyset}}^{k(p)} \int_{t_i^p}^{t_{i+1}^p} \|\nabla \dot{g}(s)\|_{L^2(\Omega)} \, ds \\
&+ 2 \sup_t \|\nabla u(t)\|_{L^2(\Omega)} \sum_{\substack{i=0 \\ [t_i^p, t_{i+1}^p] \cap A \neq \emptyset}}^{k(p)} \int_{t_i^p}^{t_{i+1}^p} \|\nabla \dot{g}(s)\|_{L^2(\Omega)} \, ds \\
\le\; & (\varepsilon + \sqrt{\varepsilon}) \int_0^t \|\nabla \dot{g}(s)\|_{L^2(\Omega)} \, ds \;+\; 2\varepsilon \sup_t \|\nabla u(t)\|_{L^2(\Omega)},
\end{aligned} \tag{3.30}$$

where we have used (3.29) in the second to last inequality and the second property of our partition $[t_i, t_{i+1}]$ in the last inequality.

But, from Lemma 3.8, $\sup_t \|\nabla u(t)\|_{L^2(\Omega)} \leq C < \infty$, so that by letting ε tend to 0, the above string of inequalities implies (3.27).

We now continue where we left off, and show (3.28). Consider

$$J := \inf_v \Big\{ \int_\Omega |\nabla v|^2 \, dx : v \in SBV(\Omega');\ v = g(s) \text{ on } \Omega' \setminus \overline{\Omega};\ \mathcal{H}^{N-1}(S(v) \backslash \Gamma(t)) = 0 \Big\}.$$

This infimum is attained. Indeed, take a minimizing sequence $\{v_n\}$. Since $\mathcal{H}^{N-1}(\Gamma(t)) < \infty$ (see Lemma 3.8), and since $g \in L^\infty(\Omega')$, we can apply the SBV–compactness theorem to a subsequence of $\{v_n\}$, still indexed by n. Thus v_n $SBV(\Omega')$–converges to v. Now an argument identical to that which led to (3.23) would demonstrate that $\mathcal{H}^{N-1}(S(v) \backslash \Gamma(t)) = 0$.

We denote that minimum by $w(s,t)$. Then $u(t) - w(s,t)$ is the solution to

$$K := \inf_v \Big\{ \int_\Omega |\nabla v|^2 \, dx : v \in SBV(\Omega');\ v = g(t) - g(s) \text{ on } \Omega' \setminus \overline{\Omega};\ \mathcal{H}^{N-1}(S(v) \backslash \Gamma(t)) = 0 \Big\}.$$

Indeed, if $\zeta \in SBV(\Omega')$ is such that $\zeta = 0$ on $\Omega' \setminus \overline{\Omega}$, while $\mathcal{H}^{N-1}(S(\zeta) \backslash \Gamma(t)) = 0$, then, since $u(t)$ satisfies (3.19),

$$\int_\Omega \nabla u(t) \cdot \nabla \zeta \, dx = 0,$$

and since $w(s,t) + \zeta$ is an admissible competitor in J,

$$\int_\Omega \nabla w(s,t) \cdot \nabla \zeta \, dx = 0. \tag{3.31}$$

Thus,

$$\int_\Omega |\nabla (u(t) - w(s,t))|^2 \, dx \leq \int_\Omega |\nabla (u(t) - w(s,t) + \zeta)|^2 \, dx.$$

Taking $g(t) - g(s)$ as a test function in K yields

$$\|\nabla (u(t) - w(s,t))\|_{L^2(\Omega)} \leq \|\nabla (g(t) - g(s))\|_{L^2(\Omega)}. \tag{3.32}$$

Now,

$$\|\nabla u(s)\|^2_{L^2(\Omega)} - \|\nabla w(s,t)\|^2_{L^2(\Omega)} = \int_\Omega \nabla (u(s) - w(s,t)) \cdot \nabla (u(s) + w(s,t)) \, dx.$$

But, since $\mathcal{H}^{N-1}(S(u(s)) \backslash \Gamma(t)) = 0$, $u(s)$ is an admissible test function in J, so that, according to (3.31) with $\zeta \equiv u(s) - w(s,t)$,

$$\int_\Omega \nabla w(s,t) \cdot \nabla (u(s) - w(s,t)) \, dx = 0,$$

hence

$$\begin{aligned} \|\nabla u(s)\|^2_{L^2(\Omega)} - \|\nabla w(s,t)\|^2_{L^2(\Omega)} &= \int_\Omega \nabla (u(s) - w(s,t)) \cdot \nabla (u(s) - w(s,t)) \, dx \\ &= \|\nabla u(s) - \nabla w(s,t)\|^2_{L^2(\Omega)}. \end{aligned}$$

We now invoke the minimality property (3.19) of $u(s)$ which implies, since $w(s,t) = g(s)$ on $\Omega' \setminus \overline{\Omega}$, that

$$\|\nabla u(s)\|^2_{L^2(\Omega)} - \|\nabla w(s,t)\|^2_{L^2(\Omega)} \leq \gamma(s,t),$$

or still, in view of the preceding equality,

$$\|\nabla u(s) - \nabla w(s,t)\|^2_{L^2(\Omega)} \leq \gamma(s,t). \tag{3.33}$$

Collecting (3.32,3.33), we conclude to (3.28). □

Recalling Lemmas 3.8 and 3.9, we note that there is nothing special about the interval of study, namely $[0,1]$. We have therefore established the existence result given in Theorem 1.1 for the model of fracture evolution introduced in [14].

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